III. Hyperelastic Materials

Hyperelasticity, also known as nonlinear elasticity, is an essential branch of continuum mechanics that provides a mathematical framework for describing materials that undergo large elastic deformations. Unlike linear elastic materials, which exhibit small, proportional deformations under applied forces, hyperelastic materials can experience large strains and still recover their original shape once the applied load is removed. This makes hyperelasticity particularly important for modeling rubber-like materials, biological tissues, polymers, and foams, which experience significant deformations in real-world applications.

In classical continuum mechanics, materials are often modeled as *linear elastic* material using Hooke's Law, where stress is linearly related to strain. This approach, however, is only valid for small deformations. For larger deformations, the linear approximation fails to capture the complex, nonlinear relationship between stress and strain. In Chapter 2, we address the *simple elastic material*, where the stress only depends on the deformation gradient **F**. In this chapeter, we introduce Hyperelasticity, which addresses this limitation by using a more general approach to describe materials under finite (large) deformations, where the stress-strain relationship is nonlinear and time-independent.

At the core of hyperelasticity is the concept of the strain energy density function, $W(\mathbf{F})$. The strain energy density function represents the amount of energy stored in the material per unit volume as it deforms. For a given deformation, this function describes how the material resists deformation and returns to its original configuration when the external forces are removed. Different forms of the strain energy function correspond to different material models, which can be used to capture the behavior of specific materials. Generally, we use a functional relationship as a constitutive equation, enabling us to specify the stress components in terms of other field functions, such as strain and temperature. A constitutive equation determines the state of stress at any point x of a continuum body at time t and is necessarily different for different types of continuous bodies.

For most cases, we follow the so-called phenomenological approach, describing the macroscopic nature of materials as continua. The phenomenological approach is mainly concerned with fitting mathematical equations to experimental data and is particularly successful in solid mechanics (such as simple elastic materials and elastoplasticity). However, phenomenological modeling cannot relate the mechanism of deformation to the underlying physical (microscopic) structure of the materials and the multiple field coupling effects in many advanced materials.

Hyperelasticity provides a powerful theoretical framework for understanding the mechanical behavior of materials that experience large deformations. By using a strain

energy density function and constitutive models, hyperelasticity allows engineers and material scientists to model and predict the stress-strain behavior of complex materials under a wide range of loading conditions. This makes it an indispensable tool in fields ranging from automotive and aerospace engineering to biomechanics and soft tissue modeling. There are several important references served as the milestone of the development of mechanics of hyperelasticity.

- **Ogden, R. W.** (1972). *Large Deformation Isotropic Elasticity: On the Correlation of Theory and Experiment for Incompressible Rubberlike Solids*. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 326(1567), 565-584.
- **Mooney, M.** (1940). *A Theory of Large Elastic Deformations*. Journal of Applied Physics, 11(9), 582-592.
- **Rivlin, R. S.** (1948). *Large Elastic Deformations of Isotropic Materials. IV. Further Developments of the General Theory*. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 241(835), 379-397.
- **Arruda, E. M., & Boyce, M. C.** (1993). *A Three-Dimensional Constitutive Model for the Large Stretch Behavior of Rubber Elastic Materials.* Journal of the Mechanics and Physics of Solids, 41(2), 389-412.

Overview: Hyperelasticity material (Green elastic materials)

A free-energy function exists for the deformation(finite) of the material

❒ Simple elastic material $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{F})$—Frame indifference + Material symmetry

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{B}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2$$

$\alpha_0, \alpha_1, \alpha_2$ are the function of $\mathrm{I}_{\mathbf{B}}, \mathrm{II}_{\mathbf{B}}, \mathrm{III}_{\mathbf{B}}$

❒ Hyperelastic materials—Helmholtz free energy function

Conservation of energy: $\boldsymbol{\sigma} : \mathbf{D} + r - \mathrm{div}\boldsymbol{q} = \dot{e}$

(*e*: internal energy, *s*: entropy, *r:* heat source, $\boldsymbol{q}$*:* heat flux)

Clausius-Duhem Inequality: $\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\left(\frac{\boldsymbol{q}}{T}\right)$

Combining the equation above, we have $\left(T - \frac{\partial e}{\partial s}\right)\dot{s} + \left(\boldsymbol{\sigma} : \mathbf{D} - \frac{\partial e}{\partial \mathbf{F}} : \dot{\mathbf{F}}\right) - \frac{1}{T}\boldsymbol{q} \cdot \nabla T \geqslant 0$

1. Coleman-Noll procedure

Constitutive relations cannot be selected arbitrarily. They must conform to certain

constraints imposed on them by physical laws, and they must be consistent with the structure of the material. These constraints will help immensely to reduce the set of possible forms from which all constitutive equations must be chosen. However, the relations that we will obtain are still quite general.

In this section, we start from the expression in terms of the internal energy $e(s, \mathbf{F})$, derived from the Clausius–Duhem inequality, and follow the Coleman-Noll procedure. By doing so, we can obtain the specific functional dependence of the temperature and heat flux response functions. In addition, we will show that the stress tensor can be divided into two parts: a conservative elastic part and an irreversible viscous part. The procedure followed here is due to Coleman and Noll [1] and Ziegler and McVean [2].
(*Ref:* [1] *D. Coleman and W. Noll. The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Ration. Mech. Anal., 13:167–178, 1963.*
[2] *H. Ziegler and D. McVean. On the notion of an elastic solids. In B. Broberg, J. Hult, and F. Niordson, editors, Recent Progress in Applied Mechanics, 1967.*)

The internal energy $e=e(s, \mathbf{F})$ is the function of s and $\mathbf{F}$. By taking the derivative of the internal energy, the Clausius-Duhem inequality is written as:

$$\left(T-\frac{\partial e}{\partial s}\right)\dot{s}+\left(\boldsymbol{\sigma}:\mathbf{F}^{-\mathrm{T}}-\frac{\partial e}{\partial \mathbf{F}}\right):\dot{\mathbf{F}}-\frac{1}{T}\boldsymbol{q}\cdot\nabla T\geqslant 0 \tag{3-1}$$

The argument made by Coleman and Noll is that Eqn. (3-1) must be satisfied for every admissible process. Selecting special cases gives insight into the relation between the different continuum fields. This line of thinking is referred to as the Coleman–Noll procedure (very similar to the BAT in thermodynamics). We apply it below to obtain the functional forms for the constitutive relations of temperature, heat flux, and stress.

1) Temperature constitutive relation

Consider a process where the deformation is constant in time $\dot{\mathbf{F}}=0$, and the temperature is uniform across the body, so that $\nabla T=0$. In this case, Eq. (3-1) reduces to

$$\left(T-\frac{\partial e}{\partial s}\right)\dot{s}\geqslant 0 \tag{3-2}$$

The rate of change of entropy $\dot{s}$ can be assigned arbitrarily (e.g., by modifying an external heat source r. The sign of $\dot{s}$ is arbitrary ($\dot{s}=0, \dot{s}>0, \dot{s}<0$), Eq. (3-2) can only be satisfied every process if

$$T=\hat{T}(s,\mathbf{F})=\frac{\partial e}{\partial s} \tag{3-3}$$

We see that the derivative of the internal energy w.r.t the entropy has the same relation to the one derived from the thermal system in Chapter 2.

2) Heat flux constitutive relation

Substituting Eqn. (3-3) in Eqn. (3-1), the second law of inequality reduces to

$$\left(\boldsymbol{\sigma}:\mathbf{F}^{-\mathrm{T}}-\frac{\partial e}{\partial \mathbf{F}}\right):\dot{\mathbf{F}}-\frac{1}{T}\boldsymbol{q}\cdot\nabla T\geqslant 0$$

Again, considering a process where the deformation is constant ($\dot{\mathbf{F}}=0$), we have

$$-\frac{1}{T}\boldsymbol{q}\cdot\nabla T\geqslant 0 \tag{3-4}$$

This **heat condition inequality** is consistent with our physical intuition: heat flows from hot to cold, flowing against a temperature gradient. For Fourier's law of heat conduction, we have $\boldsymbol{q}=-k\nabla\mathrm{T}$ ($k>0$), which guarantees the existence of the above inequality $\boldsymbol{q}\cdot\nabla\mathrm{T}\leqslant 0$. For the gradient of temperature, we normally define the direction where temperature increases as positive.

According to the heat conduction inequality condition in Eq. (3-4), the heat flux depends on the spatial temperature gradient. Combined with the **Clausius-Duhem inequality**, we can obtain an alternative stronger form of the second law of thermodynamics, often referred to as the **Clausius-Planck inequality**, i.e.,

$$D_{int}=\boldsymbol{\sigma}:\mathbf{D}-\dot{e}+T\dot{s}=\mathbf{P}:\dot{\mathbf{F}}-\dot{e}+T\dot{s}$$

with the internal dissipation or local production of entropy $D_{int}>0$, which is required to be non-negative at all times at any particle of a body.

The non-negative internal dissipation D_{int} consists of three terms: the work conjugate pair $\boldsymbol{\sigma}:\mathbf{D}$ or $\mathbf{P}:\dot{\mathbf{F}}$, i.e., the rate of internal mechanical work (or stress power) per unit current volume, then the rate of internal energy $\dot{e}$, and the absolute temperature multiplied by the rate of entropy $T\dot{s}$. The internal dissipation is zero for reversible processes while the inequality holds for irreversible processes.

