Distributed and Truncated Reduced-Order Observer Based Output Feedback Consensus of Multi-Agent Systems

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Abstract—This technical note is concerned with output feedback consensus of both continuous-time and discrete-time multi-agent systems (MASs) characterized by high-order linear systems with directed communication topologies. Distributed reduced-order observer based protocols are established by only using the relative outputs and inputs of neighboring agents. It is shown that consensusability by the proposed output feedback protocols is equivalent to the consensusability by state feedback protocol. Under the condition that the open-loop dynamics of the MASs is not exponentially unstable, a truncated reduced-order observer based protocol utilizing only the relative outputs of neighboring agents is also established. A numerical example is given to illustrate the effectiveness of the proposed approaches.

Index Terms—Consensus, multi-agent systems, output feedback, reduced-order observer, truncation.

I. INTRODUCTION

Cooperative control has broad applications in formation control, flocking, and complex networks [4], [11]. A group of agents working cooperatively can accomplish some complex missions that cannot be accomplished by a single one [12]. Hence, cooperative control of multi-agent systems (MASs) has received significant research attention in recent years (see [1], [3], [7], [12], [19] and the references therein). Consensus is a fundamental approach to achieve cooperative control. Consensus means the agreement of a group of agents on their common states via local interaction. Consensus of MASs by relative state feedback has been extensively studied in the literature (see [9], [12], [13] and the references cited there). For continuous-time linear MASs, it is known that state feedback consensus can be achieved if and only if the communication topology contains a spanning tree and each agent is stabilizable (see, for example, [18]). However, for discrete-time linear MASs, the consensusability condition is considerably harder to derive than its continuous counterpart [5], [17]. For single input MASs, the consensusability condition can be expressed by Mahler measure and the eigen-ratio of the graph [5], [17]. Since the relative states between the neighboring agents are not always available while the relative outputs are most accessible, consensus by output feedback is more realistic than state feedback in practice. However, analogously to the traditional control systems design, consensus by static output feedback is generally quite restrictive (see [16] for more detailed explanation). An alternative is to use observer based, especially, full-order observer based output feedback. Existing results on this topic can be found in [6], [8], [9], [18] and the references therein. On the other hand, reduced-order based output feedback is theoretically appealing as it requires less additional dynamics and is thus more easy to implement. Recently, by extending the traditional reduced-order observer-based controller for a single system to those for MASs, reduced-order observer based protocols were established in [10].

In this technical note we will propose some new reduced-order observer based protocols. We first establish a new distributed reduced-order observer based protocol that only utilizes the relative outputs and inputs of the neighboring agents. It is proven that the consensus of the agents can be achieved by the proposed reduced-order observer based protocols as long as it can be achieved by state feedback protocols. The separation principle is shown to be true for the proposed reduced-order observer based scheme. In addition, under the condition that the open-loop dynamics of the MAS is not exponentially unstable, a truncated reduced-order observer based protocol which only utilizes the relative outputs of the neighboring agents is established. The proposed two approaches are applicable to both continuous-time and discrete-time MASs. A numerical example is worked out to illustrate the effectiveness of the established approaches. We emphasize that, compared with the reduced-order observer based protocols given in [10], the proposed protocols possess several appealing features. On the other hand, compared with the reduced-order observer based protocol in [10] where the absolute output measurement is required, the one proposed here is distributed, which is more practical since absolute output measurement is not available in many cases. On the other hand, different from the reduced-order observer in [10] where the states of the observers diverge to infinity if a non-trivial consensus is achieved, the states of the observers proposed in this technical note are proven to be asymptotically stable. Finally, for discrete-time MASs, different from [10], the approaches proposed in this technical note not only enable us to find explicit observer gains, but also allow the dynamics of the agents to contain exponentially unstable poles.

The remainder of this technical note is organized as follows. A distributed reduced-order observer based protocol is firstly proposed in Section II. A truncated version is then introduced in Section III. In Section IV, a numerical example is worked out to illustrate the theoretical results. Finally, Section V concludes the technical note.

