Observer-based output feedback control of discrete-time linear systems with input and output delays

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In this paper, we study observer-based output feedback control of discrete-time linear systems with both multiple input and output delays. By generalising our recently developed truncated predictor feedback approach for state feedback stabilisation of discrete-time time-delay systems to the design of observer-based output feedback, two types of observer-based output feedback controllers, one being memory and the other memoryless, are constructed. Both full-order and reduced-order observer-based controllers are established in both the memory and memoryless schemes. It is shown that the separation principle holds for the memory observer-based output feedback controllers, but does not hold for the memoryless ones. We further show that the proposed observer-based output feedback controllers solve both the $l_2$ and $l_\infty$ semi-global stabilisation problems. A numerical example is given to illustrate the effectiveness of the proposed approaches.

Keywords: output feedback; truncated predictor feedback; time-delay systems; separation principle; $l_\infty$ and $l_2$ semi-global stabilisation

1. Introduction

Time-delay systems can be utilised to model many practical physical systems, especially those systems influenced by the effect of transportation, inertia phenomena and long transmission such as pneumatic systems, rolling mills, nuclear reactors, hydraulic systems, manufacturing processes, digital control systems and systems that are controlled remotely (see, for example, Campos-Delgado, Martin Luna-rivera, & Bonilla, 2013; Chen & Zheng, 2010; Hale, 1977; Mendez-Barrios, Niculescu, Chen, & Maya-Mendez, 2014; Richard, 2003). Time delay as a primary source of instability and performance degradation makes practical control systems hard to control since a system in the presence of time delay is of infinite dimension in the continuous-time setting (Hale, 1977). As a result, in many applications, effort should be made to overcome the effect of time delay. However, existing methods that have been well developed for conventional delay-free control systems modelled by ordinary differential or difference equations are generally not directly applicable. As a result, control of timedelay systems has received much attention for several decades and a large number of research results have been reported in the literature that deal with various analysis and design problems (see, for instance, Ahn, 2013; Chen & Latchman, 1995; Cong, 2013; Deaecto, Geromel, & Galbusera, 2013; Efimov, Perruquet, & Richard, 2013; Gu, 2000; Lam, Gao, & Wang, 2007; Wu, He, & She, 2004; Xia, Liu, Shi, Chen, & Rees, 2008; Zhang, Zhang, & Xie, 2004; Zhang, Zheng, & Xu, 2012; and the references therein).

Discrete-time time-delay systems have many practical applications such as network-based control systems (Xiong & Lam, 2007) and sampled process control (Gu, Kharitonov, & Chen, 2003). A lot of results have been reported in the literature for discrete-time time-delay systems, especially, for stability analysis and stabilisation of discrete-time time-delay systems. In those studies, different kinds of Lyapunov–Krasovskii functional are proposed to derive less-conservative stability conditions which are expressed in linear matrix inequalities. The related results can be found in Xu, Lam, and Zou (2005, 2006), Zhou, Li, and Lin (2013b) and the references therein. Predictor feedback approaches that were well built for continuous-time time-delay systems (see, for example, Kosugi & Suyama, 2012; Lombardi, Olaru, Niculescu, & L. Hetel, 2012; Zhou, Li, & Lin, 2013a) were also developed for discrete-time time-delay systems (Karafyllis & Krstic, 2013). Discrete-time time-delay system has a very specific feature that it can be converted into an augmented linear system without time delay. Therefore, by applying the standard stability analysis tool for discrete-time linear system on the augmented system, the stability analysis and stabilisation controller design can be achieved. That is to say, corresponding results can be obtained without any conservatism which cannot be achieved by using the Lyapunov–Krasovskii functional approach. However, such an approach is applicable only when the delays are known exactly and constant.

Recently, different from the above-mentioned augmentation approach, we proposed, in Zhou and Lin (2011)
and Zhou et al. (2013b), a truncated predictor feedback (TPF) approach for stabilisation and semi-global stabilisation of discrete-time linear systems with multiple input delays and constraints. The idea of TPF is to safely neglect the terms containing the history information of the control signals in the traditional predictor-based controller, so that the proposed TPF only utilises the current state for control. Moreover, we have shown that the proposed TPF can also solve the semi-global stabilisation when the open-loop system is simultaneously subject to delays and constraints. The idea of TPF is to safely neglect the terms containing the history information of the control signals in the traditional predictor-based controller, so that the proposed TPF only utilises the current state for control.

The remainder of this paper is organised as follows. The problem formulation and some preliminary results are given in Section 2. The memory and memoryless observer-based output feedback controllers are then established in Sections 3 and 4, respectively. A numerical example is provided in Section 5 to illustrate the effectiveness of the proposed approaches and Section 6 concludes the paper.

2. Problem formulation and preliminaries

2.1 Problem formulation

Consider the following discrete-time linear system with multiple input and output delays:

\[
\begin{align*}
x(k+1) &= Ax(k) + \sum_{i=0}^{p} B_i u(k - r_i), \\
y(k) &= \sum_{j=0}^{q} C_j x(k - l_j),
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}, i \in I[0, p], \) and \( C_j \in \mathbb{R}^{r \times m}, j \in I[0, q], \) are constant matrices, and \( r_i, i \in I[0, p] \), and \( l_j, j \in I[0, q] \), are some constant nonnegative integers denoting the delays in the actuator. Without loss of generality, we assume that

\[
0 = r_0 < r_1 < r_2 < \cdots < r_p = r < \infty, \quad 0 = l_0 < l_1 < l_2 < \cdots < l_q = l < \infty.
\]

In this paper, we are interested in the stabilisation of the discrete-time linear system (3) by using observer-based output feedback controller. For easy reference, the problems to be studied are formally stated as follows.

Problem 1 (Observer based output feedback stabilisation): For the linear system (3) with input and output delays that are arbitrary large, bounded and exactly known, find a full-order observer-based output feedback controllers, we have identified a class of discrete-time linear systems with multiple input and output delays (Zhou et al., 2013a), we further study observer-based output feedback control of discrete-time linear systems with both multiple input and output delays, if the open-loop system is polynomially unstable. Two types of observer-based output feedback controllers, one being memory and the other memoryless, are constructed. Both full-order and reduced-order observer-based controllers are established in both the memory and memoryless schemes. It is shown that the separation principle holds for the memory observer-based output feedback controllers, but does not hold for the memoryless ones. We further show that the proposed observer-based output feedback controllers solve both the \( l_1 \) and \( l_\infty \) semi-global stabilisation problems. A numerical example is given to illustrate the effectiveness of the proposed approaches. We emphasise that, differently from the continuous-time setting considered in Zhou et al. (2013a) where only full-order observer is considered, both full-order and reduced-order observer-based output feedback controllers will be established in the present paper.

The merit of the proposed approaches in the present paper is that the resulting controllers are either memoryless or memory with only the information of the inputs on the delayed instants being utilised. Therefore, the proposed controllers are very easy to implement. Moreover, by proposing the memoryless observer-based output feedback controllers, we have identified a class of discrete-time linear systems with input and output delays that can be stabilised by memoryless controllers. Possible future extensions of the reported results are the relaxation of the assumptions imposed on the open-loop systems and the allowance of time-varying parameters (delays and/or the coefficient matrices).

Notation. Throughout this paper, we will use standard notation. We use \( A^T, \text{tr}(A), \det(A) \) and \( \lambda(A) \) to denote the transpose, the trace, the determinant and the eigenvalue set of matrix \( A \), respectively, \( R^l \) to denote the \( n \)-dimensional Euclidean space, and \( N = \{0, 1, 2, \ldots \} \). For a vector \( u \), we use \( \|u\|_\infty \) and \( \|u\|_2 \) to denote its \( \infty \)-norm and 2-norm, respectively. For two integers \( n_1 \) and \( n_2 \) with \( n_1 \leq n_2 \), we use \( I[n_1, n_2] \) to denote the integer set \( \{n_1, n_1 + 1, \ldots, n_2\} \).

Let \( \mathcal{D}_{n,r} = \{\psi : [-r, 0] \rightarrow \mathbb{R}^n \} \) denote the Banach space of continuous-time vector valued functions mapping the set \( I[-r, 0] \) into \( \mathbb{R}^n \) with the topology of uniform convergence. Denote

\[
u^m_\infty = \left\{ u(k) : N \rightarrow \mathbb{R}^m \left| \frac{\|u\|_\infty}{\sup_{k \geq 0} \|u(k)\|_\infty} \leq 1 \right. \right\}, \]

\[
u^m_2 = \left\{ u(k) : N \rightarrow \mathbb{R}^m \left| \frac{\|u\|_2}{\sum_{k=0}^{\infty} \|u(k)\|_2^2} \leq 1 \right. \right\}. \]
feedback controller,
\[
\begin{align*}
z(k+1) &= f_f(z_k, y_k, u_k), \\
u(k) &= g_f(z_k, y_k, u_{k-1}),
\end{align*}
\tag{6}
\]
where \(z \in \mathbb{R}^n\) is the observer state, \(f_f: \mathbb{R}_{n, d_z} \times \mathbb{R}_{d_z} \times \mathbb{R}_{m, d_s} \rightarrow \mathbb{R}^n\), and \(g_f: \mathbb{R}_{n, d_z} \times \mathbb{R}_{d_z} \times \mathbb{R}_{m, d_s} \rightarrow \mathbb{R}^m\) are some functions with \(d_z, d_s\) and \(d_m\) being some integers, or a reduced-order observer-based output feedback controller
\[
\begin{align*}
w(k+1) &= f_r(w_k, y_k, u_k), \\
u(k) &= g_r(w_k, y_k, u_{k-1}),
\end{align*}
\tag{7}
\]
where \(w \in \mathbb{R}^{n-s}\) is the observer state, \(f_r: \mathbb{R}_{n-s,d_z} \times \mathbb{R}_{d_z} \times \mathbb{R}_{m, d_s} \rightarrow \mathbb{R}^{n-s}\), and \(g_r: \mathbb{R}_{n-s,d_z} \times \mathbb{R}_{d_z} \times \mathbb{R}_{m, d_s} \rightarrow \mathbb{R}^m\) are some functions with \(d_w, d_z\) and \(d_s\) being some integers, such that the closed-loop system is (globally) asymptotically stable at the origin.

We are also interested in the following observer-based \(l_\infty\) and \(l_2\) semi-global stabilisation problems.

**Problem 2 (Observer-based \(l_\infty\) and \(l_2\) semi-global stabilisation):** Let \(\Omega_x \subset \mathbb{R}_{n,l_\infty}\) and \(\Omega_y \subset \mathbb{R}_{m,l_\infty}\) be two prescribed bounded sets that can be arbitrarily large. For the linear system (3), find a full-order observer-based output feedback controller (6) or a reduced-order observer-based output feedback controller (7) such that, for any bounded sets \(\Omega_x \subset \mathbb{R}_{n,d_z}\) and \(\Omega_w \subset \mathbb{R}_{n-s,d_z}\), the closed-loop system is asymptotically stable, and
\[
(x_0, u_0, z_0) \in (\Omega_x \times \Omega_u \times \Omega_z) \implies u \in u_\infty(u_\infty^2),
\tag{8}
\]
in the full-order case, and
\[
(x_0, u_0, w_0) \in (\Omega_x \times \Omega_u \times \Omega_w) \implies u \in u_\infty(u_\infty^2),
\tag{9}
\]
in the reduced-order case.

The aim of Problem 2 is to design both full-order and reduced-order observer-based output feedback controllers such that the closed-loop system is asymptotically stable, and the \(l_\infty\) and \(l_2\) norms of the control \(u\) do not exceed 1 for any initial conditions within any prescribed sets that are bounded yet can be arbitrarily large. This is why we name this problem as \(l_\infty\) and \(l_2\) semi-global stabilisation problem.

### 2.2 Predictor and truncated predictor feedback
In this subsection, we give a brief introduction on the predictor and TPF for discrete-time time-delay systems. To this end, we first present the following lemma regarding the reduction of the discrete-time linear time-delay system (3) into a delay-free linear system. The proof is provided in Section A.1 for the sake of clarity.