We express the Helmholtz free energy function in terms of the internal energy e, and the entropy s, $\psi=e-Ts$. We can rewrite the Clausius-Planck inequality

$$D_{int}=\boldsymbol{\sigma}:\mathbf{D}-\dot{\psi}-s\dot{T}=\mathbf{P}:\dot{\mathbf{F}}-\dot{\psi}-s\dot{T}$$

3) Cauchy stress constitutive relation

Returning to Eqn. (3-1), consider the case where the temperature is uniform across the body ($\nabla T=0$). In this case, the second law of inequality is,

$$\left(\boldsymbol{\sigma}:\mathbf{F}^{-\mathrm{T}}-\frac{\partial e}{\partial \mathbf{F}}\right):\dot{\mathbf{F}}\geqslant 0 \tag{3-5}$$

This equation must hold for any choice of $\dot{\boldsymbol{F}}$, and it is only satisfied for all $\dot{\boldsymbol{F}}$ if

$$\boldsymbol{\sigma} : \mathbf{F}^{-\mathrm{T}} = \frac{\partial e}{\partial \mathbf{F}} \tag{3-6}$$

Equation (3-6) implies that all irreversibility enters through the heat flux term in Eqn. (3-4),

$$-\frac{1}{T}\boldsymbol{q} \cdot \nabla T \geqslant 0$$

Consequently, no irreversibility is possible under uniform temperature conditions ($\nabla T = 0$). *This is not consistent with experimental observation, where the homogeneous temperature does not guarantee the reversible process.*

To proceed, we partition σ into two parts: an elastic reversible part that is a state variable and a "viscous", or dissipative part that is irreversible: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)}$,

$$\left(\boldsymbol{\sigma}^{(\mathbf{e})} : \mathbf{F}^{-\mathrm{T}} - \rho \frac{\partial e}{\partial \mathbf{F}}\right) : \dot{\mathbf{F}} + \boldsymbol{\sigma}^{(\mathbf{v})} : \boldsymbol{\Gamma} - \frac{1}{\mathrm{T}}\boldsymbol{q} \cdot \nabla \mathrm{T} \geqslant 0 \tag{3-7}$$

If we now assume that $\boldsymbol{\sigma}^{(v)}$ represents an irreversible process, then its entropy production is always positive (the dissipated energy is converted into heat, which causes the entropy to increase),

$$\boldsymbol{\sigma}^{(\mathbf{v})} : \boldsymbol{\Gamma} = \boldsymbol{\sigma}^{(\mathbf{v})} : \mathbf{D} \geqslant 0 \tag{3-8}$$

Since the last two terms in Eqn. (3-7) have a fixed sign (always positive) and (by choice of $\dot{\boldsymbol{F}}$) the first term can take on any value, the inequality can only be guaranteed to be satisfied if

$$\boldsymbol{\sigma}^{(\mathbf{e})} = \hat{\boldsymbol{\sigma}}^{(\mathbf{e})}(s, \mathbf{F}) = \frac{\partial e}{\partial \mathbf{F}} \mathbf{F}^{\mathrm{T}} \tag{3-9}$$

Furthermore, we require that the inequality in Eqn. (3-8) must be satisfied for every process. Similar to the heat flux, this inequality is insufficient to obtain an explicit form for $\boldsymbol{\sigma}^{(\mathbf{v})}$. However, it does indicate that the viscous stress must depend on the rate of deformation tensor,

$$\boldsymbol{\sigma}^{(\mathbf{v})} = \hat{\boldsymbol{\sigma}}^{(\mathbf{v})}(s, \mathbf{F}, \mathbf{D}) \tag{3-10}$$

A material for which $\boldsymbol{\sigma}^{(v)} = 0$, and for which an energy function exists, such that the stress is entirely determined by Eqn. (3-9) is called a **hyperelastic material**.

4) Entropy change in reversible and irreversible processes

Following the definition of the elastic stress in Eqn. (3-9), the Clausius–Duhem

inequality is reduced to its final form:

$$T\dot{s}^{\text{ir}} = \boldsymbol{\sigma}^{(\text{v})} : \boldsymbol{\Gamma} - \frac{1}{T}\boldsymbol{q}\cdot\nabla T \geq 0 \tag{3-11}$$

If the process is reversible, then each of the terms in Eqn. (3-11) is zero (since a sum of two positive terms is zero only if they are both zero):

$$\boldsymbol{\sigma}^{(\text{v})} : \boldsymbol{\Gamma} = 0, \qquad -\frac{1}{T}\boldsymbol{q}\cdot\nabla T = 0 \tag{3-12}$$

The rate of entropy is obtained as,

$$\dot{s} = \dot{s}^{r} = \frac{r}{T} - \frac{1}{T}\text{div}(\boldsymbol{q}) \tag{3-13}$$

5) Strain energy function of hyperelastic materials

A so-called perfectly elastic material is, by definition, a material that produces locally no entropy. We will only consider perfectly elastic hyperelastic materials in this chapter. In other words, the material should have reversible deformation with $\boldsymbol{\sigma}^{(\text{v})} = 0,\ \nabla T = 0$. In addition, we consider the process without entropy generation, $r = 0,\ q_{k,k} = 0$. In this case, the internal energy equals the strain energy.

$$\dot{e} = \dot{\varepsilon} = \boldsymbol{\sigma} : \mathbf{D} \tag{3-13}$$

The strain energy $\varepsilon = \varepsilon(\mathbf{F})$ generated by the deformation gradient $\mathbf{F}$ should be objective (frame-indifference) to construct the constitutive model for hyperelastic materials. This means for different frames in the current configuration, the amount of energy is unchanged. For a rotation tensor $\mathbf{Q}$, where $\forall \mathbf{Q} \in$ the orthogonal group,

$$\varepsilon(\mathbf{F}) = \varepsilon(\mathbf{QF}) \tag{3-14}$$

To obtain the equivalent form of Eqn. (3-14), we take a special choice for $\mathbf{Q}$, namely the transpose of the proper orthogonal tensor, $\mathbf{R}^{\mathbf{T}}$. We find that

$$\varepsilon(\mathbf{F}) = \varepsilon(\mathbf{R}^{\mathbf{T}}\mathbf{F}) = \varepsilon(\mathbf{R}^{\mathbf{T}}\mathbf{RU}) = \varepsilon(\mathbf{U}) \tag{3-15}$$

which holds for arbitrary $\mathbf{F}$. It is important to note that the relation $\varepsilon(\mathbf{F}) = \varepsilon(\mathbf{U})$ specifies the necessary and sufficient condition for the strain energy to be objective during superimposed rigid body motions.

We can also express the strain energy as a function of the left stretch tensor $\mathbf{V}$,

$$\varepsilon(\mathbf{F}) = \varepsilon(\mathbf{R}^{\mathbf{T}}\mathbf{VR}) = \varepsilon(\mathbf{V}) \tag{3-16}$$

We can see that by taking the left stretch tensor in a different frame $\overset{\star}{\mathbf{V}} = \mathbf{QVQ}^{\mathbf{T}}$. Given

that the strain energy is objective, $\overset{\star}{\varepsilon}(\overset{\star}{\mathbf{V}}) = \varepsilon(\mathbf{V})$, and the strain energy is a scalar, $\overset{\star}{\varepsilon} = \varepsilon$. Therefore, we have $\varepsilon(\mathbf{Q}^{\mathrm{T}}\mathbf{V}\mathbf{Q}) = \varepsilon(\mathbf{V})$.

The right Cauchy-Green tensor and left Cauchy-Green tensor are given by $\mathbf{C}=\mathbf{U}^2$ and $\mathbf{B}=\mathbf{V}^2$. Based on the Cayley-Hamilton theorem, any second-order tensor $\mathbf{T}$ satisfies: $\mathbf{T}^3 = \mathrm{I}_{\mathbf{T}}\mathbf{T}^2 - \mathrm{II}_{\mathbf{T}}\mathbf{T} + \mathrm{III}_{\mathbf{T}}\mathbf{I}$, where I, II, III are the invariants of $\mathbf{T}$. We may express the strain energy as

$$\varepsilon(\mathbf{F}) = \hat{\varepsilon}(\mathbf{C}) = \hat{\varepsilon}(\mathbf{B})$$

2. Isotropic hyperelastic materials

We now restrict the strain energy function by a particular property the material may possess, namely isotropy. This property is based on the physical idea that the response of materials, when studied in a stress-strain experiment, is the same in all directions. Rubber is a class of isotropic materials. In this section, we will develop the constitutive model for isotropic hyperelastic materials.