II. CONSENSUS BY REDUCED-ORDER OBSERVER BASED OUTPUT FEEDBACK

We consider a MAS of $N$ identical linear agents with dynamics

\[ x_i^+ = A x_i + B u_i, \quad y_i = C x_i, \quad i \in \{1, N\} \Delta \{1, 2, \ldots, N\} \]

where $(A, B, C) \in (\mathbb{R}^{n \times n}, \mathbb{R}^{n \times m}, \mathbb{R}^{r \times n})$ is a given matrix triple and such that $(A, B)$ is stabilizable and $(A, C)$ is detectable. Without loss of generality, we assume that $C$ has full row rank. Here $x^+$ denotes $x$ in the continuous-time setting and denotes $x(t + 1)$ for the discrete-time setting in this case $t \in \mathbb{Z} \equiv \{0, 1, 2, \ldots\}$. Let the communication topology among these agents be characterized by a weighted directed graph $G(\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N}$ is the node set, $\mathcal{E}$ is the edge set, and $\mathcal{A} = \{a_{ij}\} \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix. Denote the corresponding Laplacian by $L = [l_{ij}] \in \mathbb{R}^{N \times N}$.

Assume that Agent $i$ collects the relative outputs of its neighbors for feedback, namely, the signal

\[ z_i = \sum_{j \in \mathcal{N}_i} a_{ij} (y_j - y_i) = \sum_{j=1}^{N} l_{ij} y_j, \]

where $\mathcal{N}_i$ is the set of neighbors of agent $i$.
is available for Agent $i$, $\forall i \in [1, N]$, where $N_i = \{j: a_{ij} \neq 0\}$ is the set of neighbors for Agent $i$. Output feedback consensus refers to designing a (dynamic) protocol $u_i = u_i(x_i^r, \cdot, \cdot, \cdot)$, $i \in [1, N]$, such that the states of the MAS (1) satisfies

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \forall i, j \in [1, N]. \tag{3}$$

While only the relative output $z_i$ is available for feedback and the static output feedback protocol $u_i = K z_i$ is quite restrictive for solving the consensus problem (see, for example, [16], for explanation), observer based (or dynamic) output feedback protocol is a good choice. In this case, some full-order observer based output feedback protocols have been established in the literature (see [6], [8], [9], [18] and the references therein). Just as traditional observer-based output feedback control of a single linear system, reduced-order observer based output feedback is more appealing than full-order one since it is in lower order which saves operational calculations in the protocol implementation. Hence, in this technical note, we are interested in the design of reduced-order observer based output feedback protocol in the form of $\mathcal{F}(\cdot, z_i, t)$, $u_i = K(z_i, \xi_i, \cdot, \cdot, \cdot, \cdot)$, $i \in [1, N]$, such that the consensus of the MAS (1) is achieved.

To derive our reduced-order observer based protocol, we first consider a new state vector $r_i = \sum_{j=1}^{N} a_{ij} x_j - x_i$, on which the dynamics in (1) can be represented as

$$r_i^+ = Ar_i + B \sum_{j=1}^{N} l_{ij} u_j, \quad z_i = Cr_i, \quad i \in [1, N]. \tag{4}$$

It follows that, if $r_i$ and $u_i = \sum_{j=1}^{N} l_{ij} u_j$, are respectively considered as the state and input of a new Agent $i^*$, then the relative output $z_i$ in (2) now represents the absolute output of Agent $i^*$ (in comparison with the relative output of Agent $i$). This observation, together with the traditional reduced-order observer design for a single linear system, motivate us to establish the following reduced-order observer based output feedback protocol

$$\begin{bmatrix} \xi_i^+ = F \xi_i + H z_i + TB \sum_{j=1}^{N} l_{ij} u_j; \\ u_i = K_1 z_i + K_2 \xi_i, \end{bmatrix}, \quad i \in [1, N], \tag{5}$$

where $F \in \mathbb{R}^{(n-1) \times (n-1)}$ is Hurwitz (Schur stable for the discrete-time case), $[T, H]$ satisfies

$$TA - FT = HC \tag{6}$$

and is such that $W = [C^r]^\top$ is nonsingular, and $K_1$ and $K_2$ are gains to be specified. It is well known that, if $[F, H]$ is stabilizable, $(A, C)$ is detectable and $\lambda(A) \cup \lambda(F) = \emptyset$ (here $\lambda(A)$ denotes the eigenvalue set of $A$), the probability for $W$ to be nonsingular is 1 [2], [10].