**Lemma 1:** Consider the discrete-time linear system (3) with multiple input and output delays. Let
\[
\begin{align*}
\chi(k) &= x(k) + \sum_{i=1}^{p} \sum_{j=1}^{r_i} A^{i-r_i-1} B_i u(k-j), \\
\omega(k) &= y(k) + \sum_{i=0}^{q} \sum_{j=0}^{r_{i+1}} \sum_{v=0}^{r_i} C_j A^{v-r_i-l_i-1} B_i u(k-v).
\end{align*}
\tag{10}
\]
Then, the original time-delay system (3) can be rewritten as the following delay-free system:
\[
\begin{align*}
\chi(k+1) &= A \chi(k) + B u(k), \\
\omega(k) &= C \chi(k),
\end{align*}
\tag{11}
\]
where \(B = B(r_0, r_1, \ldots, r_p)\) and \(C = C(l_0, l_1, \ldots, l_q)\) are, respectively, defined as
\[
\begin{align*}
B &= A^{-r_0} B_0 + A^{-r_1} B_1 + \cdots + A^{-r_{p-1}} B_{p-1} + A^{-r_p} B_p, \\
C &= C_0 A^{-l_0} + C_1 A^{-l_1} + \cdots + C_{q-1} A^{-l_{q-1}} + C_q A^{-l_q}.
\end{align*}
\tag{12}
\]
Lemma 1 is a generalisation of the predictor feedback approach for systems with a single input delay to systems with both (multiple) input and output delays. In the case that the states are available for feedback, it is easy to see that the controller
\[
u(k) = F \chi(k) = F \left( x(k) + \sum_{i=1}^{p} \sum_{j=1}^{r_i} A^{i-r_i-1} B_i u(k-j) \right),
\tag{13}
\]
where \(F\) is such that \(A + BF\) is Schur stable, makes the closed-loop system \(\chi(k+1) = (A + BF) \chi(k)\) be asymptotically stable. The controller in Equation (15) is known as predictor feedback, which is memory since it involves the past control signals.

Recently, based on the predictor feedback (15), we proposed a TPF approach for stabilisation of discrete-time linear system with (multiple) input delays (Zhou et al., 2013b) by using the current state vector only. The idea of TPF is briefly explained as follows. If the feedback gain \(F = F(\gamma): (0, 1) \rightarrow \mathbb{R}^{n \times n}\) is properly designed such that
\[
\lim_{\gamma \downarrow 0} F(\gamma) = 0, \quad \lim_{\gamma \downarrow 0} \|F(\gamma)\| < \infty, \tag{14}
\]
namely, \(u(k)\) is ‘of order 1’ with respect to \(\gamma\), then the memory terms in Equation (15) are at least ‘of order 2’ with respect to \(\gamma\) and can thus be neglected if \(\gamma\) is sufficiently
small. In this case, the predictor feedback (15) is truncated as

$$u(k) = Fx(k), \quad (17)$$

which is termed as TPF.

It is well known that the necessary and sufficient conditions guaranteeing the existence of a feedback $F(y)$ satisfying Equation (16) is that $(A, B)$ is asymptotically null controllable by bounded controls (ANCBC), namely, $(A, B)$ is stabilisable in the ordinary sense and all the poles of $A$ are located within the unit circle (Yang, Sontag, & Sussmann, 1997). Since the stable poles of $A$ do not affect the solvability of the output feedback stabilisation problem, we assume, without loss of generality, the following assumption.

**Assumption 1:** The matrix pair $(A, B) \in (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m})$ is controllable with all the eigenvalues of $A$ being on the unit circle and the matrix pair $(A, C)$ is observable.

**Remark 1:** We point out that, even in the absence of input delay, Problem 2 is solvable if and only if $(A, B)$ is ANCBC (see, for example, Yang et al., 1997). This fact indicates that Assumption 1 is not restrictive at all when Problem 2 is concerned. Moreover, this assumption can be relaxed by assuming that $(A, B)$ is stabilisable, $(A, C)$ is detectable and all the eigenvalues of $A$ are on the closed unit disk.

Under the above assumption, the feedback gain $F$ satisfying Equation (16) can be designed as

$$F(y) = -((I_m + B^TPB)^{-1}B^TPA), \quad (18)$$

where $P = P(y)$ is the unique positive definite solution to the following parametric discrete-time algebraic Riccati equation (DARE):

$$A^TPA - P - A^TPB(I_m + B^TPB)^{-1}B^TPA = -\gamma P, \quad (19)$$

where $\gamma \in (0, 1)$ is a design parameter. Some additional properties of such a feedback gain as well as $P$ have been collected in Lemma A.0 in Section A.2. With $F$ designed as above, we have been able to prove in Zhou et al. (2013b) the following result regarding stabilisation of system (3) by the TPF (17).

**Lemma 2:** Let $(A, B)$ satisfies Assumption 1 and $F$ be related with Equation (18). Then, there exists a positive scalar $\gamma^* = \gamma^*([r_i]_{i=1}^p) \in (0, 1)$ such that the TPF (17) stabilises system (3) for all $\gamma \in (0, \gamma^*)$, namely, the following discrete-time time-delay system is asymptotically stable for all $\gamma \in (0, \gamma^*)$:

$$x(k+1) = Ax(k) + \sum_{i=0}^p B_i Fx(k-r_i). \quad (20)$$

### 2.3 Observer design by predictor feedback

If only the output variables rather than the state variables are available for feedback, observer-based output feedback is a good choice for controllers design. According to the standard linear systems theory (Kailath, 1980), it follows from Lemma 1 that the conventional full-order observer-based output feedback controller for the original time-delay system (3) can be constructed easily as Kailath (1980):

$$\begin{align*}
  z(k+1) &= Az(k) + Bu(k) - L(\omega(k) - Cz(k)), \\
  u(k) &= Fz(k),
\end{align*} \quad (21)$$

where $F$ and $L$ are such that $A + BF$ and $A + LC$ are both Schur stable. Similarly, the reduced-order observer-based output feedback controller can be constructed as Kailath (1980):

$$\begin{align*}
  w(k+1) &= Dw(k) + TBu(k) + E\omega(k), \\
  u(k) &= Gw(k) + H\omega(k),
\end{align*} \quad (22)$$

where $w \in \mathbb{R}^{n-1}$ is the state of the observer. The coefficient matrices in Equation (22) should satisfy the following equations (Kailath, 1980):

$$TA - DT = EC, \quad (23)$$

$$GT + HC = F, \quad (24)$$

where $D \in \mathbb{R}^{(n-s) \times (n-s)}$ is a given matrix that is Schur stable, $F$ is such that $A + BF$ is Schur stable and $E$ is chosen such that the solution $T$ of Equation (23) satisfies

$$\text{rank}(U) = n, \quad U = \begin{bmatrix} T \\ C \end{bmatrix} \in \mathbb{R}^{n \times n}. \quad (25)$$

It is well known that, if $(A, C)$ is observable, then for almost all matrix $E$ such that $(D, E)$ is observable, the matrix $U$ defined in Equation (25) is nonsingular (Kailath, 1980). Consequently, the matrices $G$ and $H$ can be solved from Equation (24) as

$$[G \ H] = FU^{-1}. \quad (26)$$

However, both the full-order and reduced-order observer-based output feedback in Equations (21) and (22) are memory since the new output $\omega(k)$ involves all the past controls in the interval $[k - l - r, k]$, namely, $u(k + s), s \in \mathbb{I}[-l - r, 0]$. In this paper, we will present both memory and memoryless observer-based output feedback controllers that only use the past control and output signals at some discrete-time points and the current input and output signals of the system, respectively.
3. Memory observer-based output feedback

In this section, we will propose a memory observer-based output feedback controller for the stabilisation of the discrete-time time-delay system (3) with multiple input and output delays by using the past control and output signals at some discrete-time points.

3.1 Multiple output delays

We first consider the case that there are multiple output delays in system (3). In this case, we construct the following memory observer-based output feedback controller:

\[
\begin{align*}
    z(k + 1) &= Ax(k) + \sum_{i=0}^{p} B_i u(k - r_i) \\
    &\quad - L \left( y(k) - \sum_{j=0}^{q} C_j z(k - l_j) \right), \\
    u(k) &= Fz(k),
\end{align*}
\]  

(27)

where \( F \) is related with Equation (18) and

\[
L = L(\rho)
\]

\[
= -AQ(\rho)CT(L + CQ(\rho)C^T)^{-1} : (0, 1) \to \mathbb{R}^{n\times s},
\]

(28)

where \( Q(\rho) = Q \) is the unique positive definite solution to the following DARE:

\[
AQAT - Q - AQCT(L + CQCT)^{-1}CQA^T = -\rho Q.
\]

(29)

**Remark 2:** It is easy to see that the maximal length of delay in the memory observer-based output feedback controller (27) is \( \{r, l\} \) while the maximal length of delay in the predictor-based output feedback controller (21) is \( r + l \), which indicates that the former one is relatively easier to implement than the latter. Moreover, different from the predictor-based output feedback controller (21) where the past control signals \( u(k) \) in the interval \([k - l - r, k]\) are used, the memory observer-based output feedback controller (27) only use the past controls at the discrete-time points \( k - r_i, i \in I[0, \rho] \).

We can then prove the following result regarding the stability of the closed-loop system consisting of Equation (3) and (27).

**Theorem 1:** Assume that \((A, B, C)\) satisfies Assumption 1. Let \( F \) and \( L \) be designed as Equations (18) and (28), respectively. Then, for any given arbitrarily large yet bounded delays \( \{r_i\}_{i=1}^{p} \) and \( \{l_j\}_{j=1}^{q} \), there exist two scalars,

\[
\gamma^* = \gamma^* (\{r_i\}_{i=1}^{p}) \in (0, 1), \quad \rho^* = \rho^* (\{l_j\}_{j=1}^{q}) \in (0, 1),
\]

(30)

such that the memory observer-based output feedback (27) solves Problem 1 for all \( \gamma \in (0, \gamma^*) \) and \( \rho \in (0, \rho^*) \).

**Proof:** Let \( e(k) = x(k) - z(k) \). Then, it follows from Equations (3) and (27) that the closed-loop system can be written as

\[
\begin{align*}
x(k + 1) &= Ax(k) + \sum_{i=0}^{p} B_i F x(k - r_i) \\
&\quad - \sum_{i=0}^{p} B_i F e(k - r_i), \\
\end{align*}
\]

(31)

\[
\begin{align*}
e(k + 1) &= A e(k) + \sum_{j=0}^{q} LC_j e(k - l_j),
\end{align*}
\]

which is asymptotically stable if and only if the following two uncoupled time-delay systems are

\[
\begin{align*}
x(k + 1) &= Ax(k) + \sum_{i=0}^{p} B_i F x(k - r_i), \\
e(k + 1) &= A e(k) + \sum_{j=0}^{q} LC_j e(k - l_j).
\end{align*}
\]

The stability of these two time-delay systems follows from Lemma 2 as well as the dual principle. The proof is finished. \( \square \)

**Remark 3:** It is easy to see that a separation principle exists in the design of the memory observer-based output feedback controller (27), namely, the feedback gains \( F \) and \( L \) can be designed separately. Moreover, both the gains \( L \) and \( F \) need to approach zero as does, namely, larger values of \( \{r_i\}_{i=1}^{p} \) and \( \{l_j\}_{j=1}^{q} \) allow only smaller values of \( \gamma \) and \( \rho \), and consequently, smaller values of \( \|F\| \) and \( \|L\| \).

**Remark 4:** In Equation (27), not the current observer state \( z(k) \) but the delayed observer states \( z(k - l_j) \), \( j \in I[1, q] \), are fed back, which is not desirable since, on the one hand, the delay effect will degrade the performances of the observer, and on the other hand, it makes the implementation of this observer expensive.