The material is isotropic as a motion of an elastic body superimposed on any particularly rotated reference configuration leads to the same strain energy function at time *t*. However, if a superimposed rigid-body rotation changes the strain energy function, the hyperelastic material is said to be anisotropic (Section 5). With this, we can express the strain energy function for isotropic hyperelastic material as,

$$\varepsilon(\mathbf{F}) = \varepsilon(\overset{\star}{\mathbf{F}}) = \varepsilon(\mathbf{F}\mathbf{H}),\ \mathbf{H} \in \text{orthogonal group (all rotations)} \qquad (3\text{-}17)$$

We now consider the equivalent form of the strain energy function $\varepsilon(\mathbf{F}) = \hat{\varepsilon}(\mathbf{C})$. For the isotropic hyperelastic material, we require the strain energy function $\hat{\varepsilon}(\mathbf{C}) = \hat{\varepsilon}(\overset{\star}{\mathbf{C}})$, with $\overset{\star}{\mathbf{C}} = \overset{\star}{\mathbf{F}}^{\mathbf{T}}\overset{\star}{\mathbf{F}}$. With reference to Eqn. (3-17),

$$\hat{\varepsilon}(\mathbf{C}) = \hat{\varepsilon}(\overset{\star}{\mathbf{F}}^{\mathbf{T}}\overset{\star}{\mathbf{F}}) = \hat{\varepsilon}(\mathbf{H}^{\mathbf{T}}\mathbf{F}^{\mathbf{T}}\mathbf{F}\mathbf{H}) = \hat{\varepsilon}(\mathbf{H}^{\mathbf{T}}\mathbf{C}\mathbf{H}) \qquad (3\text{-}18)$$

If the requirement for isotropy holds for all symmetric tensors $\mathbf{C}$ and orthogonal tensors $\mathbf{H}$, we say that the strain energy function $\hat{\varepsilon}(\mathbf{C})$ is a scalar-valued isotropic tensor function of variable $\mathbf{C}$ or simply an invariant of the symmetric tensor $\mathbf{C}$, which cannot change with rotation. We can also prove this for the left Cauchy-Green tensor $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathrm{T}}$, $\hat{\varepsilon}(\mathbf{B}) = \hat{\varepsilon}(\mathbf{H}\mathbf{B}\mathbf{H}^{T})$, which is an invariant of the symmetric tensor $\mathbf{B}$.

1) Constitutive equations in terms of invariants

If a scalar-valued tensor function is invariant under a rotation, it may be expressed in terms of the principal invariants of its arguments (e.g., ***C*** and ***B***), which is known as the representation theorem for invariants [Gurtin, 1981].

Having this in mind, the strain energy function established in Eqn. (3-18) may be expressed as a set of independent strain invariants of the symmetric Cauchy-Green tensors **C** (or **B**), namely, through I_C, II_C, III_C (or I_B, II_B, III_B). With reference to Eqn. (3-18), we can write the strain energy function,

$$\varepsilon = \varepsilon(\mathrm{I}_C, \mathrm{II}_C, \mathrm{III}_C) = \varepsilon(\mathrm{I}_B, \mathrm{II}_B, \mathrm{III}_B) \tag{3-19}$$

Since **C** and **B** have the same eigenvalues, which are the squares of the principal stretches, $\lambda_1^2, \lambda_2^2, \lambda_3^2$, we can conclude that

$$\varepsilon = \varepsilon(\lambda_1, \lambda_2, \lambda_3) \tag{3-20}$$

In order to determine constitutive equations for isotropic hyperelastic materials in terms of strain invariants, consider a differentiation of $\varepsilon(\mathbf{B}) = \varepsilon(\mathrm{I}_B, \mathrm{II}_B, \mathrm{III}_B)$ with respect to tensor **B**. We assume that $\varepsilon(\mathbf{B})$ has continuous derivatives with respect to the principal invariants I_B, II_B, III_B. Using the chain rule of differentiation, we find,

$$\frac{\partial \varepsilon(\mathbf{B})}{\partial B_{mn}} = \frac{\partial \varepsilon}{\partial \mathrm{I}_B}\frac{\partial \mathrm{I}_B}{\partial B_{mn}} + \frac{\partial \varepsilon}{\partial \mathrm{II}_B}\frac{\partial \mathrm{II}_B}{\partial B_{mn}} + \frac{\partial \varepsilon}{\partial \mathrm{III}_B}\frac{\partial \mathrm{III}_B}{\partial B_{mn}} \tag{3-21}$$

The derivative of the invariants I_B, II_B, and III_B with respect to B have the form,

$$\frac{\partial \mathrm{I}_B}{\partial B_{mn}} = \frac{\partial B_{kk}}{\partial B_{mn}} = \delta_{mn}$$

$$\frac{\partial \mathrm{II}_B}{\partial B_{mn}} = \frac{1}{2}\frac{\partial}{\partial B_{mn}}(B_{kk}B_{ll} - B_{kl}B_{lk}) = \mathrm{I}_B\delta_{mn} - B_{mn} \tag{3-22}$$

$$\frac{\partial \mathrm{III}_B}{\partial B_{mn}} = \frac{\partial}{\partial B_{mn}}(\det(\mathbf{B})) = \det(\mathbf{B})\mathbf{B}^{-T} = \mathrm{III}_B\mathbf{B}^{-1} = \mathrm{II}_B\mathbf{I} - \mathrm{I}_B\mathbf{B} + \mathbf{B}^2$$

where $\mathrm{I}_B = \mathrm{tr}(\mathbf{B})$, $\mathrm{II}_B = \frac{1}{2}[(\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}\mathbf{B}^2]$, $\mathrm{III}_B = \det(\mathbf{B})$. Substituting Eqn. (3-21) and (3-22) into (3-9), the Cauchy stress for isotropic hyperelastic materials in finite strain are,

$$\sigma_{kl} = \frac{\partial \varepsilon}{\partial x_{k,K}}x_{l,K} = \left(\frac{\partial \varepsilon}{\partial \mathrm{I}_\mathbf{B}}\frac{\partial \mathrm{I}_\mathbf{B}}{\partial B_{mn}} + \frac{\partial \varepsilon}{\partial \mathrm{II}_\mathbf{B}}\frac{\partial \mathrm{II}_\mathbf{B}}{\partial B_{mn}} + \frac{\partial \varepsilon}{\partial \mathrm{III}_\mathbf{B}}\frac{\partial \mathrm{III}_\mathbf{B}}{\partial B_{mn}}\right)\frac{\partial B_{mn}}{\partial x_{k,K}}x_{l,K} \tag{3-23}$$

Given that the left Cauchy-Green tensor is symmetric $B_{ij} = B_{ji}$, its derivative is expressed as,

$$\frac{\partial B_{mn}}{\partial x_{k,K}} = \frac{\partial}{\partial x_{k,K}}(x_{m,M}x_{n,M}) = (\delta_{mk}\delta_{MK}x_{n,M} + \delta_{nk}\delta_{MK}x_{m,M}) = \delta_{mk}x_{n,K} + \delta_{nk}x_{m,K} \quad (3\text{-}24)$$

$$\frac{\partial B_{mn}}{\partial x_{k,K}}x_{l,K} = \delta_{mk}x_{n,K}x_{l,K} + \delta_{nk}x_{m,K}x_{l,K} = \delta_{mk}B_{nl} + \delta_{nk}B_{ml}$$

Combining Eqn. (3-23) and (3-24), we deduce the Cauchy tensor as,

$$\sigma_{kl} = \left[\frac{\partial \varepsilon}{\partial \mathrm{I}_{\mathbf{B}}}\delta_{mn} + \frac{\partial \varepsilon}{\partial \mathrm{II}_{\mathbf{B}}}(\mathrm{I}_{\mathbf{B}}\delta_{mn} - B_{mn}) + \frac{\partial \varepsilon}{\partial \mathrm{III}_{\mathbf{B}}}(\mathrm{II}_{\mathbf{B}}\delta_{mn} - \mathrm{I}_{\mathbf{B}}B_{mn} + B_{mp}B_{pn})\right](\delta_{mk}B_{nl} + \delta_{nk}B_{ml})$$

$$= 2\left(\frac{\partial \varepsilon}{\partial \mathrm{I}_{\mathbf{B}}} + \frac{\partial \varepsilon}{\partial \mathrm{II}_{\mathbf{B}}}\mathrm{I}_{\mathbf{B}} + \frac{\partial \varepsilon}{\partial \mathrm{III}_{\mathbf{B}}}\mathrm{II}_{\mathbf{B}}\right)B_{kl} - 2\left(\frac{\partial \varepsilon}{\partial \mathrm{I}_{\mathbf{B}}} + \frac{\partial \varepsilon}{\partial \mathrm{I}_{\mathbf{B}}}\mathrm{I}_{\mathbf{B}}\right)B_{kn}B_{nl} + 2\frac{\partial \varepsilon}{\partial \mathrm{III}_{\mathbf{B}}}B_{kp}B_{pn}B_{nl}$$

$$= 2\left[\left(\frac{\partial \varepsilon}{\partial \mathrm{I}_{\mathbf{B}}} + \frac{\partial \varepsilon}{\partial \mathrm{II}_{\mathbf{B}}}\mathrm{I}_{\mathbf{B}} + \frac{\partial \varepsilon}{\partial \mathrm{III}_{\mathbf{B}}}\mathrm{II}_{\mathbf{B}}\right)\mathbf{B} - 2\left(\frac{\partial \varepsilon}{\partial \mathrm{II}_{\mathbf{B}}} + \frac{\partial \varepsilon}{\partial \mathrm{III}_{\mathbf{B}}}\mathrm{I}_{\mathbf{B}}\right)\mathbf{B}^2 + \frac{\partial \varepsilon}{\partial \mathrm{III}_{\mathbf{B}}}\mathbf{B}^3\right]$$

Adopting the Cayley-Hamilton theorem with $\mathbf{B}^3 = \mathrm{I}_{\mathbf{B}}\mathbf{B}^2 - \mathrm{II}_{\mathbf{B}}\mathbf{B} + \mathrm{III}_{\mathbf{B}}\mathbf{I}$,

$$\boldsymbol{\sigma} = 2\left[\frac{\partial \varepsilon}{\partial \mathrm{III}_{\mathbf{B}}}\mathrm{III}_{\mathbf{B}}\mathbf{I} + \left(\frac{\partial \varepsilon}{\partial \mathrm{I}_{\mathbf{B}}} + \frac{\partial \varepsilon}{\partial \mathrm{II}_{\mathbf{B}}}\mathrm{I}_{\mathbf{B}}\right)\mathbf{B} - \frac{\partial \varepsilon}{\partial \mathrm{II}_{\mathbf{B}}}\mathbf{B}^2\right] = \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2 \quad (3\text{-}25)$$