The purpose of the remainder of this section is to show that the reduced-order observer based output feedback protocol (5) can indeed solve the consensus problem. To this end, the following assumption on the communication topology among the agents is necessary (see, for example, [12]).

**Assumption 1:** The communication topology $\mathcal{G}(N, \mathcal{E}, \mathcal{A})$ contains a directed spanning tree.

**Lemma 1:** [12] Under Assumption 1, there exists an $N \times N$ nonsingular matrix $U$ characterized by

$$U = [I_N, U_{12}], \quad U^{-1} = \begin{bmatrix} r_1^\top \\ V_{21} \end{bmatrix} \tag{7}$$

where $I_N \triangleq [1, 1, \ldots, 1]^\top \in \mathbb{R}^N$, $r_1^\top I_N = 1$ and $r_1^\top L = 0$, such that

$$U^{-1} LU = \begin{bmatrix} 0 & 0 \\ 0 & D_r \end{bmatrix} \triangleq J_L, \tag{8}$$

where $D_r \in \mathbb{C}^{(N-1) \times (N-1)}$, is an upper-triangular matrix whose diagonal elements are $\lambda_i, i \in [2, N]$ satisfying $\Re\{\lambda_i\} > 0, i \in [2, N]$.

For future use, we define the augmented state vector for the closed-loop dynamics as

$$\chi = [x_1^r, x_2^r, \ldots, x_N^r]^\top, \quad \chi = [x_i^\top, \xi_i^\top]^\top. \tag{9}$$

Then the main result in this section can be stated as follows.

**Theorem 1:** For the MAS (1) which satisfies Assumption 1, the reduced-order observer based protocol (5) solves the output feedback consensus problem if there exists a matrix $K$ such that $A + \lambda_i BK, i \in [2, N]$, are Hurwitz (Schur stable for the discrete-time case). In this case,

$$[K_1, K_2] = KW^{-1}. \tag{10}$$

Moreover, the states of the observers satisfy

$$\lim_{t \to \infty} \|\xi_i(t)\| = 0, \quad \forall i \in [1, N], \tag{11}$$

and the states of the agents satisfy, for all $i \in [1, N]$,

$$z_i(t) \overset{t \to \infty}{\longrightarrow} \begin{cases} \chi(0) & \text{continuous-time case;} \\ A_3 \left[ \begin{array}{c} \int_0^t e^{A(t-s)} B K_2 F \chi(0) \, ds \end{array} \right] & \text{discrete-time case.} \end{cases} \tag{12}$$

**Proof:** The closed-loop network dynamics consisting of (1) and (5) can be written as

$$\chi^+ = \left( \sum_{k=0}^{2} (L^k \otimes A_k) \right) \chi \tag{13}$$

where $A_k, k = 0, 1, 2$, are related with

$$\begin{bmatrix} A_3 & BK_2 \\ 0 & F \end{bmatrix}, \begin{bmatrix} B K_2 C & 0 \\ HC & TB K_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & TB K_1 C \end{bmatrix}. \tag{14}$$

Consider the following new augmented state vector

$$\omega = [(I_N - 1_N R_1) \otimes I_{2^{n-1}}] \chi. \tag{15}$$

Thus, by the definition of $\omega$, $\chi_1 - \chi_2 - \cdots - \chi_N$ if $\omega = 0$ (see, for example, [10]), namely, the consensus of (13) is achieved if

$$\lim_{t \to \infty} \|\omega(t)\| = 0. \tag{16}$$

Notice that, for any non-negative integer $k$, there holds (by Lemma 1 and $11_N = 0$)

$$\begin{bmatrix} I_N - 1_N R_1 \end{bmatrix} L^k = L^k [I_N - 1_N R_1^k]. \tag{16}$$