3.2 A single output delay

In the case that there is only a single delay in the output, namely, the system in Equation (3) becomes

\[
\begin{align*}
x(k + 1) &= Ax(k) + \sum_{i=0}^{p} B_i u(k - r_i), \\
y(k) &= C_1 x(k - l_{1}),
\end{align*}
\]

(33)

if we set \( \xi(k) = x(k - l_{1}) \), then system (33) can be converted into

\[
\begin{align*}
\xi(k + 1) &= A \xi(k) + \sum_{i=0}^{p} B_i u(k - r_i - l_{1}), \\
y(k) &= C_1 \xi(k).
\end{align*}
\]

(34)
The time-delay system in Equation (34) now has no delay in the output. This special structure allows us to give the following new solutions to Problem 1.

**Theorem 2:** Assume that \((A, B, C)\) satisfies Assumption 1 where \(C\) is replaced by \(C_1\). Then, there exists a \(\gamma^* = \gamma^* (\{r_i\}_{i=1}^p, l) \in (0, 1)\), such that the full-order observer-based output feedback controller

\[
\begin{aligned}
z(k + 1) &= A z(k) + \sum_{i=0}^{p} B_i u(k - l_i - r_i) \\
u(k) &= F A^{l_1} z(k) + \mathcal{F} z(k),
\end{aligned}
\]  

(35)
solves Problem 1 associated with system (33) for all \(\gamma \in (0, \gamma^*)\), where \(L \in \mathbb{R}^{m \times p}\) is chosen such that \(A + LC_1\) is asymptotically stable and \(F\) is given by Equation (18) with \(P(\gamma)\) being the unique positive definite solution to the parametric DARE (19).

**Proof:** Let \(e(k) = \xi(k) - z(k), \forall k \geq l_1\). Then it follows from Equations (34) and (35) that

\[
e(k + 1) = (A + LC_1) e(k), \forall k \geq l_1,
\]

and

\[
\xi(k + 1) = A \xi(k) + \sum_{i=0}^{p} B_i \mathcal{F} \xi(k - l_i - r_i) \\
- \sum_{i=0}^{p} B_i e(k - r_i), \forall k \geq l_1.
\]

(37)

Since \(A + LC_1\) is Hurwitz, it follows from Equation (36) that system (37) is asymptotically stable if and only if

\[
\xi(k + 1) = A \xi(k) + \sum_{i=0}^{p} B_i \mathcal{F} \xi(k - l_i - r_i), \forall k \geq l_1,
\]

(38)

It follows from Lemma 2 that, if \(\mathcal{F} = -(I_m + \mathcal{B}^T \mathcal{P} \mathcal{B}^{-1} \mathcal{B}^T \mathcal{P} A)\), where

\[
\mathcal{B} \triangleq \sum_{i=0}^{p} A^{-(r_i + l_i)} B_i = A^{-l_1} B,
\]

(39)

and \(\mathcal{P}\) is the unique positive definite solution to the following DARE:

\[
A^T \mathcal{P} A - \mathcal{P} - A^T \mathcal{P} \mathcal{B} (I_m + \mathcal{B}^T \mathcal{P} \mathcal{B})^{-1} \mathcal{B}^T \mathcal{P} A
= -\gamma \mathcal{P},
\]

(40)

then there exists a \(\gamma^* = \gamma^* (\{r_i\}_{i=0}^p, l)\) such that Equation (38) is asymptotically stable for all \(\gamma \in (0, \gamma^*)\). Notice that the DARE in Equation (40) is equivalent to the DARE in Equation (19) by denoting \(P = (A^{-l_1})^T \mathcal{P} A^{-l_1}\). Consequently,

\[
\mathcal{F} = -(I_m + \mathcal{B}^T \mathcal{P} \mathcal{B})^{-1} \mathcal{B}^T \mathcal{P} A = FA^{l_1}.
\]

(41)

The proof is completed. \(\Box\)

**Remark 5:** The advantage of Equation (35) over Equation (27) is that the observer gain \(L\) in Equation (35) needs not to satisfy \(\lim_{\rho \rightarrow 0} L(\rho) = 0\).

We next propose a reduced-order observer-based output feedback solution to Problem 1 associated with system (33).

**Theorem 3:** Assume that \((A, B, C)\) satisfies Assumption 1 where \(C\) is replaced by \(C_1\). Consider the following reduced-order observer-based output feedback controller:

\[
\begin{aligned}
\w(k + 1) &= D w(k) + T \left( \sum_{i=0}^{p} B_i u(k - l_i - r_i) \right) \\
&+ E \xi(k),
\end{aligned}
\]

\[
u(k) = G w(k) + H \xi(k),
\]

(42)

where \(D \in \mathbb{R}^{(n-a) \times (n-a)}, E, T, G, H\) and \(F\) are related with Equations (23)–(26), where \(C\) is replaced with \(C_1\) and \(F\) is replaced with \(\mathcal{F}\) defined in Theorem 2. Then, for any given arbitrarily large but bounded input delays \(\{r_i\}_{i=1}^p\) and output delay \(l_1\), there exists a scalar \(\gamma^* = \gamma^* (\{r_i\}_{i=1}^p, l_1) \in (0, 1)\) such that Equation (42) solves Problem 1 associated with system (33) for all \(\gamma \in (0, \gamma^*)\).

**Proof:** Let \(e(k) = w(k) - T \xi(k), \forall k \geq l_1\). Then, by virtue of Equations (34) and (23), we get

\[
e(k + 1) = w(k + 1) - T \xi(k + 1)
= Dw(k) + T A \xi(k) + E \xi(k)
= De(k) + (DT - TA + EC_1) \xi(k)
= De(k), \forall k \geq l_1.
\]

(43)

which implies that the dynamics of \(e(k)\) is exponentially stable. On the other hand, by Equations (42) and (24), we have

\[
\xi(k + 1) = A \xi(k) + \sum_{i=0}^{p} B_i (Gw(k - r_i - l_i) \\
+ H \xi(k - r_i - l_i))
= A \xi(k) + \sum_{i=0}^{p} B_i ((GT + HC_1) \xi(k - l_i - r_i) \\
+ Ge(k - r_i - l_i))
= A \xi(k) + \sum_{i=0}^{p} B_i \mathcal{F} \xi(k - l_i - r_i) \\
+ \sum_{i=0}^{p} B_i Ge(k - r_i - l_i).
\]

(44)
The stability of the above system is equivalent to the stability of system (38). The remaining of the proof is similar to the proof of Theorem 2 and is omitted for brevity. □

Remark 6: From the proofs of Theorems 2 and 3, we can see that the error dynamics $e(k)$ and the system dynamics $x(k)$ are decoupled. Hence the separation principle also holds for these two classes of observer-based output feedback controllers. Consequently, the determination of $\gamma$ in $F$ is independent of the other parameters in the observer.

3.3 Determination of the design parameters

In this subsection, we give a brief discussion on the determination of the parameters $\gamma$ and $\rho$ in the observer-based output feedback controllers. We take the full-order memory observer-based output feedback controller (27), for example.

It follows from Equation (32) that the $x$-system is asymptotically stable if and only if all the zeros of the characteristic equation,

$$0 = \alpha_b(\gamma, z) \triangleq \det \left( zI_n - A - \sum_{i=0}^{p} B_i F(\gamma) z^{-n_i} \right),$$

(45)

are located in the closed unit circle. Clearly, there exist two positive integers $n_i$ and $n_z$ and functions $\alpha_i(\gamma), i \in \{0, n_z\}$, $\alpha_0(\gamma) \neq 0$, such that

$$\alpha_b(\gamma, z) = z^{-n_z} \sum_{i=0}^{n_z} \alpha_i(\gamma) z^i \triangleq z^{-n_z} \beta_b(\gamma, z).$$

(46)

Since $z = 0$ is not a zero of $0 = \alpha_b(\gamma, z)$, Equation (45) reduces to $\beta_b(\gamma, z) = 0$, which is a polynomial equation when $\gamma$ is fixed. Then, the maximal value of $\gamma$ can be computed easily as

$$\gamma_{\text{sup}} = \min\{\gamma > 0 : \max\{|z| : \beta_b(\gamma, z) = 0\} = 1\}. \quad (47)$$

Similarly, the maximal value of $\rho$, denoted by $\rho_{\text{sup}}$, for the $e$-system in Equation (32) can be obtained similarly as

$$\rho_{\text{sup}} = \min\{\rho > 0 : \max\{|\rho| : \beta_e(\rho, z) = 0\} = 1\}, \quad (48)$$

where $\beta_e(\rho, z)$ is defined in a similar way.

For given $\gamma$ and $\rho$, the convergence speeds of the states $x(k)$ and $e(k)$ are, respectively, proportional to

$$z_{\max}^b(\gamma) = \max\{|z| : \beta_b(\gamma, z) = 0\}$$

and

$$z_{\max}^e(\gamma) = \max\{|z| : \beta_e(\rho, z) = 0\}. \quad (49)$$

Then, it follows from $z_{\max}^b(\gamma_{\text{opt}}) = z_{\max}^b(0) = 1$ that there is an optimal value $\gamma_{\text{opt}}$ such that $z_{\max}^b(\gamma)$ is minimised. Denote such an optimal value by $z_{\max}^b(\gamma_{\text{opt}}) = z_{\max}^b(\gamma_{\text{opt}})$. Then, $z_{\max}^b(\gamma_{\text{opt}})$ can be obtained by computing $z_{\max}^b(\gamma)$ for discrete values of $\gamma$ and then choosing the one corresponding to the minimal value of $z_{\max}^b(\gamma)$. Similarly, there is an optimal value $\rho_{\text{opt}}$ such that $z_{\max}^e(\rho)$ is minimised with the minimal value $z_{\max}^e(\rho_{\text{opt}})$. Consequently, the maximal decay rate of the closed-loop system is given by

$$z_{\max} = \max\{z_{\max}^b(\gamma_{\text{opt}}), z_{\max}^e(\rho_{\text{opt}})\}.$$

(51)

4. Memoryless observer-based output feedback

4.1 Observer design by TPF

To introduce our memoryless observer-based output feedback controllers by using the idea of TPF, we first consider the distributed term in $\omega(k)$, namely,

$$\eta(k) = \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{v=1}^{r_i+j} C_j A^{v-r_i-j-1} B_i u(k - v). \quad (52)$$

If the feedback gain $F = F(\gamma)$ is designed such that Equation (16) is satisfied, as $u(k) = F_X(k)$, it is possible to reduce the value of $\gamma$ such that $\|\eta(k)\|$ is reduced to a sufficiently ‘small’ level, and, consequently, the effect of $\eta(k)$ on the first equation of Equation (21) can be ignored safely. As a result, the predictor-based full-order memory observer (21) can be truncated as

$$\begin{cases} z(k+1) = Az(k) + Bu(k) - L(y(k) - Cz(k)), \\ u(k) = Fz(k), \end{cases} \quad (53)$$

where $F$ and $L$ are such that $A + BF$ and $A + LC$ are Schur-stable. It follows that Equation (53) is memoryless.

Similarly, if $F$ is such that Equation (16) is satisfied, the predictor-based reduced-order memory observer (22) may also be truncated as

$$\begin{cases} w(k+1) = Dw(k) + Tu(k) + Ey(k), \\ u(k) = Gw(k) + Hy(k), \end{cases} \quad (54)$$

where the matrices $E$, $T$, $E$, $G$ and $H$ are the same as in Equation (22). This observer-based output feedback controller is also memoryless.

Compared with Equations (21) and (22), both the full-order and reduced-order observer-based output feedback controllers in Equations (53) and (54) only use the current output signal $y(k)$ for feedback, which turn out to be very easy to implement. The aim of the remaining part of this section is to show that these two classes of memoryless observer-based output feedback controllers can indeed solve Problems 1 and 2.

Remark 7: As will be made clear in the proof of the main theorems given later, we can see that the resulting observer
error dynamics associated with both Equations (53) and (54) and the system dynamics are coupled with each other, which indicates that the separation principle does not hold for these two classes of memoryless observer-based output feedback controllers.

### 4.2 Stability analysis: the full-order case

Regarding the stability of the closed-loop system under the full-order memoryless observer-based output feedback controller (53), we can prove the following result.