2) Constitutive equations in terms of principal stretches

Since the strain energy $\varepsilon(\mathbf{B})$ is defined as the unit mass in the current configuration, we can also write the strain energy per unit reference volume. Given the density in the reference and current configuration is related as $dv = jdV$ and $\rho_0 = j\rho$, the strain energy per unit reference volume is

$$\Psi(\mathbf{B}) = j\varepsilon(\mathbf{B}) \quad (3\text{-}26)$$

Combining Eqn. (3-25) and (3-26), the Cauchy stress tensor can be expressed as,

$$\boldsymbol{\sigma} = 2j^{-1}\left[\frac{\partial \Psi}{\partial \mathrm{III}_{\mathbf{B}}}\mathrm{III}_{\mathbf{B}}\mathbf{I} + \left(\frac{\partial \Psi}{\partial \mathrm{I}_{\mathbf{B}}} + \frac{\partial \Psi}{\partial \mathrm{II}_{\mathbf{B}}}\mathrm{I}_{\mathbf{B}}\right)\mathbf{B} - \frac{\partial \Psi}{\partial \mathrm{II}_{\mathbf{B}}}\mathbf{B}^2\right] \quad (3\text{-}27)$$

Since the strain energy function $\Psi(\mathbf{B})$ is an invariant, we may regard it as a function of the principal stretches, $\lambda_1, \lambda_2, \lambda_3$. We can represent the invariants of $\boldsymbol{B}$ as,

$$\mathrm{I}_{\mathbf{B}} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \mathrm{II}_{\mathbf{B}} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 \quad \mathrm{III}_{\mathbf{B}} = \lambda_1^2\lambda_2^2\lambda_3^2 \quad (3\text{-}28)$$

For a stress-free reference configuration, the strain energy function takes on the form $\Psi(1,1,1)=0$. We can also express the Cauchy stress tensor in terms of principle stretches,

$$j\sigma_\alpha = \frac{\partial \Psi}{\partial \lambda_\alpha}\lambda_\alpha \tag{3-29}$$

For example, in Neo-Hookean materials, the strain energy per unit reference volume is, $\Psi = \frac{1}{2}\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)$. Therefore, the Cauchy stress tensor is

$$j\sigma_\alpha = \frac{\partial \Psi}{\partial \lambda_\alpha}\lambda_\alpha = \mu\lambda_\alpha{}^2 \tag{3-30}$$

3. Incompressible Hyperelastic materials

Numerous polymeric materials can sustain finite strains without noticeable volume changes. Such type of material may be regarded as incompressible so that only *isochoric deformation* (等体积变形) is possible. In many cases, this is a common idealization and accepted assumption often invoked in continuum and computational mechanics. In this section, we present the constitutive foundation of incompressible hyperelastic materials.

1) Incompressible hyperelasticity

Materials that keep the volume constant throughout deformation are characterized by the incompressibility constraint.

$$j=\det(\mathbf{F})=1 \tag{3-31}$$

Generally, a material subjected to an internal constraint, of which incompressibility is the most common, is referred to as a constrained material. This is an internal constraint applied to the material in every deformation. To ensure the internal constraint is considered in the deformation, we introduce the Lagrange multiplier for the strain energy function.

$$\Pi = \psi(\mathbf{F}) - p(j-1) \tag{3-32}$$

where the strain energy Π is defined for $j=\det(\mathbf{F})=1$. The scalar p introduced in Eqn. (3-32) serves as an indeterminate Lagrange multiplier, which can be identified as a hydrostatic pressure. Note that the scalar P may only be determined from equilibrium equations and the boundary conditions. It represents a workless reaction to the kinematic constraint on the deformation field.

Differentiating Eqn. (3-32) for the deformation gradient **F**, we arrive at a general

constitutive equation for the Cauchy stress tensor,

$$j\boldsymbol{\sigma} = \frac{\partial \Pi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} = \frac{\partial \psi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} - jp\mathbf{F}^{-\mathbf{T}}\mathbf{F}^{\mathbf{T}} = \frac{\partial \psi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} - jp\boldsymbol{I} \tag{3-33}$$

Written in terms of the index notation, the Cauchy stress for incompressible hyperelastic material is,

$$\sigma_{ij} = \frac{\partial \psi}{\partial F_{iK}} F_{jK}^{T} - p\delta_{ij}$$

Multiplying the $\mathbf{F}^{-1}$ from the left-hand side, we conclude that the first Piola-Kirchhoff stress tensor takes on the form,

$$\begin{aligned} \mathbf{P} &= j\mathbf{F}^{-1}\boldsymbol{\sigma} = \mathbf{F}^{-1}\left(\frac{\partial \psi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} - jp\mathbf{I}\right) = \mathbf{F}^{-1}\frac{\partial \psi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} - jp\mathbf{F}^{-1} \\ &= \mathbf{F}^{-1}\mathbf{F}\left(\frac{\partial \psi}{\partial \mathbf{F}}\right)^{T} - jP\mathbf{F}^{-1} \\ &= \left(\frac{\partial \psi}{\partial \mathbf{F}}\right)^{T} - P\mathbf{F}^{-1} \end{aligned} \tag{3-34}$$

Noticing the first P-K stress tensor **P** (also known as the Nominal stress in the finite deformation) is the derivative of the strain energy function with respect to the deformation gradient, **F**. Expressing **P** with the index notation, where one index in the current configuration, the other in the reference configuration, namely,

$$P_{Kl} = \left(\frac{\partial \psi}{\partial F_{lK}}\right)^{T} - pF_{lK}^{-1}, \tag{3-35}$$

You might have learned a different definition of the first P-K stress tensor, which is the transpose of Eqn. (3-35). In our course, I follow strictly the derivative of the strain energy function with respect to the deformation gradient, namely,

$$\mathbf{P}^{\mathbf{T}} = \frac{\partial \psi}{\partial \mathbf{F}} - p\mathbf{F}^{-\mathbf{T}}$$

To obtain the second P-K stress tensor **S**, we multiply Eqn. (3-34) the $\mathbf{F}^{-\mathrm{T}}$ from the right-hand side,

$$\mathbf{S} = \mathbf{P}\mathbf{F}^{-\mathbf{T}} = j\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathbf{T}} = \mathbf{F}^{-1}\frac{\partial \psi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}}\mathbf{F}^{-\mathbf{T}} - jp\mathbf{F}^{-1}\mathbf{F}^{-\mathbf{T}} = \mathbf{F}^{-1}\frac{\partial \psi}{\partial \mathbf{F}} - p\mathbf{C}^{-1} \tag{3-36}$$

Written in terms of the index notation, the second P-K stress for incompressible hyperelastic material is,

$$S_{KL} = F_{iK}^{-1}\frac{\partial \psi}{\partial F_{iL}} - PC_{KL}^{-1}$$

The fundamental constitutive equations in Eqn. (3-33), (3-34), and (3-36) are the

most general forms used to define incompressible hyperelastic materials at finite strains.

2) Incompressible isotropic hyperelasticity

For the case of isotropy, we have already pointed out that the dependent of Ψ on the Cauchy-Green tensor **C** or **B** may be expressed by their invariants. However, for the incompressible case, we consider the kinematic constraint, namely, $\mathrm{III}_{\mathbf{C}}=\mathrm{III}_{\mathbf{B}}=\det(\mathbf{C})=\det(\mathbf{B})=1$. Therefore, the two principal invariants $\mathrm{I}_{\mathbf{B}}$ and $\mathrm{II}_{\mathbf{B}}$ are the only independent deformation variables.

A suitable strain energy function for incompressible isotropic hyperelastic materials is given by,

$$\Pi = \psi(\mathrm{I}_{\mathbf{B}}, \mathrm{II}_{\mathbf{B}}) - \frac{1}{2}p(\mathrm{III}_{\mathbf{B}} - 1) \tag{3-37}$$

where $P/2$ serves as an indeterminate Lagrange multiplier.

To examine the associated constitutive equation in terms of the two principal strain invariants I_B and II_B, we proceed by deriving the Cauchy stress tensor. Analogous to the procedure that led to Eqn. (3-33), we find,

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{\partial \Pi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} = \frac{\partial \psi}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} - jp\frac{\partial j}{\partial \mathbf{F}}\mathbf{F}^{\mathbf{T}} \\ &= 2\left(\frac{\partial \psi}{\partial\, \mathrm{I}_{\mathbf{B}}}\frac{\partial\, \mathrm{I}_{\mathbf{B}}}{\partial \mathbf{F}} + \frac{\partial \psi}{\partial\, \mathrm{II}_{\mathbf{B}}}\frac{\partial\, \mathrm{II}_{\mathbf{B}}}{\partial \mathbf{F}}\right)\mathbf{F}^{\mathbf{T}} - pj^2\mathbf{I} \\ &= 2\left(\frac{\partial \psi}{\partial\, \mathrm{I}_{\mathbf{B}}} + \frac{\partial \psi}{\partial\, \mathrm{II}_{\mathbf{B}}}\mathrm{I}_{\mathbf{B}}\right)\mathbf{B} - 2\frac{\partial \Psi}{\partial\, \mathrm{II}_{\mathbf{B}}}\mathbf{B}^2 - p\mathbf{I}\end{aligned} \tag{3-38}$$

which is the constitutive equation in Eqn. (3-27), in which the term $\mathrm{III}_{\mathbf{B}}\,\partial\Psi/\partial\mathrm{III}_{\mathbf{B}}$ is substituted by $-p/2$.