Hence it follows from (13) and (15) that

$$\omega^+ = \left( \sum_{k=0}^{2} (L^k \otimes A_k) \right) \omega. \tag{17}$$

Let $U^{-1} \otimes I_{2^{n-1}} \omega \triangleq \omega' = [w_1^\top, w_2^\top, \ldots, w_N^\top]^\top$, where $w_i \in \mathbb{C}^{2^{n-1}}$, $i \in [1, N]$. Then, by using (8), we have

$$\omega^+ = \left( \sum_{k=0}^{2} (J_L^k \otimes A_k) \right) \omega'. \tag{18}$$
By the definition of $\omega$ and Lemma 1, it is easy to see that

$$\pi_i = (r_i^T \otimes I_{N-1})\omega = (r_i^T \otimes I_{n-1})(I_N - 1_{Nr}) \otimes I_{n-1}) \chi = 0.$$  \hfill (19)

Hence we need only to prove $\lim_{t \to -\infty} \|\pi_i(t)\| = 0$, $i \in [2, N]$. On the other hand, by recognizing the special structure of $J_i^T$, $\lim_{t \to -\infty} \|\pi_i(t)\| = 0$, $i \in [2, N]$, hold true if and only if the following series of linear systems

$$\pi_i^+ = A_i \pi_i, \ i \in [2, N]$$  \hfill (20)

are asymptotically stable, where

$$A_i = \sum_{k=0}^{2} \lambda_i^k A_k = \begin{bmatrix} A + \lambda_i BK_i & BK_i \\ \lambda_i HC + \lambda_i^2 TK_i & F + \lambda_i TBK_i \end{bmatrix}.$$

For any $i \in [1, N]$, consider a nonsingular matrix $Q_i$ and its inverse

$$Q_i = \begin{bmatrix} I_n & 0 \\ -\lambda_i T & I_{n-1} \end{bmatrix}, \ \ \ \ Q_i^{-1} = \begin{bmatrix} I_n & 0 \\ \lambda_i T & I_{n-1} \end{bmatrix}.$$

Then direct manipulation gives

$$Q_i A_i Q_i^{-1} = \begin{bmatrix} A + \lambda_i BK_i & BK_i \\ 0 & F \end{bmatrix}$$  \hfill (23)

where we have used (6) and (10). Hence $A_i$, $i \in [2, N]$, are Hurwitz (Schrur stable for the discrete-time case) if and only if $A + \lambda_i BK_i, i \in [2, N]$ are. This completes the proof of the first part of this theorem.

Next we prove (11) and (12). We first consider the continuous-time case. It follows from (8) and (13) that

$$\chi(t) = \exp \left( \sum_{k=0}^{2} (L_k \otimes A_k) t \right) \chi(0)$$

$$= \mathcal{U} \exp \left( \sum_{k=0}^{2} (J_k^T \otimes A_k) t \right) \mathcal{U}^{-1} \chi(0)$$

$$= \mathcal{U} \left[ \exp \left( \sum_{k=0}^{2} \left( \frac{D_k^T \otimes A_k \mathcal{U}^{-1}}{\lambda_k^t} \right) t \right) \right] \mathcal{U}^{-1} \chi(0).$$  \hfill (24)

From the above proof we have known that $\sum_{k=0}^{2} (D_k^T \otimes A_k)$ is Hurwitz. Hence

$$\chi(t) = \exp \left( 1_{Nn} \otimes I_{n-1} \right) \exp \left( \sum_{k=0}^{2} (L_k \otimes A_k) t \right) \exp \left( 1_{Nn} \otimes I_{n-1} \right)^{-1} \chi(0)$$

$$= \left( 1_{Nn} \otimes I_{n-1} \right) \exp \left( \sum_{k=0}^{2} \left( C_k \otimes A_k \mathcal{U}^{-1} \right) t \right) \exp \left( 1_{Nn} \otimes I_{n-1} \right)^{-1} \chi(0).$$  \hfill (25)

As $F$ is Hurwitz, the above relation implies (11) and (12) exactly.

For the discrete-time case, the result can be shown in a similar way as the above or that in [10] by noting that

$$A_{0i}^t = \begin{bmatrix} A^t & \sum_{i=0}^{t} A^{t-i-1} BK_i F^t \\ 0 & F^t \end{bmatrix}, \ 1 \leq t \in \mathbb{Z}.$$  \hfill (26)

The proof is complete.

Several remarks for Theorem 1 are given in order.