**Theorem 4:** Assume that \((A, B, C)\) satisfies Assumption 1. Let \(L\) be such that \(A + LC\) is Hurwitz and \(F\) be designed as in Equation (18). Then, for any given arbitrarily large yet bounded delays \([r_i]_i^{p} \) and \([l_i]_i^{q}\), there exists a

\[
\gamma^* = \gamma^* \left( L, [r_i]_i^{p}, [l_i]_i^{q} \right) \in (0, 1),
\]

such that the memoryless full-order observer-based output feedback controller (53) solves Problem 1 for all \(\gamma \in (0, \gamma^*)\).

**Proof:** Let \(e(k) = \chi(k) - z(k)\). It follows from Equations (12) and (53) that

\[
\begin{align*}
\chi(k + 1) &= (A + BF)\chi(k) - BF e(k), \\
e(k + 1) &= (A + LC) e(k) - L \eta(k), \quad k \geq r + 1.
\end{align*}
\]

Associated with the unique positive definite solution \(P\) to the DARE (19), and the solution \(Q > 0\) to the following discrete-time Lyapunov equation,

\[
(A + LC)^TQ(A + LC) - Q = -I_n,
\]

we define

\[
V_1(\chi(k)) = \chi^T(k) P \chi(k), \quad V_2(e(k)) = e^T(k) Q e(k).
\]

The time-shift of \(V_1(\chi(k))\) along the trajectories of the first equation in system (56) is given by

\[
\nabla V_1(\chi(k)) \triangleq V_1(\chi(k + 1)) - V_1(\chi(k)) = \chi^T(k)((A + BF)^T P (A + BF) - P) \chi(k)
\]

\[
- 2\chi^T(k)(A + BF)^T PB F e(k)
\]

\[
+ e^T(k) F^T B^T P B F e(k)
\]

\[
\leq -\gamma \chi^T(k) P \chi(k) - \chi^T(k) F^T F \chi(k)
\]

\[
+ 2\chi^T(k) F^T F e(k) + e^T(k) F^T B^T P B F e(k)
\]

\[
\leq -\gamma \chi^T(k) P \chi(k) + e^T(k) F^T (L_m + B^T PB) F e(k)
\]

\[
\leq -\gamma \chi^T(k) P \chi(k) + \frac{1 - (1 - \gamma)^p}{(1 - \gamma)^p} e^T(k) P e(k),
\]

where we have used Lemma 3, Equations (160) and (161) in Section A.1. Similarly, the time-shift of \(V_2(e(k))\) along the trajectories of the second equation in system (56) can be computed as

\[
\nabla V_2(e(k)) \triangleq V_2(e(k + 1)) - V_2(e(k))
\]

\[
= e^T(k)((A + LC)^T Q (A + LC) - Q) e(k)
\]

\[
+ 2e^T(k)(A + LC)^T Q L \eta(k) + \eta^T(k) L^T Q L \eta(k)
\]

\[
\leq -\|e(k)\|^2 + 2e^T(k)(A + LC)^T Q L \eta(k)
\]

\[
+ \eta^T(k) L^T Q L \eta(k)
\]

\[
\leq -\|e(k)\|^2 + \alpha \eta^T(k) \eta(k),
\]

where

\[
\alpha = 2 \left( (A + LC)^T Q L \right)^T + \|L^T Q L\|
\]

is a constant independent of \(\gamma\). In the following, we will simplify the term \(\eta^T(k) \eta(k)\) in Equation (60). Let \(C_j A^{-n_i} B_j = z_{ij}\). By Lemma 41, we have

\[
\eta^T(k) \eta(k)
\]

\[
\leq (p + 1) \left( \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r_i+l_i} z_{ij} u(k - v) \right)^T
\]

\[
\times \left( \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r_i+l_i} z_{ij} u(k - v) \right)
\]

\[
\leq (p + 1) (q + 1) \left( \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r_i+l_i} u^T(k - v) \right)^T \left( \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r_i+l_i} z_{ij} u(k - v) \right)
\]

\[
\leq (p + 1) (q + 1) (l + r)
\]

\[
\times \left( \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r_i+l_i} u^T(k - v) \right)^T \left( \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r_i+l_i} z_{ij} u(k - v) \right)
\]

\[
\leq (p + 1) (q + 1) (l + r)
\]

\[
\times \left( \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r_i+l_i} u^T(k - v) z_{ij} u(k - v) \right).
\]
Let $c_0 = \max_{i \in [0, p]} \max_{j \in [0, q]} ||z_{ij}||^2$, which is a constant independent of $\gamma$. Then, Equation (62) can be continued as

$$
\eta^T(k) \eta(k) \leq c_0 (p + 1)(q + 1)(l + r) 
\times \sum_{i=0}^{p} \sum_{j=0, j+i>0}^{q} \sum_{v=1}^{r+l} u^T(k - v) u(k - v)
\leq c_0 (p + 1)^2 (q + 1)^2 (l + r) 
\times \sum_{i=1}^{r+l} (\chi^T(k - v) F^T F \chi(k - v)
+ e^T(k - v) F^T F e(k - v))
= c_1 \sum_{i=1}^{r+l} (\chi^T(k - v) F^T F \chi(k - v)
+ e^T(k - v) F^T F e(k - v)),
\tag{63}
$$

where $c_1 = 2c_0(p + 1)^2(q + 1)^2(l + r)$ is also independent of $\gamma$.

Consider another Lyapunov functional,

$$
V_3(\chi_k, e_k) = \sum_{j=-(l+r)}^{-(l+r)+1} \sum_{i=0}^{k-1} (\chi^T(i) F^T F \chi(i)
+ e^T(i) F^T F e(i)).
\tag{65}
$$

Then, it is easy to verify that

$$
\nabla V_3(\chi_k, e_k) = (l + r) \left( \chi^T(k) F^T F \chi(k)
+ e^T(k) F^T F e(k) \right)
\sum_{i=1}^{r+l} \chi^T(k - v) F^T F \chi(k - v)
+ e^T(k - v) F^T F e(k - v).
\tag{66}
$$

Now choose the total Lyapunov functional as

$$
V(\chi_k, e_k) = V_1(\chi(k)) + \sqrt{P} ||V_2(e(k))
+ ac_1 V_3(\chi_k, e_k)).
\tag{67}
$$

Invoking of Equations (59), (60), (64) and (66) yields

$$
\nabla V(\chi_k, e_k) \leq -\gamma \chi^T(k) P \chi(k) + \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \|P\| \|e(k)\|^2
- \frac{1}{2} \sqrt{\|P\|} \|e(k)\|^2 + \sqrt{\|P\| ac_1, (l + r)(\chi^T(k) F^T F \chi(k)
+ e^T(i) F^T F e(k))}
\leq -\gamma \chi^T(k) P \chi(k) + \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \|P\| \|e(k)\|^2
- \frac{1}{2} \sqrt{\|P\|} \|e(k)\|^2 + \sqrt{\|P\| ac_1, (l + r) \times (\chi^T(k) P \chi(k) + \|P\| \|e(k)\|^2)
- \gamma f(\gamma) \chi^T(k) P \chi(k) - \frac{1}{2} \sqrt{\|P\| g(\gamma) \|e(k)\|^2},
\tag{68}
$$

where Lemma 3 has been used, and $f(\gamma)$ and $g(\gamma)$ are two scalar functions defined as

$$
f(\gamma) = 1 - \alpha c_1 (l + r) \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1} \sqrt{\|P\|}}.
\tag{69}
g(\gamma) = 1 - \frac{2(1 - (1 - \gamma)^n)}{(1 - \gamma)^{n-1} \sqrt{\|P\|}}
- 2\alpha c_1 (l + r) \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1} \|P\|}. \tag{70}
$$

Since $\lim_{\gamma \to 0} \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} = n$ and $\lim_{\gamma \to 0} P = 0$, we know that there exists a scalar $\gamma^* = \gamma^*(L, r, l) \in (0, 1)$ such that

$$
\min\{f(\gamma), g(\gamma)\} \geq \frac{1}{2}, \quad \forall \gamma \in (0, \gamma^*). \tag{71}
$$

Consequently, inequality (68) implies

$$
\nabla V(\chi_k, e_k) \leq -\gamma \chi^T(k) P \chi(k) - \frac{1}{4} \sqrt{\|P\| \|e(k)\|^2},
\forall \gamma \in (0, \gamma^*), \forall k \geq l + r. \tag{72}
$$

By Lyapunov stability theorem, we have from Equation (72) that the coupled $(\chi, e)$-system in Equation (56) is asymptotically stable. Consequently, the vectors $z$ and $u$ decay exponentially and thus $x$ does in view of Equation (10). The proof is finished.

**Remark 8:** For a fixed $L$, let $\gamma_{\text{sup}}$ be the maximal value of $\gamma$ such that the closed-loop system (56) is asymptotically stable for all $\gamma \in \langle 0, \gamma_{\text{sup}} \rangle$. Then, the scalar $\gamma^*$ stated in Theorem 4 is an estimate of $\gamma_{\text{sup}}$ and can be computed by carefully checking the proof of this theorem. However, it is possible to compute $\gamma_{\text{sup}}$ exactly. To this end, we write the
closed-loop system as

\[
\sigma(k + 1) = \mathcal{A} \sigma(k) + \sum_{i=0}^{p} \mathcal{B}_i \sigma(k - r_i) + \sum_{i=0}^{q} \mathcal{G}_i \sigma(k - l_i),
\]

where \(\sigma(k) = [x^T(k), z^T(k)]^T\), and

\[
\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & A + BF + LC \end{bmatrix}, \quad \mathcal{B}_i = \begin{bmatrix} 0 & B_i \end{bmatrix},
\]

\[
\mathcal{G}_i = \begin{bmatrix} 0 \\ -LC_i \end{bmatrix}.
\]

As the closed-loop system (73) is time invariant, then similarly to the discussion in Section 3.3, \(\gamma_{\text{opt}}\) can be computed by calculating the roots of the associated characteristic equation for Equation (73), which, as done in Section 3.3, is equivalent to a polynomial equation. By using this method, it is also possible to get the optimal value of \(\gamma\) (denoted by \(\gamma_{\text{opt}}\)) such that the decay rate of system (73) is maximised.

### 4.3 Stability analysis: the reduced-order case

In this subsection, we aim to show that the reduced-order memoryless observer-based output feedback controller (54) also solves Problem 1.

**Theorem 5:** Assume that \((A, B, C)\) satisfies Assumption 1, and the matrices \(D, E, G\) and \(H\) are as in Equation (22) where \(F\) is designed in Theorem 4. Then, for any given arbitrarily large but bounded delays \(r_i\), \(i=1, \ldots, p\), and \(l_j\), \(j=1, \ldots, q\), there exists a scalar \(\gamma_*(D, E, [r_i], [l_j]) \in (0, 1)\), such that the Equation (54) solves Problem 1 for all \(\gamma \in (0, \gamma^*)\).