We can also replace $\mathbf{B}^2$ by $\mathbf{B}^{-1}$ using the Cayley-Hamilton theorem, where $\mathbf{B}^2 = \mathrm{I}_{\mathbf{B}}\mathbf{B} - \mathrm{II}_{\mathbf{B}}\mathbf{I} + \mathrm{III}_{\mathbf{B}}\mathbf{B}^{-1}$. It yields a different form of **σ**, namely,

$$\boldsymbol{\sigma} = 2\frac{\partial \psi}{\partial\, \mathrm{I}_{\mathbf{B}}}\mathbf{B} - 2\frac{\partial \psi}{\partial\, \mathrm{II}_{\mathbf{B}}}\mathbf{B}^{-1} - p\mathbf{I} \tag{3-39}$$

If we express as a function of the three principal stretches $\lambda_1, \lambda_2, \lambda_3$, we can write

$$\Pi = \psi(\lambda_1, \lambda_2, \lambda_3) - p(j-1) \tag{3-40}$$

Using the relation $\partial j/\partial \lambda_\alpha = j\lambda_\alpha^{-1}$, α=1, 2, 3, the stress tensors in Eqn. (3-33), (3-34), and (3-36) can be expressed as,

$$\sigma_\alpha = \lambda_\alpha \left(\frac{\partial \Psi}{\partial \lambda_\alpha} - p \frac{\partial j}{\partial \lambda_\alpha} \right) = \lambda_\alpha \frac{\partial \Psi}{\partial \lambda_\alpha} - p$$

$$P_\alpha = \lambda_\alpha^{-1} \sigma_\alpha = \frac{\partial \Psi}{\partial \lambda_\alpha} - \frac{p}{\lambda_\alpha} \tag{3-41}$$

$$S_\alpha = \lambda_\alpha^{-2} \sigma_\alpha = \frac{1}{\lambda_\alpha} \frac{\partial \Psi}{\partial \lambda_\alpha} - \frac{p}{\lambda_\alpha^2}$$

With the three principal Cauchy stresses σ_α, the first P-K stresses P_α, and the second P-K stresses S_α. These relations incorporate the unknown scalar P, which must be determined from the equilibrium equations and the boundary conditions. The incompressibility constraint j=1 takes the form

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{3-42}$$

leaving two independent stretches as the deformation measures.

4. Some forms of strain-energy function

From the previous sections, we have learned that the stress responses of hyperelastic materials are derived from the given strain energy function Ψ. Numerous specific forms of strain energy functions to describe the elastic properties of incompressible materials have been proposed in the literature.

This section aims to specify some forms of strain energy functions that are well established within the constitutive theory of finite elasticity and frequently employed in the literature. In particular, we present a selection of representative examples of Ψ known from rubber elasticity describing isotropic hyperelastic materials within the isothermal regime. We start by presenting suitable strain energy functions for incompressible materials and continue with some particular forms that are able to describe compressibility.

1) Mooney-Rivlin model for incompressible hyperelastic material

The classical strain energy of the Mooney-Rivlin model is often employed in the description of the behavior of isotropic rubber-like materials. Mooney derived it based on mathematical arguments employing considerations of symmetry. Using the strain invariants I_B and II_B, with the incompressible constraint $III_{\mathbf{B}} = \lambda_1^2 \lambda_2^2 \lambda_3^2 = 1$, we find,

$$\psi = C_1(\mathrm{I}_{\mathbf{B}} - 3) + C_2(\mathrm{II}_{\mathbf{B}} - 3) = C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3) \tag{3-43}$$

with $C_1=\mu_1/2$ and $C_2=\mu_2/2$, and the small strain shear modulus of hyperelastic materials is $\mu=\mu_1-\mu_2$.

The Neo-Hookean model results from Eqn. (3-43) by setting C_2=0, and using the

first principal invariant I_B, we find that,

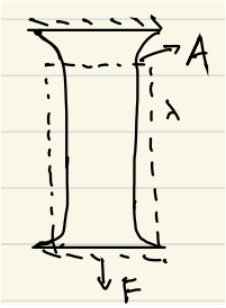
$$\psi = C_1(\mathrm{I_B} - 3) = C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \qquad (3\text{-}44)$$

with $C_1=\mu_1/2$ and the shear modulus $\mu=\mu_1$.

The Neo-Hookean model involves a single parameter only and provides a mathematically simple and reliable constitutive model for the nonlinear deformation behavior of isotropic rubber-like materials. It relies on phenomenological considerations and includes typical effects from nonlinear elasticity within the small strain domain. The important stain energy function Eqn. (3-44) can be motivated by the statistical theory in which vulcanized rubber is regarded as a three-dimensional network of long-chain molecules connected at a few points (See Appendix). In addition, the material parameter C_1 can be obtained from experimental measurements.

i) Uniaxial tensile experiment to characterize the Neo-Hookean model

For a simple with cross-section area A, we perform a uniaxial tensile experiment by stretching the sample with direction 1, leaving 2 and 3 free. The stretches in the sample are homogeneous and follow $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$.

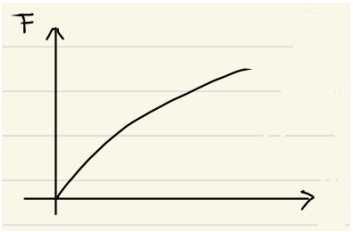


The Cauchy stresses (true stresses) in the deformed sample are $\sigma_1 = F/a = \lambda F/A,\ \sigma_2 = \sigma_3 = 0$, where a is the cross-section area in the current state. The strain energy function of the Neo-Hookean material is $\Pi(\lambda_1,\lambda_2,\lambda_3) = C_1\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right) - P\left(\lambda_1\lambda_2\lambda_3 - 1\right)$, and the Cauchy stresses are derived as,

$$\sigma_1 = 2C_1\lambda_1^2 - p = \lambda F/A \qquad (3\text{-}45)$$

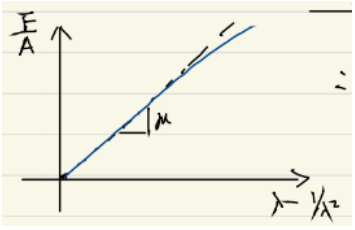
$$\sigma_2 = 2C_1\lambda_2^2 - p = 0 \qquad (3\text{-}46)$$

$$\sigma_3 = 2C_1\lambda_3^2 - p = 0 \qquad (3\text{-}47)$$



From Eqn. (3-46) or (3-47), we can determine the Lagrange multiplier $p = 2C_1\lambda_2^2 = 2C_1\lambda^{-1}$. Substituting p into Eqn. (3-45), we can have

$$\sigma_1 = 2C_1(\lambda^2 - \lambda^{-1}) \qquad (3\text{-}48)$$



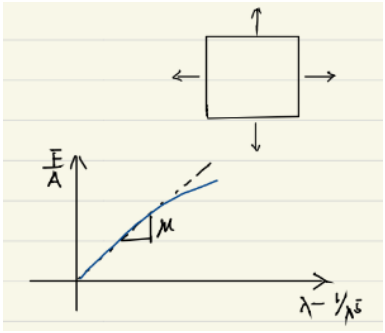
In addition, the first P-K stress (nominal stress) is,

$$P_1 = \frac{F}{A} = \frac{\sigma_1}{\lambda} = 2C_1\left(\lambda - \frac{1}{\lambda^2}\right) \qquad (3\text{-}49)$$

The ratio F/A and the stretch λ can be obtained from experiments directly. When we plot the y-axis as F/A and the x-axis as λ-$1/\lambda^2$, the slope

should be the shear modulus $\mu = 2C_1$. (Why do we use λ-$1/\lambda^2$ as the x-axis? No need to perform a nonlinear curve fitting).

ii) Biaxial tensile experiment to characterize the Neo-Hookean model



$$\lambda_1 = \lambda_2 = \lambda, \lambda_3 = \frac{1}{\lambda^2}$$

$$\Pi(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) - p\left(\lambda_1\lambda_2\lambda_3 - 1\right)$$

$$\sigma_1 = \mu\lambda_1^2 - p = \mu\lambda^2 - p \qquad \sigma_1 = \lambda F / A$$

$$\sigma_2 = \mu\lambda_2^2 - p = \mu\lambda^2 - p \qquad \sigma_2 = \lambda F / A$$

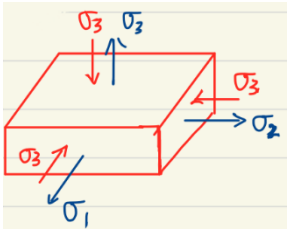
$$\sigma_3 = \mu\lambda_3^2 - p = \mu\lambda^{-4} - p \qquad \sigma_3 = 0$$

$$\therefore \lambda\frac{F}{A} = \mu\lambda^2 - \mu\lambda^{-4} = \mu(\lambda^2 - \frac{1}{\lambda^4})$$

$$\rightarrow \quad \frac{F}{A} = \mu(\lambda - \frac{1}{\lambda^5})$$

Remark: we sometimes put $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$ into the strain energy function first,

$$\psi(\lambda_1, \lambda_2, \lambda_3) = \psi(\lambda_1, \lambda_2) = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3)$$



$$\rightarrow \quad \lambda_1\frac{\partial\psi}{\partial\lambda_1} = \mu(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}) = \sigma_1 - \sigma_3$$

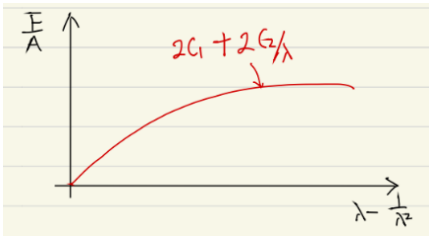
$$\rightarrow \quad \lambda_2\frac{\partial\psi}{\partial\lambda_2} = \mu(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}) = \sigma_2 - \sigma_3$$

Do we still need Lagrange multiplier P?