Remark 1: There exists a $K$ such that $A + \lambda_i BK_i, i \in [2, N]$, are Hurwitz (Schrur stable for the discrete-time case) if and only if the consensus of the MAS (1) can be achieved by state feedback (see, for example, [18]).

$$u_i = \sum_{j \in N} \alpha_{ij} K \langle x_i - x_j \rangle = \sum_{j=1}^{N} \alpha_{ij} K x_j,$$  \hfill (27)

Hence Theorem 1 indicates that the reduced-order observer based protocol (5) recovers the ability of the state feedback protocol (27). Moreover, we notice from (23) that the separation principle holds for the reduced-order observer based protocol (5).

Remark 2: In the continuous-time setting, there exists a $K$ such that $A + \lambda_i BK_i, i \in [2, N]$, are Hurwitz if and only if $(A, B)$ is stabilizable (see, for example, [9] and [18]). In fact, a feedback satisfying this condition can be chosen as $K = -\mu B^TP, \ \mu > 1/2 \Re \{\lambda_i\}, \ \forall i \in [2, N]$, and $P$ solves the algebraic Riccati equation (ARE) $A^TP + PA - PBB^TP = -Q_d, \ Q_d > 0$. However, in the discrete-time setting, additional assumptions should be imposed on $A$ to guarantee the existence of a $K$ such that $A + \lambda_i BK_i, i \in [2, N]$, are Schur stable [5], [17]. For example, if $m = 1$, the communication graph is undirected, and $(A, B)$ is controllable, then a $K$ exists if and only if $\prod_{i} |\lambda_i^+(A)| < 1 + (\lambda_2/\lambda_N)(/1 - (\lambda_2/\lambda_N))$, where $\lambda_i^+(A)$ represents an unstable eigenvalue of $A$ [17].

Remark 3: A reduced-order observer based protocol in the form of

$$\xi_t = F \xi_t + H y_t + TB u_t, \ \xi_t \in \mathbb{R}^{n-\gamma},$$

$$u_t = K_1 z_t + K_2 \sum_{j \in N} \alpha_{ij} (\xi_t - \xi_j), \ i \in \{1, N\},$$  \hfill (28)

was proposed recently in [10]. Different from (28), the reduced-order observer in the protocol (5) is asymptotically stable. This property benefits the implementation of the protocol (5) since the states (internal variables of the protocol) of the observers keep bounded even when $A$ is unstable. Moreover, both the observer and the protocol in (5) are distributed as they only require the relative information of neighboring agents, which is different from (28) where the observer for each agent requires its absolute output measurement $y_i$, which may be impractical in many cases (see the example given in [9] and [14]). Finally, in the discrete-time setting, differently from Theorem 4.3 in [10] where the matrix $A$ should have no eigenvalues with magnitude larger than 1, Theorem 1 is applicable to MASs whose dynamics contain exponentially unstable poles.

### III. CONSENSUS BY TRUNCATED REDUCED-ORDER OBSERVER BASED OUTPUT FEEDBACK

The reduced-order observer based protocol in (5) also requires the relative control signals of the neighboring agents, namely, the distributed term $\sum_{j=1}^{N} \alpha_{ij} u_j$. In this section, we show that such a distributed term can be neglected by imposing some assumptions on the dynamics of the agents.

Assume that the stabilizing gain $K$ in Theorem 1 is parameterized as $K = K(\gamma) : [0, 1) \to \mathbb{R}^{n \times \gamma}$ and is such that

$$\lim_{\gamma \to 0^+} K(\gamma) = 0.$$  \hfill (29)

Then, by decreasing the value of $\gamma$, the amount of the control signal $u_i = K_1 z_t + K_2 \xi_t, i \in [1, N]$, in the observer dynamics in (5) can be neglected. Consequently, the protocol (5) can be truncated as

$$\xi_t = F \xi_t + H z_t, \ u_t = K_1 z_t + K_2 \xi_t, \ i \in [1, N],$$  \hfill (30)

where $K_1, K_2$ and $K$ are related with (10).
Remark 4: The most significant advantage of the truncated reduced-order observer based output feedback protocol (30) is that it requires the relative outputs information of the neighboring agents and needs to exchange neither the information of the control signal $u_i$ nor the information of the observer state $\xi_j$ between the agents. To guarantee that $K$ satisfying (29) exists, it is necessary to assume that all the eigenvalues of $A$ are on the closed-left half plane (closed unit circle for the discrete-time case) [20], [21]. Since the stable poles of $A$ do not affect the solvability of the consensus problem (see Remark 6 in [15] for details), for simplicity, while without loss of generality, we assume that all the eigenvalues of $A$ are on the imaginary axis (on the unit circle for the discrete-time case).