**Proof:** Since the closed-loop system is linear, we can assume, without loss of generality, that \(k \geq l + r\). Define the error vector \(e(k) = w(k) - TX(k), \forall k \geq l + r\). Then, it is easy to see from Equations (12) and (54) that

\[
e(k + 1) = De(k) - E\eta(k), \quad \forall k \geq l + r.
\]

Moreover, system (12) can be simplified as

\[
\chi(k + 1) = (A + BF)\chi(k) + B(Ge(k) - H\eta(k)), \quad \forall k \geq l + r.
\]

We choose \(V_1(\chi(k)) = \chi^T(k)P\chi(k)\) where \(P\) solves the DARE (19). Then, by using Equations (A.4) and (A.5), we can obtain

\[
\nabla V_1(\chi(k)) = \chi^T(k + 1)P\chi(k + 1) - \chi^T(k)P\chi(k) = \chi^T(k)((A + BF)^T P(A + BF) - P)\chi(k) + 2\chi^T(k)(A + BF)^T PB(Ge(k) - H\eta(k)) + (Ge(k) - H\eta(k))^T B^TPB(Ge(k) - H\eta(k)) = -\gamma \chi^T(k)P\chi(k) - \chi^T(k)F^TF\chi(k) + 2\chi^T(k)F^TP(Ge(k) - H\eta(k)) + (Ge(k) - H\eta(k))^T B^TPB(Ge(k) - H\eta(k)) \leq -\gamma \chi^T(k)P\chi(k) + (Ge(k) - H\eta(k))^T R(Ge(k) - H\eta(k)) \leq -\gamma \chi^T(k)P\chi(k) + 2\varepsilon \chi^T(k)G^TRGe(k) + 2\eta^T(k)H^TRH\eta(k),
\]

(78)

where \(R = I_m + B^TPB\). It follows from Equation (26) that

\[
G = FU^{-1} \begin{bmatrix} I_{n-s} \\ 0 \end{bmatrix}, \quad H = FU^{-1} \begin{bmatrix} 0 \\ I_s \end{bmatrix},
\]

(79)

by which and using inequality (163) in the appendix, we obtain

\[
G^T G = \left(U^{-1} \begin{bmatrix} I_{n-s} \\ 0 \end{bmatrix}\right)^T F^TRF\left(U^{-1} \begin{bmatrix} I_{n-s} \\ 0 \end{bmatrix}\right) \leq \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \|U^{-1}\|^2 \|P\| \left\| \begin{bmatrix} I_{n-s} \\ 0 \end{bmatrix} \right\|^2 I_{n-s} = \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \|P\| \|U^{-1}\|^2 I_{n-s},
\]

(80)

and, similarly,

\[
H^T H \leq \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \|P\| \|U^{-1}\|^2 I_s.
\]

(81)

Therefore, inequality (78) can be continued as

\[
\nabla V_1(\chi(k)) \leq -\gamma \chi^T(k) P\chi(k) + 2\frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \|e(k)\|^2 + 2 \|\eta(k)\|^2, \quad \forall k \geq l + r.
\]

(82)

Let us consider another function \(V_2(\chi(k)) = e^T(k)Qe(k)\), where \(Q > 0\) is the unique positive definite solution to the following discrete-time Lyapunov equation:

\[
D^T QD - Q = -I_{n-s}.
\]

(83)
Then, the time-shift of $V_2(e(k))$ along the trajectories of system (76) can be computed as

$$\nabla V_2(e(k)) = e^T(k)(D^T Q D - Q)e(k) - 2e^T(k) D^T Q E \eta(k) + \eta^T(k) E^T Q E \eta(k)$$

where $d = \|Q^2 + 2QDQD^T\|$ is a constant independent of $\gamma$. Then, from Equations (82) and (84), we obtain

$$\nabla V_1(\chi(k)) + \gamma \nabla V_2(e(k)) \leq -\gamma \chi^T(k) P \chi(k) - \gamma t_1(\gamma) \|e(k)\|^2 + \gamma t_2(\gamma) \|\eta(k)\|^2, \quad \forall k \geq l + r,$$  

where $t_1(\gamma)$ and $t_2(\gamma)$ are two scalar functions defined as

$$t_1(\gamma) = \frac{1}{2} - \frac{2}{\gamma} \frac{(1 - (1 - \gamma)^n)}{(1 - \gamma)^{n-1}} \|P\| \|U^{-1}\|^2,$$  

$$t_2(\gamma) = d + \frac{4}{\gamma} \frac{(1 - (1 - \gamma)^n)}{(1 - \gamma)^{n-1}} \|P\| \|U^{-1}\|^2.$$  

Let $\gamma^*_1 > 0$ be such that

$$t_1(\gamma) \geq \frac{1}{4}, \quad t_2(\gamma) \leq 2d, \quad \forall \gamma \in (0, \gamma^*_1).$$  

The existence of such a $\gamma^*_1$ is due to the fact that $\lim_{\gamma \downarrow 0} P(\gamma) = 0$ and $\lim_{\gamma \downarrow 0} \frac{1}{\gamma} (1 - (1 - \gamma)^n) = n$. Then, Equation (85) simplifies to, for all $\gamma \in (0, \gamma^*_1)$,

$$\nabla V_1(\chi(k)) + \gamma \nabla V_2(e(k)) \leq -\gamma \chi^T(k) P \chi(k) - \frac{1}{\gamma} \|e(k)\|^2 + 2\gamma d \|\eta(k)\|^2, \quad \forall k \geq l + r.$$  

According to the proof of Theorem 4, we have

$$\|\eta(k)\|^2 \leq c \sum_{s=1}^{l+r} u^T(k - s)u(k - s), \quad \forall k \geq l + r, \quad (90)$$

where $c$ is a constant independent of $\gamma$. Since

$$u(k) = Gw(k) + H \eta(k)$$  

$$= G(e(k) + T \chi(k)) + H(C \chi(k) - \eta(k))$$  

$$= (G + HC) \chi(k) + Ge(k) - H \eta(k)$$  

$$= F \chi(k) + Ge(k) - H \eta(k), \quad \forall k \geq l + r,$$  

we have from Equations (163), (80) and (81) that

$$\|u(k)\|^2 = (F \chi(k) + Ge(k) - H \eta(k))^\top (F \chi(k) + Ge(k) - H \eta(k))$$

$$\leq 3(\chi^T(k) F^T F \chi(k) + e^T(k) G^T Ge(k) + \eta^T(k) H^T H \eta(k))$$

$$\leq 3(\alpha \gamma \chi^T(k) P \chi(k) + \beta \gamma \|P\| (\|e(k)\|^2 + \|\eta(k)\|^2)),$$  

$$\forall k \geq l + r, \quad (92)$$

where $\alpha = \alpha(\gamma)$ and $\beta = \beta(\gamma)$ are scalar functions defined as

$$\alpha(\gamma) = \frac{1 - (1 - \gamma)^n}{\gamma (1 - \gamma)^{n-1}}, \quad \beta(\gamma) = \frac{1 - (1 - \gamma)^n}{\gamma (1 - \gamma)^{n-1}} \|U^{-1}\|^2.$$  

It follows that

$$\lim_{\gamma \downarrow 0} \alpha(\gamma) = n, \quad \lim_{\gamma \downarrow 0} \beta(\gamma) = n \|U^{-1}\|^2.$$  

Hence, inequality (90) can be continued as

$$\|\eta(k)\|^2 \leq 3c \alpha \beta \sum_{s=1}^{l+r} \chi^T(k - s) P \chi(k - s) + 3c \beta \gamma \|P\| \sum_{s=1}^{l+r} \|e(k - s)\|^2$$

$$+ 3c \beta \gamma \|P\| \sum_{s=1}^{l+r} \|\eta(k - s)\|^2, \quad \forall k \geq 2(l + r).$$  

Now choose the following two nonnegative functionals,

$$V_3(\eta_k) = 9c \beta \|P\|^2 \sqrt{\frac{l+r}{\gamma}} \sum_{s=1}^{l+r} \|\eta(k - s)\|^2,$$  

$$\forall k \geq 2(l + r), \quad (96)$$

$$V_4(\eta_k) = 3c \beta \gamma \|P\| \sum_{j=-(l+r)}^{l+r} \sum_{i=k-l+j}^{k} \|\eta(i)\|^2,$$  

$$\forall k \geq 2(l + r).$$  

Let $\gamma^*_2 \in (0, \gamma^*_1)$ be such that

$$3c \beta \gamma \|P\| (l + r) \leq 1 - 9c \beta \|P\|^2 \sqrt{\gamma} - \sqrt{\gamma} (2d + 1),$$  

$$\forall \gamma \in (0, \gamma^*_2).$$  

(98)

The existence of such a $\gamma^*_2$ is also due to the fact that $\lim_{\gamma \downarrow 0} P(\gamma) = 0$ and $\lim_{\gamma \downarrow 0} \frac{1 - (1 - \gamma)^n}{\gamma} = n$. Then, applying
Lemma A.0 on inequality (95) gives
\[
\nabla V_3(\eta_k) + \nabla V_4(\eta_k) \leq 3\alpha\gamma \sum_{s=1}^{l+r} \chi^T(k-s) P \chi(k-s) + 3\beta\gamma \| P \| \sum_{s=1}^{l+r} \| e(k-s) \| ^2 - \sqrt{\gamma} (2d+1) \| \eta(k) \| ^2,
\]
\[
\forall k \geq 2(l+r), \ \forall \gamma \in (0, \gamma_2^*). \quad (99)
\]
Choose another two nonnegative functionals \( V_5(\chi_k) \) and \( V_6(e_k) \) as
\[
V_5(\chi_k) = 3\alpha\gamma \sum_{j=-(l+r)}^{-1} \sum_{i=k-1+j}^{k-1} \chi^T(i) P \chi(i),
\]
\[
\forall k \geq 2(l+r), \quad (100)
\]
\[
V_6(e_k) = 3\beta\gamma \| P \| \sum_{j=-(l+r)}^{-1} \sum_{i=k-1+j}^{k-1} \| e(i) \| ,
\]
\[
\forall k \geq 2(l+r), \quad (101)
\]
whose time shifts are, respectively, computed as
\[
\nabla V_5(\chi_k) = 3\alpha\gamma \left( (l+r) \chi^T(k) P \chi(k) \right.
\]
\[
- \sum_{s=1}^{l+r} \chi^T(k-s) P \chi(k-s) \bigg) \quad \text{and} \quad (102)
\]
\[
\nabla V_6(e_k) = 3\beta\gamma \| P \| \left( (l+r) \| e(k) \| ^2
\]
\[
- \sum_{s=1}^{l+r} \| e(k-s) \| ^2 \right). \quad (103)
\]
Therefore, we obtain from Equations (99), (102) and (103) that
\[
\nabla V_3(\eta_k) + \nabla V_4(\eta_k) + \nabla V_5(\chi_k) + \nabla V_6(e_k)
\]
\[
\leq 3\alpha\gamma (l+r) \chi^T(k) P \chi(k) + 3\beta\gamma \| P \| (l+r) \| e(k) \| ^2 - \sqrt{\gamma} (2d+1) \| \eta(k) \| ^2,
\]
\[
\forall k \geq 2(l+r), \ \forall \gamma \in (0, \gamma_2^*). \quad (104)
\]
Choose the total Lyapunov functional \( V(\chi_k, e_k, \eta_k) \) as
\[
V(\chi_k, e_k, \eta_k) = V_1(\chi(k)) + \gamma V_2(e(k)) + \sqrt{\gamma} (V_3(\eta_k)
\]
\[
+ V_4(\eta_k) + V_5(\chi_k) + V_6(e_k) \). \quad (105)
\]
Then, it follows from Equations (99) and (104) that
\[
\nabla V(\chi_k, e_k, \eta_k) = \nabla V_1(\chi(k)) + \gamma \nabla V_2(e(k))
\]
\[
+ \sqrt{\gamma} (\nabla V_1(\eta_k) + \nabla V_4(\eta_k)
\]
\[
+ \nabla V_5(\chi_k) + \nabla V_6(e_k)) \leq -\gamma \chi^T(k) P \chi(k) - \frac{1}{4} \gamma \| e(k) \| ^2
\]
\[
+ 2\gamma d \| \eta(k) \| ^2 + \sqrt{\gamma} (3\alpha\gamma (l+r) \chi^T(k) P \chi(k) + 3\beta\gamma \| P \| (l+r) \| e(k) \| ^2
\]
\[
- \gamma (2d+1) \| \eta(k) \| ^2 \
\]
\[
\leq -\gamma \chi^T(k) P \chi(k) (1 - 3\alpha\gamma \sqrt{l+r}) \| e(k) \| ^2
\]
\[
- \gamma \| \eta(k) \| ^2, \ \forall k \geq 2(l+r). \quad (106)
\]
Let \( \gamma^* \in (0, \gamma_2^*) \) be such that
\[
1 - 3\alpha\gamma\sqrt{l+r} \geq \frac{1}{8}, \quad (107)
\]
\[
\frac{1}{4} - 3\beta \| P \| \sqrt{\gamma} (l+r) \geq \frac{1}{8}, \quad (108)
\]
are satisfied for all \( \gamma \in (0, \gamma^*) \). The existence of such a \( \gamma^* \) is again due to the fact that \( \lim_{\gamma \to 0} P(\gamma) = 0 \). Then, Equation (106) simplifies to
\[
\nabla V(\chi_k, e_k, \eta_k) \leq -\frac{\gamma}{8} (\chi^T(k) P \chi(k) + \| e(k) \| ^2
\]
\[
+ \| \eta(k) \| ^2), \ \forall \gamma \in (0, \gamma^*),
\]
\[
\forall k \geq 2(l+r). \quad (109)
\]
Next, we use Lemma 5 to prove the stability of the closed-loop system. Let
\[
W(\chi_k, e_k, \eta_k) = V(\chi_k, e_k, \eta_k) + \frac{\gamma}{16} \sum_{s=0}^{k-1} (\chi^T(s) P \chi(s)
\]
\[
+ \| e(s) \| ^2 + \| \eta(s) \| ^2) \leq V(\chi_k, e_k, \eta_k) + S(k), \ \forall k \geq 2(l+r), \quad (110)
\]
where \( S(k) \) is a nondecreasing function of \( k \). Then, by using Equation (109), the time shift of \( W(\chi_k, e_k, \eta_k) \) along the closed-loop systems (76) and (77) satisfies
\[
\nabla W(\chi_k, e_k, \eta_k) = \nabla V(\chi_k, e_k, \eta_k) + \nabla S(k)
\]
\[
\leq -\frac{\gamma}{16} (\chi^T(k) P \chi(k) + \| e(k) \| ^2
\]
\[
+ \| \eta(k) \| ^2), \quad (111)
\]
which indicates that
\[ W(\chi_k, e_k, \eta_k) \leq W(\chi_{2l+r}, e_{2l+r}, \eta_{2l+r}), \forall k \geq 2(l + r). \]
Hence
\[
\sum_{s=0}^{\infty} (\chi^T(s) P \chi(s) + \|e(s)\|^2 + \|\eta(s)\|^2)
\]
\[
= \lim_{k \to \infty} \sum_{s=0}^{k-1} (\chi^T(s) P \chi(s) + \|e(s)\|^2 + \|\eta(s)\|^2)
\]
\[
= \frac{16}{\gamma} \lim_{k \to \infty} \sum_{s=0}^{\infty} W(\chi_k, e_k, \eta_k)
\]
\[
\leq \frac{16}{\gamma} W(\chi_{2l+r}, e_{2l+r}, \eta_{2l+r})
\]
\[
< \infty.
\] (112)