Answers: No. $\lambda_1\lambda_2\lambda_3 = 1$ indicates that $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$, and we have applied it to the strain energy function. In addition, since the material is incompressible, adding the hydrostatic pressure -σ_3 will not change the deformation.

iii) How to identify C_1 and C_2 for the Mooney-Rivlin model in experiments?

The Mooney-Rivlin model is characterized by two material properties, C_1 and C_2, and for a uniaxial tensile experiment, the stretches in the sample follow $\lambda_1 = \lambda$, and $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$. The Nominal stresses in the sample are $T_1 = F / A, T_2 = T_3 = 0$. Based on the constitutive equations of the Mooney-Rivlin model, the nominal stresses are expressed as, $P_\alpha = \frac{\partial \Psi}{\partial \lambda_\alpha} - \frac{p}{\lambda_\alpha}$.



$$\rightarrow \quad P_1 = 2C_1\lambda_1 - 2C_2\lambda_1^{-3} - \frac{p}{\lambda_1} = 2(C_1\lambda - C_2\lambda^{-3}) - p\lambda^{-1} = \frac{F}{A}$$

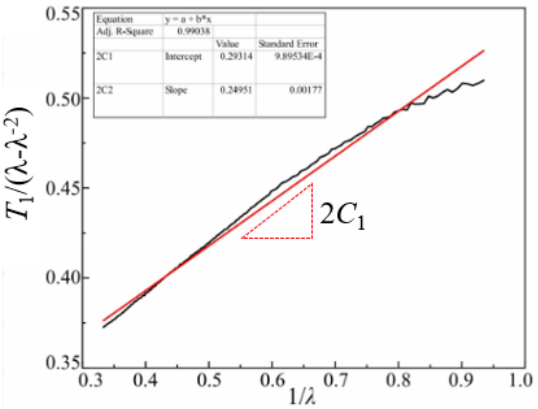
$$\rightarrow \quad P_2 = 2C_1\lambda_2 - 2C_2\lambda_2^{-3} - \frac{p}{\lambda_2} = 2(C_1\lambda^{-\frac{1}{2}} - C_2\lambda^{\frac{3}{2}}) - p\lambda^{\frac{1}{2}} = 0$$

From $P_2 = 0$, the Lagrange multiplier is $p = 2(C_1\lambda^{-1} - C_2\lambda)$. Substituting into the expression of P_1,

$$P_1 = 2C_1\left(\lambda - \lambda^2\right) - 2C_2\left(\lambda^{-3} - 1\right) = \left(2C_1 + 2\frac{C_2}{\lambda}\right)\left(\lambda - \frac{1}{\lambda^2}\right) = \frac{F}{A}$$

When we plot the y-axis as F/A and the x-axis as λ-$1/\lambda^2$, the slope is $2C_1+2C_2/\lambda$, which changes with the stretch and is no longer a constant, compared to the Neo-Hookean material.

To obtain the material parameters C_1 and C_2 directly, without using nonlinear curve fitting. We take the y-axis as $P_1/(\lambda-\lambda^{-2})$ and the x-axis as $1/\lambda$, where $S=F/A$. The plot below shows that the slope should be C_1, and the interception is C_2, which can be obtained directly from the line.

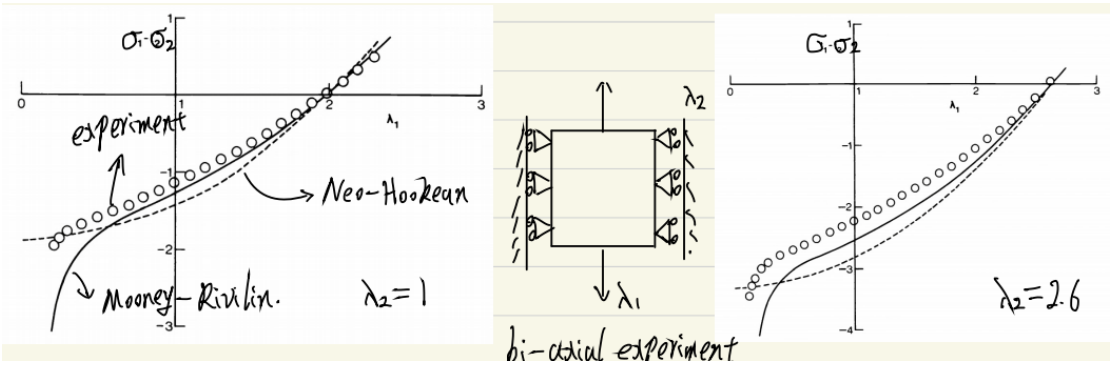


2) Ogden model (Phenomenological model: fitting, not in terms of invariants)

The Ogden model is suitable to describe materials undergoing finite strains relative

to an equilibrium state, such as biological soft tissues and solid polymers of rubber-like materials. Both of them undergo very small volume changes subjected to high hydrostatic pressures. Changing the shape of a piece of rubber is much easier than changing its volume. For the purpose of computational analyses, rubber is often regarded as incompressible with the constraint $j=\lambda_1\lambda_2\lambda_3=1$.

The Mooney-Rivlin model (or the simplified Neo-Hookean model), built upon the microscopic three-dimensional network of long-chain molecules, shows a stress-stretch relation in biaxial loading conditions unaffected by the λ_2. This pitfall is due to the simplified molecular pictures adopted in developing the model, which clearly does not capture the experimental results (see below).



We need to provide a strain function representing the effect of bi-axial loading on the stress. A very sophisticated development for simulating incompressible (rubber-like) materials in the phenomenological context is due to Ogden. The postulated strain energy is a function of the principal stretches λ_α (α=1, 2, 3), which is computationally simple and plays a crucial role in the theory of finite elasticity. Built upon the Valanis-Landel Hypothesis, which postulated that the strain energy may be written as the sum of three separate functions,

$$\psi(\lambda_1,\lambda_2,\lambda_3)=\varphi(\lambda_1)+\varphi(\lambda_2)+\varphi(\lambda_3) \tag{3-50}$$

which depend on the principal stretches. According to Ogden, such a separation may also be motivated by data obtained from biaxial experiments of Jones and Treloar. The Ogden model describes the changes of the principal stretches from the reference to the current configuration and has the form,

$$\psi(\lambda_1,\lambda_2,\lambda_3)=\psi_1(\lambda_1)+\psi_2(\lambda_2)+\psi_3(\lambda_3)=\sum_{p=1}^{N}\frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p}+\lambda_2^{\alpha_p}+\lambda_3^{\alpha_p}-3) \tag{3-51}$$

In comparison with the linear theory, we obtain the consistency condition

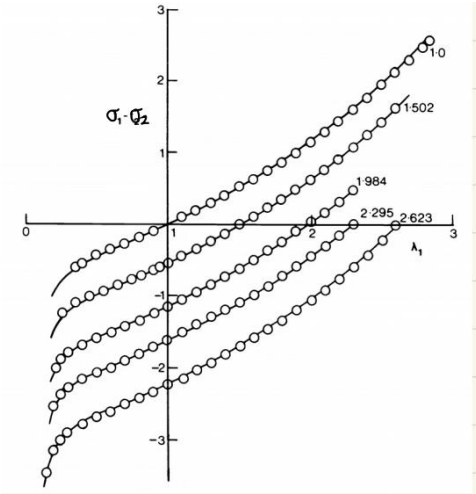
$$2\mu=\sum_{p=1}^{N}\mu_p\alpha_p \quad \text{with} \quad \mu_p\alpha_p>0, \quad p=1,....,N \tag{3-52}$$

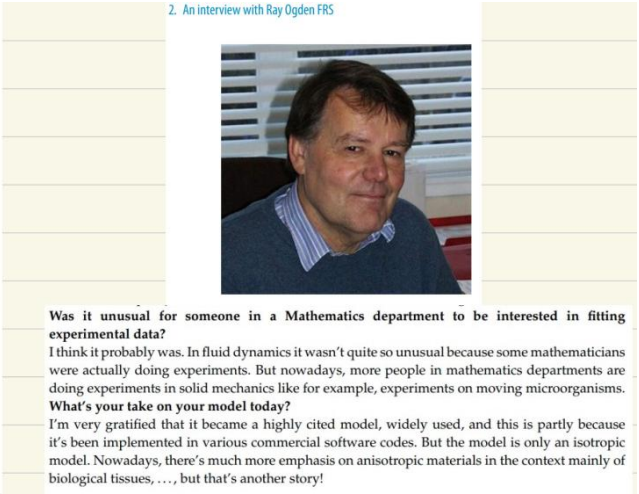
where the parameter μ denotes the small-strain shear modulus in linear theory, α_p is dimensionless constants, and μ_p is the shear moduli in the Ogden model. For most case, we take N=3, and adopt the following set of material parameters.

$$\begin{aligned} a_1 &= 1.3 \qquad \mu_1 = 1.491\mu; \\ a_2 &= 5.0 \qquad \mu_2 = 0.003\mu; \\ d_3 &= -2.0 \qquad \mu_3 = -0.0237\mu \end{aligned} \tag{3-53}$$

For the biaxial tensile experiments, the Ogden results in $\sigma_1 - \sigma_2 = \lambda_1 \frac{\partial \psi}{\partial \lambda_1} - \lambda_2 \frac{\partial \psi}{\partial \lambda_2}$,

which shows very good fit for large stretches in rubber-like materials.