Remark 5: Notice that MASs characterized by a chain of integrators or a single oscillator (see, for example, [1], [7] and [13]) satisfy the above assumption on $A$ automatically. We however emphasize that $A$ is allowed to have eigenvalues on the imaginary axis (on the unit circle for the discrete-time setting) with algebraic and geometric multiplicities larger than 1, which is weaker than the condition that $A$ is neutrally stable imposed in some papers (for example, Assumption 1 in [15]). This assumption on $A$ is also necessary for guaranteeing that the consensus value reached by the agents will not tend to infinity exponentially [9].

The main result of this section is presented as follows.

Theorem 2: For the MAS (1) which satisfies Assumption 1, assume that all the eigenvalues of $A$ are on the imaginary axis (on the unit circle for the discrete-time case), $[A, B]$ is controllable and $(A, C)$ is observable. Then there exists a number $\xi$ and a matrix $V$ such that the reduced-order observer based protocol (30) solves the output feedback consensus problem. In this case, $\xi$ and $V$ are related with (10) and the states of the observers and the agents satisfy (11) and (12), respectively.

To prove this result, we need the following technical lemma.

Lemma 2: Assume that $[A, B] \in (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n})$ is controllable, all the eigenvalues of $A$ are on the imaginary axis (on the unit circle for the discrete-time case), $F \in \mathbb{R}^{p \times p}$ is Hurwitz (Schur stable for the discrete-time case), $E_i$, $i = 1, 2, 3, 4$, are some real matrices of appropriate dimensions, and $\lambda$ is a complex number such that $\Re \{\lambda\} > 0$. Then there exists a scalar $\gamma^* = \gamma^*(\lambda) \in (0, 1)$ and a matrix $K(\gamma) : (0, 1) \rightarrow \mathbb{R}^{n \times k}$ such that the matrix

$$
A = \begin{bmatrix}
A + \lambda BK & BKE_i \\
\lambda^2 E_3 BK & F + \lambda E_4 E_4
\end{bmatrix}
$$

(31)

is Hurwitz (Schur stable for the discrete-time case) for all $\gamma \in (0, \gamma^*)$.

In particular:

- For the continuous-time case, $K$ can be designed as $K = -\mu B^TP$, where $\mu$ is a real number satisfying $\mu \geq 1/\Re \{\lambda\}$, and $P = P(\gamma) > 0$ is the unique positive definite solution to the ARE

$$
A^TP + PA - PBB^TP = -\gamma P.
$$

(32)

- For the discrete-time case, $K$ can be designed as $K = -\mu (I_m + B^TPB)^{-1}B^TPA$, where $\mu$ is a real number satisfying $\mu \geq 1/\Re \{\lambda\}$, and $P = P(\gamma) > 0$ is the unique positive definite solution to the ARE

$$
A^TPA - P - A^TPB(I_m + B^TPB)^{-1}B^TPA = -\gamma P.
$$

(33)

Proof: Clearly, we only need to prove the stability of the following coupled linear system

$$
\begin{cases}
\varphi^+ = (A + \lambda BK)\varphi + BKE_i \psi, \\
\psi^+ = (F + \lambda E_3 E_4)\psi + \lambda^2 E_2 BK \varphi.
\end{cases}
$$

(34)

The idea of our proof is as follows. Since both the systems $\varphi^+ = (A + \lambda BK)\varphi + \lambda^2 E_2 BK \varphi$ and $\psi^+ = F\psi$ are asymptotically stable, they possess respectively Lyapunov functions $V_1(\varphi)$ and $V_2(\psi)$. Hence, as the coupling terms between these two systems in (34) are linear functions of $K$, they can be weakly decoupled by reducing $\|K\|$, indicating that $V(\varphi, \psi) = V_1(\varphi) + \|K\|V_2(\psi)$ may be a possible Lyapunov function for the overall system.