Then, by Lemma 5 in the appendix, we have
\[
\lim_{k \to \infty} \left\| \begin{bmatrix} \chi(k) \\ e(k) \\ \eta(k) \end{bmatrix} \right\| = 0.
\] (113)

By \(w(k) = e(k) + T \chi(k)\), we obtain \(\lim_{k \to \infty} \|w(k)\| = 0\), and by Equation (92), we get \(\lim_{k \to \infty} \|u(k)\| = 0\). Hence, according to Equation (10), we get \(\lim_{k \to \infty} \|x(k)\| = 0\). In conclusion, we have
\[
\lim_{k \to \infty} \left\| \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \right\| = 0.
\] (114)

In view of the fact that, for linear time-invariant systems, attractivity implies asymptotic stability, the proof is completed.

**Remark 9:** In the reduced-order case, since the output \(y(k)\) of the system enters the input \(u(k)\) of the system directly through the gain \(H\) (see the second equation in Equation (54)), a difference inequality in the form of
\[
\|\eta(k)\| \leq \|v(k)\| + \sum_{s=1}^{l+r} J(k, s) \|\eta(k-s)\|
\] (115)

is encountered (see Equation (95)), where \(r\) and \(l\) are positive scalars, and \(v(k)\) and \(J(k, s)\) are some piecewise continuous functions. Such an integral delay difference inequality is coupled with the system dynamics as well as the observer dynamics, making the analysis of the stability very complicated. To handle such a difference inequality, we have to introduce some additional Lyapunov functionals \(V_3(\eta_k)\) and \(V_4(\eta_k)\) in Equations (96) and (97)) and a technical lemma (Lemma 6) so that it can be incorporated in the Lyapunov analysis of the overall time-delay systems.

**Remark 10:** Similarly to Remark 8, for fixed matrices \(D\) and \(E\), the maximal and optimal values of \(\gamma\), denoted by \(\gamma_{\text{opt}}\), respectively, in the controller (54) can be obtained by computing the zeros of the characteristic equation of the closed-loop system that can be written as
\[
\sigma(k + 1) = \mathcal{A} \sigma(k) + \sum_{i=0}^{p} \mathcal{B}_i \sigma(k - r_i)
\]
\[
+ \sum_{j=0}^{q} \mathcal{E}_{ij} \sigma(k - l_j),
\] (116)

where \(\sigma(k) = [x^T(k), w^T(k)]^T\), and
\[
\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & D + TBG \end{bmatrix}, \quad \mathcal{B}_i = \begin{bmatrix} 0 & B_i G \end{bmatrix}, \quad \mathcal{E}_{ij} = \begin{bmatrix} 0 & B_i H C_j \end{bmatrix}.
\] (117)

\[
\mathcal{E}_{ij} = \begin{bmatrix} 0 & (TBH + E) C_j \end{bmatrix}, \quad \mathcal{E}_{ij} = \begin{bmatrix} 0 & B_i H C_j \end{bmatrix}.
\] (118)

### 4.4 Semi-global stabilisation

In this subsection, we further show that both the full-order and reduced-order memoryless observer-based output feedback controllers (53) and (54) semi-globally stabilise system (3) in both the \(l_\infty\) and \(l_2\) senses.

**Theorem 6:** Let \((A, B, C)\) satisfies Assumption 1, \(L\) be such that \(A + LC\) is Hurwitz and \(F\) be as designed in Equation (18). Assume that the initial conditions for systems (3) and (53) satisfy \(\chi_0 \in \Omega_a \subset \mathcal{D}_{a,l}, u_0 \in \Omega_d \subset \mathcal{D}_{m,r}\), and \(z_0 \in \Omega_z \subset \mathcal{D}_z\), where \(\Omega_a\), \(\Omega_d\) and \(\Omega_z\) are arbitrarily large yet bounded. Then, for any given arbitrarily large yet bounded delays \(\{r_i\}^p_{i=1}\) and \(\{l_j\}^q_{j=1}\), there exists a
\[
\gamma^* = \gamma^* \left( L, \{r_i\}^p_{i=1}, \{l_j\}^q_{j=1} \right) \in (0, 1),
\] (119)
such that the full-order memoryless observer-based output feedback (53) solve Problem 2 for all \(\gamma \in (0, \gamma^*)\).

**Proof:** We first consider the \(l_\infty\) case. Let \(\gamma^+ \in (0, \gamma^*)\) be such that \(P^+ \leq Q, \gamma \in (0, \gamma^+),\) where \(\gamma^+\) is determined in the proof of Theorem 4. Then, it follows from Equations (163), (67) and (72) that, for all \(\gamma \in (0, \gamma^+),\)
\[
u^T(k) v(k)
\]
\[
= \gamma^+ \chi^T(k) F^T F \chi(k)
\]
\[
= \gamma^+ (\chi(k) - e(k))^T F^T F (\chi(k) - e(k))
\]
\[
\leq \frac{1 - \gamma^+}{1 - \gamma^+} \chi^T(k) P \chi(k) + e^T(k) Pe(k)
\] (120)
\[
\leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \left( \chi^T(k) P \chi(k) + \sqrt{\|P\|} e^T(k) e(k) \right)
\]

\[
\leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \left( \chi^T(k) P \chi(k) + \sqrt{\|P\|} \|e(k)\|^2 \right)
\]

\[
= 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \left( V_1(\chi(k)) + \sqrt{\|P\|} V_2(e(k)) \right)
\]

\[
\leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} V(\chi_k, e_k)
\]

\[
\leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} V(\chi_{l+r}, e_{l+r}), \quad \forall k \geq l + r,
\]

(121)

where \(V_1(\chi(k))\), \(V_2(e(k))\) and \(V(\chi_k, e_k)\) are defined in the proof of Theorem 4. As the set \(\Omega_1 \times \Omega_2 \times \Omega_2\) is bounded and \(l + r\) is finite, we know that \(V(\chi_{l+r}, e_{l+r})\) is bounded. It then follows from (125) that there exists \(\gamma^\ast \in (0, \gamma^+\) such that

\[
\|u(k)\|_\infty \leq 1, \quad \forall k \geq l + r, \quad \forall \gamma \in (0, \gamma^\ast).
\]

(122)

On the other hand, as \(\lim_{\gamma \to 0} F(\gamma) = 0\), \(u(k) = Fz(k)\) and the initial set is bounded, there exists a \(\gamma^\ast \in (0, \gamma^+)\) such that

\[
\|u(k)\|_\infty \leq 1, \quad \forall k \in [1, l + r], \quad \forall \gamma \in (0, \gamma^\ast).
\]

(123)

Combining Equations (122) and (123) gives the desired result.

We next consider the \(l_2\) case. Let \(\gamma^+ \in (0, \gamma^+)\) be such that \(\|P\| \leq 1, \gamma \in (0, \gamma^+)\). Then, it follows from Equations (120) and (72) that

\[
u^T(k) u(k) \leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \left( \gamma \chi^T(k) P \chi(k) + \gamma \|P\| \|e(k)\|^2 \right)
\]

\[
\leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \left( \chi^T(k) P \chi(k) + \sqrt{\|P\|} \|e(k)\|^2 \right)
\]

\[
\leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \left( \chi^T(k) P \chi(k) + \sqrt{\|P\|} e(k) \right)
\]

\[
\leq 2 \frac{1 - (1 - \gamma)^n}{(1 - \gamma)^{n-1}} \left( \chi^T(k) P \chi(k) + \sqrt{\|P\|} \|e(k)\|^2 \right)
\]

\[
\leq - \frac{8(1 - (1 - \gamma)^n)}{(1 - \gamma)^{n-1}} \gamma V(\chi_k, e_k),
\]

\(\forall k \geq l + r\).

(124)

from which we obtain

\[
\sum_{k=l+r}^\infty \|u(k)\|^2 \leq - \frac{8(1 - (1 - \gamma)^n)}{(1 - \gamma)^{n-1}} \sum_{k=l+r}^\infty \gamma V(\chi_k, e_k)
\]

\[
= - \frac{8(1 - (1 - \gamma)^n)}{(1 - \gamma)^{n-1}} \left( \lim_{k \to \infty} V(\chi_k, e_k) \right)
\]

\[
= - \frac{8(1 - (1 - \gamma)^n)}{(1 - \gamma)^{n-1}} V(\chi_{l+r}, e_{l+r}).
\]

(125)

where we have used the fact that the \((\chi, e)\)-system is asymptotically stable for all \(\gamma \in (0, \gamma^+)\) (Theorem 4). Again, as the set \(\Omega_1 \times \Omega_2 \times \Omega_2\) is bounded and \(l + r\) is finite, we know that \(V(\chi_{l+r}, e_{l+r})\) is bounded. It then follows from (125) that there exists a \(\gamma^\ast \in (0, \gamma^+)\) such that

\[
\sum_{k=l+r}^\infty \|u(k)\|^2 \leq \frac{1}{2}, \quad \forall \gamma \in (0, \gamma^\ast).
\]

(126)

On the other hand, similarly to the \(l_\infty\) case, we can see that there exists a \(\gamma^\ast \in (0, \gamma^+)\) such that

\[
\sum_{k=1}^{l+r} \|u(k)\|^2 \leq \frac{1}{2}, \quad \forall \gamma \in (0, \gamma^\ast).
\]

(127)

Combining Equation (126) with Equation (127) completes the proof.

\[
\square
\]

**Theorem 7**: Assume that \((A, B, C)\) satisfies Assumption 1 and the matrices \(D, E, G\) and \(H\) are as in Equation (22), where \(F\) is designed in Theorem 4. Assume that the initial conditions for systems (3) and (54) satisfy \(x_0 \in \Omega_1 \subset \mathbb{R}^n, u_0 \in \Omega_2 \subset \mathbb{R}^{m_r}\) and \(w_0 \in \Omega_3 \subset \mathbb{R}^{r_3}\), where \(\Omega_1, \Omega_2, \Omega_3\) and \(\Omega_4\) are arbitrarily large yet bounded. Then, for any given arbitrarily large but bounded delays \(r_i\) and \(\{r_i\}_{i=1}^\infty\), there exists a scalar

\[
\gamma^\ast \in (0, \gamma^+)
\]

such that the reduced-order memoryless observer-based output feedback controller (54) solves Problem 2 for all \(\gamma \in (0, \gamma^+)\).