2. An interview with Ray Ogden FRS

Was it unusual for someone in a Mathematics department to be interested in fitting experimental data?
I think it probably was. In fluid dynamics it wasn't quite so unusual because some mathematicians were actually doing experiments. But nowadays, more people in mathematics departments are doing experiments in solid mechanics like for example, experiments on moving microorganisms.
What's your take on your model today?
I'm very gratified that it became a highly cited model, widely used, and this is partly because it's been implemented in various commercial software codes. But the model is only an isotropic model. Nowadays, there's much more emphasis on anisotropic materials in the context mainly of biological tissues, …, but that's another story!

3) Gent model (Alan Gent, 1996. A new constitutive relation for rubber, Rubber chemistry Tech. 69. 59-61)

The Gent hyperelastic material model is a phenomenological model of rubber elasticity that is based on the concept of limiting chain extensibility. In this model, the strain energy density function is designed to have a singularity when the first invariant of the left Cauchy-Green deformation tensor (I_B or I_C) reaches a limiting value I_m.

$$\psi = -\frac{\mu J_{lim}}{2} \ln\left(1 - \frac{\mathrm{I}_{\mathbf{B}} - 3}{J_{lim}}\right) = -\frac{\mu J_{lim}}{2} \ln\left(1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_2^2 - 3}{J_{lim}}\right) \tag{3-54}$$

The strain energy density function for the Gent model is where μ is the shear modulus and $J_{lim}=I_m-3$. Here, J_{lim} is a material parameter that determines the limiting deformation in the material. For $I_B-3 \to J_{lim}$, the $1-(I_B-3)/J_{lim} \to 0$, and the strain energy of Gent model approaches ∞.

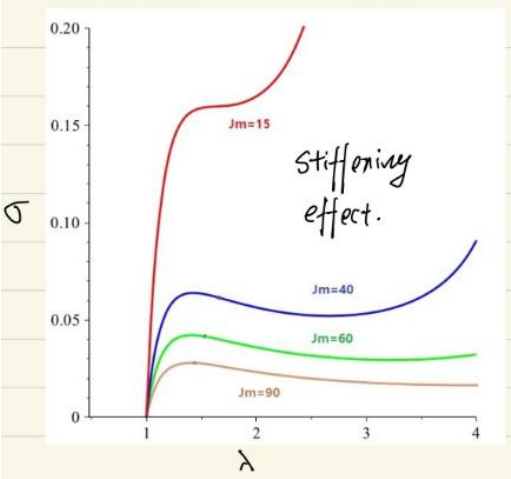
In the limit where $I_m \to \infty$, the Gent model reduces to the Neo-Hookean solid model. This can be seen by expressing the Gent model in the form of a Taylor series expansion of $\ln\left(1-\frac{I_B-3}{J_{lim}}\right)$ around $J_{lim}=\infty$, which leads to the expression for the strain energy density of a Neo-Hookean solid.

$$\sigma_1 = \lambda_1 \frac{\partial \psi}{\partial \lambda_1} - p = \frac{\lambda_1^2 \mu J_{lim}}{J_{lin} - I_B + 3.} - p$$

$$I_B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \to J_{\text{lim}}$$

$$\sigma_1 \to \infty$$



5. Compressible hyperelastic material

A material that can change volume is said to be compressible. Foamed elastomers, for example, are able to sustain finite strains with volume changes. The only restriction on this class of materials is that the volume ratio j must be positive. In this section, we introduce suitable constitutive equations to characterize compressible hyperelastic materials and discuss isotropy as a special case.

1) Compressible hyperelasticity

Since some materials behave quite differently in bulk and shear, it is most beneficial to split the deformation locally into a so-called *volumetric part* and an *isochoric part*. In particular, we consider the deformation gradient **F** and the corresponding strain measure $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathrm{T}}$. Rather than dealing directly with **F** and **B**, we perform a multiplicative decomposition of $\boldsymbol{F}$ into volume-changing (dilatational) and volume-preserving (distortional) parts, often used in elastoplasticity. We write

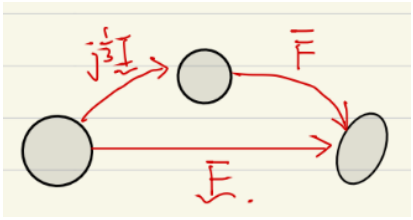
$$\mathbf{F}=(j^{\frac{1}{3}}\mathbf{I})\bar{\mathbf{F}}=j^{\frac{1}{3}}\bar{\mathbf{F}},\quad \mathbf{B}=j^{\frac{2}{3}}\bar{\mathbf{F}}\bar{\mathbf{F}}^{\mathrm{T}}=j^{\frac{2}{3}}\bar{\mathbf{B}} \tag{3-55}$$

where $j^{1/3}\boldsymbol{I}$ and $j^{2/3}\boldsymbol{I}$ are associated with the volume-changing deformations, while $\bar{\boldsymbol{F}}$ and $\bar{\boldsymbol{B}}=\bar{\boldsymbol{F}}\bar{\boldsymbol{F}}^{T}$ are associated with volume-preserving deformations of the materials, with

$$\det\mathbf{F}=j\neq 1,\quad det\bar{\mathbf{F}}=\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3=1 \tag{3-56}$$

where $\bar{\lambda}_a=j^{-\frac{1}{3}}\lambda_a,\quad a=1,2,3$ characterize the so-called modified principal stretches.

We call $\bar{\mathbf{F}}$ the **modified deformation gradient**. A material for which dilatational changes require a much higher exterior work than volume-preserving changes is called a **nearly incompressible** (or **slightly compressible**) material, for which the compressibility effects are small.



The concept of the multiplicative deformation of **F** is supported additionally by the field of computational mechanics. For example, to avoid numerical complications in the finite element analysis of slightly compressible materials, it is often advantageous to separate numerical treatments of the volumetric and isochoric parts of the deformation gradient **F** (known as the F-bar method).

We postulate a unique decoupled representation of the strain energy function per unit reference volume,

$$\psi(\mathbf{B})=\psi_{vol}(j)+\psi_{\mathrm{iso}}(\bar{\mathbf{B}}) \tag{3-57}$$

where $\psi_{vol}(j)$ and $\psi_{\mathrm{iso}}(\bar{\mathbf{B}})$ are given scalar-valued functions of j and $\bar{\mathbf{B}}$, which are

assumed to be objective. They describe the so-called volumetric (or dilatational) elastic response and the material's isochoric (or distortional) elastic response, respectively.

Additionally, we require that ψ_{vol} is a strictly convex function taking on its unique minimum at j=1, where $\psi_{vol}(j)=0$ holds if and only if j=1.

We now determine constitutive equations for compressible hyperelastic materials. The strain energy function uniquely determines the Cauchy stress tensor,

$$j\boldsymbol{\sigma} = 2\frac{\partial \psi}{\partial \mathbf{B}}\mathbf{B} = 2\left[\frac{\partial \psi_{vol}(j)}{\partial \mathbf{B}} + \frac{\partial \psi_{\text{iso}}(\bar{\mathbf{B}})}{\partial \mathbf{B}}\right]\mathbf{B} \tag{3-58}$$

The equation above is derived based on the constitutive equation of hyperelastic materials, as shown in Section 3 (Eq. (3-25)),

$$j\sigma_{kl} = \frac{\partial \Psi}{\partial x_{k,K}} x_{l,K} = \frac{\partial \Psi}{\partial B_{mn}}\frac{\partial B_{mn}}{\partial x_{k,K}} x_{l,K} = \frac{\partial \Psi}{\partial B_{mn}}\left(\delta_{mk}B_{nl} + \delta_{nk}B_{ml}\right) = 2\frac{\partial \Psi}{\partial B_{kn}}B_{nl}$$

Here, we have used the symmetric properties of **B,** B_{ij}=B_{ji}.