(Proof for the continuous-time case) By using the ARE (32), the time-derivative of the Lyapunov function $V_1(\varphi) = \varphi^H P \varphi$ can be evaluated as

$$
V_1(\varphi) = \varphi^H \left( (A + \lambda BK)^HP + P(A + \lambda BK) \right) \varphi + \varphi^H PBB^TE_i \psi + \varphi^H E_i^T K^TB^TP \varphi
$$

$$
= -\gamma \varphi^H P \varphi - (2\mu \Re \{\lambda\} - \lambda^2) \varphi^H PBB^TP \varphi + \varphi^H PBB^TE_i \psi + \varphi^H E_i^T K^TB^TP \varphi
$$

$$
\leq -\gamma \varphi^H P \varphi - \left( 2\mu \Re \{\lambda\} - \frac{3}{2} \lambda^2 \right) \varphi^H PBB^TP \varphi
$$

$$
+ 2\|K\| E_i^T \|\psi\|^2.
$$

(35)

On the other hand, the time-derivative of the Lyapunov function $V_2(\psi) = \psi^H Q \psi$, where $Q > 0$ solves $F^*Q + QF = -I_p$, satisfies

$$
V_2(\psi) = \psi^H \left( (F + \lambda E_3 K E_4)^HQ + Q(F + \lambda E_3 K E_4) \right) \psi
$$

$$
+ \lambda^2 \psi^H Q E_i BK \varphi + \lambda^2 \psi^H K^TB^TP \varphi
$$

$$
= -\|\psi\|^2 + 2\Re \{\lambda\} \psi^H Q E_i K E_4 \psi
$$

$$
+ \lambda^2 \psi^H Q E_i BK \varphi + \lambda^2 \psi^H K^TB^TP \varphi
$$

$$
\leq -\left( 1 - 2\Re \{\lambda\} \|Q E_i K E_4\| - \rho \|\lambda\|^2 \|QE_2 B^T\|^2 \right) \|\psi\|^2
$$

$$
+ \frac{1}{\rho} \psi^H K^TB^TP \varphi.
$$

(36)

Let $\rho = \min \{1, 1/2\| \lambda \|^2 \|QE_2 B^T\|^2 \}$, which is independent of $\gamma$. Then

$$
\dot{V}(\varphi, \psi) \leq -\gamma \varphi^H P \varphi - \left( 2\mu \Re \{\lambda\} - \frac{3}{2} - \frac{\rho^2}{\rho} \|K\|^2 \right) \psi^H PBB^TP \varphi
$$

$$
- \|K\| \left( 1 - 2\Re \{\lambda\} \|Q E_i K E_4\| - 2\|K\| E_i^T \|\psi\|^2 \right) \|\psi\|^2.
$$

(38)

As $\lim_{\gamma, \rho \to 0} K(\gamma) = 0$ [20] and $\mu \geq 1/\Re \{\lambda\}$, we know that there exists a scalar $\gamma^* = \gamma^*(\lambda) > 0$, such that

$$
\left\{ \begin{array}{l}
2\mu \Re \{\lambda\} - \frac{3}{2} - \frac{\rho^2}{\rho} \|K\|^2 > 0, \\
\frac{1}{2} - 2\Re \{\lambda\} \|Q E_i K E_4\| - 2\|K\| E_i^T \|\psi\|^2 \geq \frac{1}{4}
\end{array} \right.
$$

(39)

hold for all $\gamma \in (0, \gamma^*)$. Hence

$$
\dot{V}(\varphi, \psi) \leq -\gamma \varphi^H P \varphi - \frac{1}{4} \|K\| \|\psi\|^2, \forall \gamma \in (0, \gamma^*)
$$

(40)

which implies the stability of the linear system (34).