**Proof**: We first consider the \(l_\infty\) case. From Equations (92) and (95), we obtain

\[
\frac{1}{\gamma} \|u(k)\|^2 \leq 3(\alpha \chi^T(k) P \chi(k) + \beta \|P\| e(k) \|^2)
\]

\[
+ \beta \|P\| \|\eta(k)\|^2
\]

(129)
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Consider the following Lyapunov functional,

\[ U(\chi_k, e_k, \eta_k) = V(\chi_k, e_k, \eta_k) + V_7(\chi_k) + V_8(e_k), \]

\[ \forall k \geq 2(l + r), \]  

where \( \alpha \) and \( \beta \) are defined in Equation (93). Consider

\[ V_7(\chi_k) = 9\alpha c \beta \| P \| \sum_{s=1}^{\infty} \chi^T(k-s) P \chi(k-s), \]

\[ \forall k \geq 2(l + r) \]  

(131)

\[ V_8(e_k) = 9\beta^2 \| P \|^2 \sum_{s=1}^{\infty} \| e(k-s) \|^2, \]

\[ \forall k \geq 2(l + r). \]  

(132)

By noting that \( \| P \| \leq \mu \gamma, \forall \gamma \in [0, \frac{1}{2}] \), where \( \mu = \mu \left( \frac{1}{2} \right) \) is defined in Corollary 3 in the appendix, we further get

\[ \frac{1}{\gamma} \| u(k) \|^2 \leq 3\alpha V_1(\chi(k)) + \frac{3\beta \mu \gamma}{\lambda_{\min}(Q)} e^T(k) Q e(k) + V_7(\chi_k) + V_8(e_k) + V_7(\chi_k), \]

\[ \leq \delta V_1(\chi(k)) + \gamma V_2(e(k)) + V_7(\chi_k) + V_8(e_k), \]

\[ \forall k \geq 2(l + r), \]  

where \( V_1(\chi(k)), V_2(e(k)), V_3(\eta_k) \) and \( V(\chi_k, e_k, \eta_k) \) are defined in the proof of Theorem 5, and

\[ \delta = \max \left\{ 3\alpha, \frac{3\beta \mu}{\lambda_{\min}(Q)}, 1, \beta \right\}. \]  

(134)

Consider the following Lyapunov functional,

\[ U(\chi_k, e_k, \eta_k) = V(\chi_k, e_k, \eta_k) + V_7(\chi_k) + V_8(e_k), \]

\[ \forall k \geq 2(l + r). \]  

(135)

By Equations (106), (131) and (132), we can compute

\[ \nabla U(\chi_k, e_k, \eta_k) = \nabla V(\chi_k, e_k, \eta_k) + \nabla V_7(\chi_k) + \nabla V_8(e_k), \]

\[ \leq -\gamma \chi^T(k) P \chi(k) (1 - 3\alpha \sqrt{\gamma} (l + r) - 9\alpha c \beta \| P \|) \]

\[ \leq -\gamma \left( \frac{1}{4} - 3\beta \| P \| \sqrt{\gamma} (l + r) - 9\beta^2 \| P \|^2 \right) \| e(k) \|^2, \]

\[ \forall k \geq 2(l + r). \]  

(136)

Let \( \gamma^* \in (0, \gamma^*) \) be such that

\[ 1 - 3\alpha \sqrt{\gamma} (l + r) - 9\alpha c \beta \| P \| \geq \frac{1}{8}, \]

\[ \frac{1}{4} - 3\beta \| P \| \sqrt{\gamma} (l + r) - 9\beta^2 \| P \|^2 \geq \frac{1}{8}, \]

are satisfied with \( \gamma \in [0, \gamma^*] \), where \( \gamma^* \) is determined in the proof of Theorem 5. The existence of \( \gamma^* \) is due to \( \lim_{\gamma \to 0} \mathcal{P}(\gamma) = 0 \). Hence, inequality (136) becomes

\[ \nabla U(\chi_k, e_k, \eta_k) \leq -\gamma \left( \chi^T(k) P \chi(k) + \| e(k) \|^2 + \| \eta(k) \|^2 \right), \]

\[ \forall k \geq 2(l + r), \forall \gamma \in [0, \gamma^*]. \]  

(139)

The remaining of the proof is similar to that of Theorem 6 and is thus omitted.

We next consider the \( l_2 \) case. Let \( \gamma^* \in (0, \gamma^*) \) be such that \( \frac{\gamma}{\alpha} \| P \| \leq 1 \) and \( \frac{\beta}{\alpha} \| P \| \leq 1 \), \( \forall \gamma \in [0, \gamma^*] \), where \( \gamma^* \) is determined in the proof of Theorem 5. Then, it follows from Equations (129) and (109) that

\[ \frac{1}{\gamma} \| u(k) \|^2 \leq 3\alpha \chi^T(k) P \chi(k) + \frac{\beta}{\alpha} \| P \| \| e(k) \|^2 \]

\[ + \frac{\beta}{\alpha} \| P \| \| \eta(k) \|^2 \]

\[ \leq 3\alpha \chi^T(k) P \chi(k) + \| e(k) \|^2 + \| \eta(k) \|^2 \]

\[ = -\frac{24\alpha}{\gamma} \nabla V(\chi_k, e_k, \eta_k), \forall k \geq 2(l + r), \]

\[ \forall \gamma \in [0, \gamma^*]. \]  

(140)

The remaining of the proof is similar to that of Theorem 6 and thus the proof is completed. \( \square \)

**Remark 11:** We can also show that the memory observer-based output feedback controllers considered in Section 3 solve Problem 2. To this end, we need to construct some Lyapunov–Krasovskii functionals for the closed-loop system rather than using the separation principle as done in Section 3. The details are however omitted for brevity.
5. A numerical example

In this section, we use a numerical example to demonstrate the effectiveness of the proposed approach. Consider a discrete-time linear time-delay system in the form of Equation (3) with $p = 2$, $q = 3$, $r_1 = 1$, $r_2 = 2$, $l_1 = 2$, $l_2 = 3$, $l_3 = 4$ and

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

(141)

$$B_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

(142)

$$C_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -\sqrt{2} & -5 & 1 & 0 \\ -1 & -3 & 0 & -1 \end{bmatrix},$$

(143)

$$C_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. $$

(144)

Notice that $A$, $B_0$, $B_1$ and $B_2$ are taken from Zhou et al. (2013b). Direct computation gives $\lambda(A) = \{\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1, -1\}$, and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. $$

(145)

It follows that $(A, B)$ are controllable and $(A, C)$ are observable, namely, Assumption 1 is fulfilled. Consequently, solutions to the DAREs (19) and (29) can be obtained as

$$P = \begin{bmatrix} \frac{\gamma(2-3\gamma+2\gamma^2)}{(\gamma-1)^2} & 0 & -\frac{\gamma^2}{(\gamma-1)^2} & 0 \\ 0 & \frac{\gamma(\gamma-2)}{(\gamma-1)^2} & 0 & -\frac{\gamma^2}{(\gamma-1)^2} \\ -\frac{\gamma^2}{(\gamma-1)^2} & 0 & \frac{\gamma(\gamma-2)}{(\gamma-1)^2} & 0 \\ 0 & -\frac{\gamma^2}{(\gamma-1)^2} & 0 & \frac{\gamma(\gamma-2)}{(\gamma-1)^2} \end{bmatrix},$$

(146)

$$Q = \begin{bmatrix} \frac{\rho(2-3\rho+2\rho^2)}{(\rho-1)^2} & 0 & \frac{\rho^2}{(\rho-1)^2} & 0 \\ 0 & \frac{\rho(\rho-2)}{(\rho-1)^2} & 0 & -\frac{\rho^2}{(\rho-1)^2} \\ \frac{\rho^2}{(\rho-1)^2} & 0 & \frac{\rho(\rho-2)}{(\rho-1)^2} & 0 \\ 0 & -\frac{\rho^2}{(\rho-1)^2} & 0 & \frac{\rho(\rho-2)}{(\rho-1)^2} \end{bmatrix},$$

(147)

associated with which the feedback gains $F$ and $L$ defined in Equations (18) and (28) can be computed as

$$F = -\begin{bmatrix} \sqrt{2}\gamma(1-\gamma) & 0 & \sqrt{2}\gamma & 0 \\ 0 & 2\gamma & 0 & \gamma^2 \end{bmatrix},$$

(148)

$$L = -\begin{bmatrix} 0 & \frac{\sqrt{2}}{\rho^2} & \frac{\sqrt{2}}{\rho^2} & \rho (\rho - 2) \\ \frac{\rho^2}{\gamma} & 0 & 0 & -\frac{\sqrt{2}}{\rho^2} \rho (\rho - 2) \\ -\rho^2 + \rho (\rho - 2) & \frac{\sqrt{2}}{\rho^2} & \frac{\sqrt{2}}{\rho^2} & \rho (\rho - 2) \end{bmatrix}. $$

(149)

We consider three types of observer-based output feedback controllers for this system:

1. The full-order memory observer-based output feedback controller

$$z(k+1) = Ax(k) + \sum_{i=0}^{2} B_i u(k-r_i), \quad u(k) = F z(k),$$

(150)

which is associated with Equation (27). The matrices $F$ and $L$ are, respectively, given by Equations (148) and (149).

2. The full-order memoryless observer-based output feedback controller

$$z(k+1) = Az(k) + Bu(k) - L (y(k) - Cy(k)), \quad u(k) = F z(k),$$

(151)

which is associated with Equation (53). Here, for simplicity and the purpose of comparison, the matrices $F$ and $L$ are also designed as Equations (148) and (149).

3. The reduced-order memoryless observer-based output feedback controller

$$w(k+1) = Dw(k) + TBu(k) + Ey(k), \quad u(k) = G w(k),$$

(152)

which is associated with Equation (54). Here the matrices $D, E, G, H$ and $T$ are chosen as

$$D = \begin{bmatrix} 1 - \sqrt{3} & 0 \\ 0 & 1 - \sqrt{3} \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

(153)

$$G = \begin{bmatrix} 0 & -2(\sqrt{5} + 5 - \sqrt{2} - 2\sqrt{3})(\gamma - 1) \end{bmatrix},$$

(154)
Figure 1. The functions $z_{\text{max}}^b(\gamma)$ and $z_{\text{max}}^c(\rho)$ associated with the memory observer (150).

$$H = -\gamma(4 - 2\sqrt{3} + \gamma)(2\sqrt{3} + \sqrt{2} - 2)(6 + \gamma + 4\sqrt{3} - 5\sqrt{2} - 3\sqrt{6})\gamma, \quad (155)$$

$$T = \begin{bmatrix} 0 & (2\sqrt{3} + \sqrt{2} - 2)(6 + \gamma + 4\sqrt{3} - 5\sqrt{2} - 3\sqrt{6})\gamma \\ -7 - 4\sqrt{3} & 0 \end{bmatrix}, \quad (156)$$

which satisfy Equations (23) and (24) with $F$ being designed in Equation (148). It follows that $D$ is Schur stable and Equation (25) is satisfied.