Given that the strain energy is split into two parts, the stress response also consists of a purely volumetric contribution and a purely isochoric contribution, i.e., $\boldsymbol{\sigma}_{\mathbf{vol}}$ and $\boldsymbol{\sigma}_{\mathbf{iso}}$, respectively. Before proceeding to derive the constitutive equations for compressible hyperelastic materials, we obtain some kinematic variables first. Given $j^2 = det(\mathbf{B})$, using the chain rule, we have,

$$\frac{\partial j^2}{\partial \boldsymbol{B}} = \frac{\partial \det(\boldsymbol{B})}{\partial \boldsymbol{B}} = j^2\boldsymbol{B}^{-1}, \quad \frac{\partial j}{\partial \mathbf{B}} = \frac{\partial j}{\partial j^2}\frac{\partial j^2}{\partial \mathbf{B}} = \frac{1}{2}j^{-1}\cdot j^2\mathbf{B^{-1}} = \frac{j}{2}\mathbf{B^{-1}} \tag{3-59}$$

Considering the $\bar{\mathbf{B}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^{\mathbf{T}} = j^{-\frac{2}{3}}\mathbf{B}$, we have,

$$\frac{\partial \bar{\mathbf{B}}}{\partial \mathbf{B}} = \frac{\partial}{\partial \mathbf{B}}(j^{-\frac{2}{3}}\mathbf{B}) = -\frac{2}{3}j^{-\frac{5}{3}}\cdot\frac{j}{2}\mathbf{B^{-1}}\otimes\mathbf{B} + j^{-\frac{2}{3}}\mathbf{II}$$

$$= j^{-\frac{2}{3}}(\mathbf{II} - \frac{1}{3}\mathbf{B^{-1}}\otimes\mathbf{B}) \tag{3-60}$$

where $\mathbf{II}$ is a fourth-order unit tensor. Substituting Eqn. (3-59) and (3-60) to (3-58),

$$j\boldsymbol{\sigma} = 2(\frac{\partial \psi_{vol}}{\partial j}\frac{\partial j}{\partial \mathbf{B}} + \frac{\partial \psi_{\text{iso}}}{\partial \bar{\mathbf{B}}} : \frac{\partial \bar{\mathbf{B}}}{\partial \mathbf{B}})\mathbf{B}$$

$$= 2(\frac{j}{2}p\mathbf{I} + j^{-\frac{2}{3}}(\mathbf{II} - \frac{1}{3}\mathbf{B^{-1}}\otimes\mathbf{B}) : \frac{\partial \psi_{\text{iso}}}{\partial \bar{\mathbf{B}}}\mathbf{B}) \tag{3-61}$$

where $p = \partial \psi_{vol} / \partial j$ is specified by the volumetric strain energy $\psi_{vol}(j)$. We note that p is not the Lagrange multiplier, in contrast to the incompressible materials. It is obtained from the derivative of the strain energy function directly. Using the symmetry property of **B**, the Cauchy stress tensor is

$$\begin{aligned}\boldsymbol{\sigma} &= p\mathbf{I} + (j^{-\frac{2}{3}}\mathbf{II} - \frac{1}{3} j^{-\frac{2}{3}}\mathbf{B}^{-1} \otimes \mathbf{B}) : 2j^{-1} \frac{\partial \psi_{iso}}{\partial \bar{\mathbf{B}}} \mathbf{B} \\ &= p\mathbf{I} + \mathbf{B}(j^{-\frac{2}{3}}\mathbf{II} - \frac{1}{3}\mathbf{B}^{-1} \otimes \bar{\mathbf{B}})\bar{\mathbf{B}}^{-1} : 2j^{-1} \frac{\partial \psi_{iso}}{\partial \bar{\mathbf{B}}} \bar{\mathbf{B}} \\ &= p\mathbf{I} + (j^{-\frac{2}{3}}\mathbf{BII}\bar{\mathbf{B}}^{-1} - \frac{1}{3}\mathbf{B}\mathbf{B}^{-1} \otimes \bar{\mathbf{B}}\bar{\mathbf{B}}^{-1}) \cdot 2j^{-1} \frac{\partial \psi_{iso}}{\partial \bar{\mathbf{B}}} \bar{\mathbf{B}} \\ &= p\mathbf{I} + (\mathbf{II} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}) : \bar{\boldsymbol{\sigma}}\end{aligned} \tag{3-62}$$

where $\bar{\boldsymbol{\sigma}} = 2j^{-1} \frac{\partial \psi_{iso}}{\partial \bar{\mathbf{B}}} \bar{\mathbf{B}}$ is defined as the fictitious Cauchy stress tensor. It is also specified by the strain energy function $\psi_{iso}(\bar{\mathbf{B}})$ directly, which differs from the incompressible hyperelastic materials.

As we have seen from Eqn. (3-62), the scalar P and the fictitious stress $\bar{\boldsymbol{\sigma}}$ in the constitutive equations of compressible hyperelastic materials are derived directly from the strain energy function $\psi_{vol}(j)$ and $\psi_{iso}(\bar{\mathbf{B}})$, respectively. Unlike incompressible materials, they are not determined from the equilibrium equations and the boundary conditions. Therefore, the slightly compressible hyperelastic constitutive equations are widely adopted in finite element analysis.

2) Ogden model for compressible isotropic hyperelasticity

Compressibility is accounted for by adding the strain energy function $\psi_{vol}(j)$, describing the purely volumetric elastic response. For our consideration, in particular, we use the decoupled representation of the strain energy function expressed in terms of the principal stretches,

$$\psi(\lambda_1, \lambda_2, \lambda_3) = \psi_{vol}(j) + \psi_{\text{iso}}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) \tag{3-63}$$

For rubber-like materials, Ogden proposed a volumetric response function in the following forms,

$$\psi_{vol}(j)=\frac{1}{4}K(j^2-1-2\ln j) \tag{3-64}$$

The scalar-valued function in Eqn. (3-64) is a strictly convex function, and K is the bulk modulus of the material. The parameter p=0.5K(j-j^{-1}) in the constitutive equation. The second part of the decouple strain energy, i.e., $\psi_{\text{iso}}(\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3)$, describes the purely isochoric elastic response in terms of the modified principal stretches $\bar{\lambda}_a=j^{-\frac{1}{3}}\lambda_a$, $a=1,2,3$.

The Cauchy stress tensor can be expressed as,

$$j\sigma_\alpha=j(\boldsymbol{\sigma}_{\mathbf{vol}}+\boldsymbol{\sigma}_{\mathbf{iso}})=\left(\frac{\partial\psi_{\text{iso}}(j)}{\partial\lambda_\alpha}+\frac{\partial\psi_{\text{vol}}(\bar{\lambda})}{\partial\lambda_\alpha}\right)\lambda_\alpha=\left(\frac{\partial\psi_{\text{iso}}(j)}{\partial j}\frac{\partial j}{\partial\lambda_\alpha}+\frac{\partial\psi_{\text{vol}}(\bar{\lambda})}{\partial\bar{\lambda}}\frac{\partial\bar{\lambda}_\beta}{\partial\lambda_\alpha}\right)\lambda_\alpha$$

The Cauchy stress tensor can be derived as,

$$\sigma^\alpha_{\text{vol}}=j^{-1}\frac{\partial\psi_{\text{vol}}}{\partial j}\frac{\partial j}{\partial\lambda_\alpha}\lambda_\alpha=\frac{\partial\psi_{\text{vol}}}{\partial j}=p=\frac{K}{2j}(j^2-1) \tag{3-65}$$

$$\begin{aligned}\sigma^a_{iso}&=\frac{1}{j}\frac{\partial\psi_{iso}}{\partial\bar{\lambda}_\beta}\frac{\partial(j^{-\frac{1}{3}}\lambda_\beta)}{\partial\lambda_{\alpha}}\lambda_\alpha=\frac{2}{j}\frac{\partial\psi_{sio}}{\partial\lambda_\beta}\left(j^{-\frac{1}{3}}\delta_{\alpha\beta}-\frac{1}{3}j^{-\frac{4}{3}}\frac{\partial j}{\partial\lambda_{\alpha}}\lambda_\beta\right)\lambda_\alpha\\&=j^{-1}\frac{\partial\psi_{iso}}{\partial\bar{\lambda}_\beta}j^{-\frac{1}{3}}\left(\delta_{\alpha\beta}-\frac{1}{3}\bar{\lambda}_\alpha^{-1}\bar{\lambda}_\beta\right)\lambda_\alpha\end{aligned}\quad(\alpha\text{ 不求和})$$

$$=j^{-1}\left(\frac{\partial\psi_{iso}}{\partial\bar{\lambda}_\alpha}\bar{\lambda}_\alpha-\frac{1}{3}\sum_{\beta=1}^{3}\frac{d\psi_{iso}}{d\bar{\lambda}_\beta}\bar{\lambda}_\beta\right) \tag{3-66}$$

Adopting the Ogden model for the purely isochoric deformation, $\psi_{iso}(\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3)=\sum_{p=1}^{3}\frac{\mu_p}{\alpha_p}(\bar{\lambda}_1^{\alpha_p}+\bar{\lambda}_2^{\alpha_p}+\bar{\lambda}_3^{\alpha_p}-3)$, and combining it with Eqn. (3-66), we have

$$\sigma^\alpha_{iso}=j^{-1}\left(\sum_{p=1}^{3}\mu_p\bar{\lambda}_\alpha^{\alpha_p}-\frac{1}{3}\sum_{\beta=1}^{3}\sum_{p=1}^{3}\mu_p\bar{\lambda}_\beta^{\alpha_p}\right) \tag{3-67}$$

The Cauchy stress tensor for compressible hyperelastic materials with the Ogden model is expressed as,

$$\sigma_\alpha=\sigma^\alpha_{vol}+\sigma^\alpha_{iso}=\frac{K}{2j}(j^2-1)+\frac{1}{j}\left[\sum_{p=1}^{3}\mu_p j^{-\frac{\alpha_p}{2}}\lambda_\alpha^{\alpha_p}-\frac{1}{3}\sum_{\beta=1}^{3}\sum_{p=1}^{3}\mu_p j^{-\frac{\alpha_p}{2}}\lambda_\beta^{\alpha_p}\right] \tag{3-68}$$