We first discuss how to choose the parameters $\gamma$ and $\rho$ in the full-order memory observer-based output feedback controller (150). According to the discussion in Section 3.3, the functions $z_{\text{max}}^b(\gamma)$ and $z_{\text{max}}^c(\rho)$ defined in Equations (49) and (50) are shown in Figure 1 from which we see that $z_{\text{max}}^b(\gamma)$ and $z_{\text{max}}^c(\rho)$ are minimised with $\gamma = \gamma_{\text{opt}} = 0.1114$ and $\rho = \rho_{\text{opt}} = 0.0995$, respectively. Hence, according to Equation (51), the maximal decay rate of the closed-loop system under Equation (150) is given by $z_{\text{max}} = \max\{z_{\text{max}}^b(\gamma_{\text{opt}}), z_{\text{max}}^c(\rho_{\text{opt}})\} = \max(0.9435, 0.9498) = 0.9498$. In our simulation, we will choose $\gamma = \gamma_{\text{opt}}$ and $\rho = \rho_{\text{opt}}$.

We next consider the determination of the parameters in the full-order memoryless observer (151). Similarly, to the discussion in Section 3.3, we will compute $z_{\text{max}}(L)$, which is the minimised (with $\gamma$ as the variable) maximal modulus of the zeros for the characteristic equation associated with the closed-loop system for a fixed $L$. Since $L$ is parameterised by $\rho$ as shown in Equation (149), we will write, for notational simplicity, $z_{\text{max}}(L)$ as $z_{\text{max}}(\rho)$. Then, the function $z_{\text{max}}(\rho)$ is shown in Figure 2, from which we find that it is minimised with $\rho = \rho_{\text{opt}} = 0.123$. The associated optimal value of $\gamma$ is found to be $\gamma_{\text{opt}} = 0.218$. Hence the maximal convergence rate that the full-order memoryless observer (151) can achieve is $z_{\text{max}}(\rho_{\text{opt}}) = 0.9245$, and moreover, such a maximal decay rate is achieved with

Figure 2. The function $z_{\text{max}}(\rho)$ associated with the full-order memoryless observer (151).

Figure 3. The functions $z_{\text{max}}(\gamma)$ associated with the reduced-order memoryless observer (152).

Figure 4. State and observer state associated with the memory observer (150).
Figure 5. State and observer state associated with the full-order memoryless observer (151).

\[ \rho = \rho_{\text{opt}} = 0.123 \text{ and } \gamma = \gamma_{\text{opt}} = 0.218, \]

which will be chosen in the simulation.

We finally consider the determination of the parameters in the memoryless reduced-order observer (152). Similarly, for the given parameters \(D\) and \(E\) in Equation (153) and a given \(\gamma\), we denote the maximal modulus of the zeros for the closed-loop system by \(z_{\text{max}}(\gamma)\), which is shown in Figure 3. It follows that \(z_{\text{max}}(\gamma)\) is minimised with \(\gamma = \gamma_{\text{opt}} = 0.069\) and, accordingly, \(z_{\text{max}}(\gamma_{\text{opt}}) = 0.9303\), which is the maximal decay rate that Equation (152) can achieve with the parameters in Equation (153). We thus choose \(\gamma = \gamma_{\text{opt}}\) in the simulation.

For comparison purpose, the initial conditions for the open-loop system are chosen as \(x(\theta) = [-6, 6, -6, -6]^T\), \(u(\theta) = 0\) and \(\gamma(\theta) = 0\), \(\forall \theta \in [-4, 0]\), and all the initial conditions for these three classes of observers are set to be zero. Then, the state trajectories of the closed-loop systems under these three types of observer-based output feedback controllers are, respectively, shown in Figures 4–6, from which we see that all these controllers stabilise the considered time-delay system. Finally, the 2-norm of the state vectors of the closed-loop system under these three types of observers are recorded in Figure 7 from which we can see that the full-order memoryless observer-based output feedback controller outperforms the other ones.

6. Conclusion

This paper has studied observer-based output feedback control of discrete-time linear systems with both multiple input and output delays. Based on the predictor feedback theory, our recently developed TPF approach is established to design the controllers. Two classes of controllers, namely, the memory observer and memoryless observer-based controllers, are proposed. For both of these two classes of controllers, both full-order and reduced-order observers are considered. It is shown that the separation principle holds in the memory observer-based output feedback scheme and does not hold for the memoryless observer-based one. As a result, the stability of the memory observer-based output feedback control system can be proven with the help of the state feedback results established earlier and the stability of the memory observer-based output feedback control systems has to be proven via Lyapunov stability theory. Intricate Lyapunov stability analysis has been carried out for this purpose. Finally, the established results are illustrated to be effective via a numerical example.

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### Appendix

#### A.1 The proof of Lemma 1

According to the development in Zhou et al. (2013b), the first equation in Equation (3) can be simplified as the first equation in Equation (12). So we remain to show that the second equations in Equations (3) and (12) are equivalent. It follows from Equations (10) and (14) that

$$C(x(k) - y(k)) = \sum_{i=0}^{q} C_i A^{-i} \left( x(k) + \sum_{j=1}^{r_i} \sum_{j=1}^{p} A^{j-1} B_i u(k - j) \right) - \sum_{j=0}^{q} C_j x(k - l_j)$$

$$= \sum_{j=0}^{q} C_j \left( A^{(i-l)} x(k) - x(k - l_j) \right)$$

$$+ \sum_{i=0}^{q} C_i A^{-i} \sum_{j=1}^{r_i} \sum_{j=1}^{p} A^{j-1} B_i u(k - j)$$

$$= \sum_{j=1}^{q} C_j \left( A^{(i-l)} x(k) - x(k - l_j) \right)$$

$$+ \sum_{i=0}^{q} C_i A^{-i} \sum_{j=1}^{r_i} \sum_{j=1}^{p} A^{j-1} B_i u(k - j).$$

(157)
By using the first equation in Equation (3), we can compute
\[
x(k) = A^j x(k - l_j) + \sum_{i=0}^{l_j-1} A^{j-i-1} \sum_{i=0}^p B_i u(k - r_i - l_j + v),
\]
substitution of which into Equation (157) gives
\[
C \chi(k) - y(k)
= \sum_{j=1}^q C_j A^{-j} \left( \sum_{i=0}^{l_j-1} A^{j-i-1} \sum_{i=0}^p B_i u(k - r_i - l_j + v) \right)
+ \sum_{i=1}^p r_i \sum_{i=1}^q A^{j-i-1} B_i u(k - v)
+ C_0 \sum_{j=1}^q C_j A^{-j} \left( \sum_{i=1}^p r_i \sum_{i=1}^q A^{j-i-1} B_i u(k - j) \right)
= \sum_{j=1}^q C_j A^{-j} \left( \sum_{i=0}^{l_j-1} A^{j-i-1} B_i u(k - v) \right)
+ \sum_{i=1}^p r_i \sum_{i=1}^q A^{j-i-1} B_i u(k - v)
+ C_0 \sum_{j=1}^q C_j A^{-j} \left( \sum_{i=1}^p r_i \sum_{i=1}^q A^{j-i-1} B_i u(k - j) \right)
= \sum_{j=0}^q C_j A^{-j} \sum_{i=1}^{l_j} A^{j-i-1} B_i u(k - v)
+ \sum_{i=1}^p r_i \sum_{i=1}^q A^{j-i-1} B_i u(k - v)
= \sum_{i=1}^p q \sum_{i=1}^{l_j} A^{j-i-1} B_i u(k - v)
+ \sum_{i=1}^p r_i \sum_{i=1}^q A^{j-i-1} B_i u(k - v)
= \sum_{i=1}^p \sum_{i=1}^q \sum_{j=0}^{l_j} A^{j-i-1} B_i u(k - v),
\]
which completes the proof.

### A.2 Some technical lemmas

We first collect some relevant properties of solutions to the DARE (19). It is easy to verify that the following relation is true:

\[
(A + BF)P(A + BF) - P = -\gamma P - F^T F,
\]  

where $A + BF$ is the system matrix of the closed-loop system. Moreover,
\[
F = B^T P(A + BF).
\]  

Some important properties of the DARE (19) will be summarised as follows Zhou and Lin (2011).

**Lemma 3:** Assume that $(A, B)$ satisfies Assumption 1. Then, the parametric DARE (19) has a unique positive definite solution $P = W(\gamma)$ for any $\gamma \in (0, 1)$, where $W$ is the unique positive definite solution to the following parametric Lyapunov equation:
\[
W = \frac{1}{1 - \gamma} AWA^T = -BB^T,
\]

and the limit of $P$ as $\gamma$ approaches zero exits and is zero, namely, $\lim_{\gamma \to 0} P = 0$ and
\[
F^T F \leq F^T (I_m + B^T P B) F \leq \frac{1 - (1 - \gamma)^p}{(1 - \gamma)^{q+1}} P.
\]

Moreover, for any number $c \in (0, 1)$, there exists a number $\mu = \mu(c) > 0$ such that $P(\gamma) \leq \mu(\gamma) c$, $\forall \gamma \in [0, c]$.

We recall the following discrete-time Jensen inequality from Gu et al. (2003).

**Lemma 4:** For any positive definite matrix $M > 0$, two integers $r_2$ and $r_1$ with $r_2 \geq r_1$, and a vector-valued function $\omega_i = \omega(i) : [r_1, r_2) \to \mathbb{R}^r$, then
\[
\left( \sum_{i=r_1}^{r_2} \omega_i \right)^T M \left( \sum_{i=r_1}^{r_2} \omega_i \right) \leq (r_2 - r_1 + 1) \sum_{i=r_1}^{r_2} \omega_i^T M \omega_i.
\]

We also need the following result on sequences of real numbers.

**Lemma 5** (Goldberg, 1964): Let the sequence $\{a_n\}_{n=0}^\infty$ be non-negative and denote $s_n = \sum_{i=0}^n a_i$. If the sequence $\{s_n\}_{n=0}^\infty$ is bounded above, then $s_n$ converges to $s_\infty = \sum_{i=0}^\infty a_i$, and moreover, $\lim_{n \to \infty} a_n = 0$.

We finally give the following technical lemma which is essential in our proof of Theorem 5.

**Lemma 6:** Assume that $\lambda(k) : [-r, \infty) \to \mathbb{R}^+$ satisfies
\[
\lambda(k) \leq f(k) + c \sum_{i=1}^r \lambda(k - s), \quad \forall k \geq 0,
\]

where $c$ is a positive scalar, $r$ is a positive integer and $f(k) : \mathbb{N} \to \mathbb{R}^+$ is a given function. Let
\[
V_1(\lambda) = \sum_{i=r-1}^k \lambda(i), \quad V_2(\lambda) = c \sum_{j=-r}^{k-r} \sum_{i=1}^{k-1} \lambda(i), \quad \forall k \geq 0.
\]

Let $V(\lambda) = \epsilon V_1(\lambda) + V_2(\lambda)$. If there exist two scalars $\epsilon, \epsilon \in [0, 1)$ such that
\[
cr + \epsilon + \epsilon \leq 1,
\]  

\[
V(\lambda) \leq \epsilon V_1(\lambda) + V_2(\lambda).
\]
then there holds
\[ \nabla V(\lambda_k) \leq f(k) - \epsilon \lambda(k) - \epsilon \lambda(k - r) \leq f(k) - \epsilon \lambda(k), \quad \forall k \geq 0. \]  

**Proof:** By using Equation (165), the time shift of \( V_1(\lambda_k) \) satisfies
\[ \nabla V_1(\lambda_k) = \lambda(k) - \lambda(k - r) \leq f(k) + c \sum_{s=1}^{r} \lambda(k - s) - \lambda(k - r). \]  
By noting that
\[ \nabla V_2(\lambda_k) = c \left( r \lambda(k) - \sum_{s=k-r}^{k-1} \lambda(s) \right), \]  
we get from Equations (169) and (167) that
\[ \nabla V_1(\lambda_k) + \nabla V_2(\lambda_k) \leq cr \lambda(k) + f(k) - \lambda(k - r) \leq (1 - \epsilon - \epsilon) \lambda(k) - (1 - \epsilon) \lambda(k - r) - \epsilon \lambda(k) - f(k) - \epsilon \lambda(k - r). \]  
which completes the proof. \( \square \)