(Proof for the discrete-time case) Consider the Lyapunov function $V_1(\varphi) = \varphi^H P \varphi$. Then

$$
\dot{V}_1(\varphi) \leq \varphi^H (A + \lambda BK)^HP(A + \lambda BK) - P \varphi
$$

(41)

$$
= -\gamma \varphi^H P \varphi - \left( 2\mu \Re \{\lambda\} - \lambda^2 \right) \varphi^H PBB^TP \varphi
$$

$$
+ \varphi^H PBB^TE_i \psi + \varphi^H E_i^T K^TB^TP \varphi
$$

$$
\leq -\gamma \varphi^H P \varphi - \left( 2\mu \Re \{\lambda\} - \frac{3}{2} \lambda^2 \right) \varphi^H PBB^TP \varphi
$$

$$
+ 2\|K\| E_i^T \|\psi\|^2.
$$

(35)
where $\rho > 0$ is some number to be specified and
\begin{align}
\begin{aligned}
\Pi_1 \triangleq & \frac{1}{\rho} (A + \lambda BK)^T P (A + \lambda BK) - P \\
\Pi_2 \triangleq & \frac{1}{\rho} (A + \lambda BK)^T P B B^T P (A + \lambda BK).
\end{aligned}
\end{align}
(42)

By using the ARE in (33) and the expression of $K$ and some calculations, we get
\begin{align}
\Pi_1 = -\gamma P - \frac{1}{4} \mu^2 K^T \Omega_1 K \\
\Pi_2 \leq \frac{1}{2} \mu^2 K^T \Omega_2 K,
\end{align}
where $\Omega_1 = 2 \mu \Re \{ \lambda \} (I_m + B^T P B) - (I_m + B^T P B) - \mu^2 L^T B^T P B$.
(43)

\begin{align}
\Pi_2 = & -\frac{1}{\rho} (I_m + B^T P B - \mu L^T B^T P B) \\
& \times (I_m + B^T P B - \mu L^T B^T P B)^T.
\end{align}
(44)

As $\lim_{\gamma \to \gamma^*} P' (\gamma) = 0$ [21], there exists a $\gamma^*_0 \in (0, 1]$ such that $(B^T P B)^2 \leq I_m, \forall \gamma \in (0, \gamma^*_0]$. Then
\begin{align}
\Omega \geq & \left(2 \mu \Re \{ \lambda \} - 1 - \frac{1}{\rho} \mu L^T B^T P B \right) I_m \\
& + \left(2 \mu \Re \{ \lambda \} - 1 - \mu L^T B^T P B \right) B^T P B.
\end{align}
(45)

Let $\rho = \rho (\mu, \lambda) = 4 \max \{1, 1 - \mu, L^2\}$. Then, as $\mu \geq 1 / \Re \{ \lambda \}$, we have from (45) that
\begin{align}
\Omega \geq & \frac{1}{2} L_m - \mu L^T B^T P B \\
& - \frac{1}{2} \left(1 - \mu \Re \{ \lambda \} \right) B^T P B
\end{align}
(46)

which, by noting again that $\lim_{\gamma \to \gamma^*} P' (\gamma) = 0$, implies that there exists a $\gamma^*_0 \in (0, \gamma^*_1]$ such that $\Omega \geq (1/4) I_m, \forall \gamma \in (0, \gamma^*_0]$. Consequently, we obtain
\begin{align}
\Pi_1 + \Pi_2 \leq -\gamma P - \frac{1}{4} \mu^2 K^T K, \forall \gamma \in (0, \gamma^*_0]
\end{align}
(47)

by which we get from (41) that
\begin{align}
\nabla V_1 (\varphi, \psi) \leq & -\gamma \psi^T P \varphi - \frac{1}{4} \mu^2 \psi^T K^T K \psi \\
& + | \varphi |^2 \| \psi \|^2 (\rho + | B^T P B |) \| \psi \|^2.
\end{align}
(48)

Choose another Lyapunov function $V_2 (\psi) = \psi^T Q \psi$, where $Q > 0$ solves $F^T Q F = Q - I_\psi$. Then
\begin{align}
\nabla V_2 (\psi) = & \psi^T \left( (F + \lambda E_3 K E_4) Q (F + \lambda E_3 K E_4) - Q \right) \psi \\
& + (\lambda E_3^T B K^T \varphi)^T Q \lambda E_3^T B K^T \varphi \\
& + (\lambda E_3 K E_4^T \varphi)^T Q \lambda E_3 K E_4^T \varphi \\
& + \lambda \psi^T Q (F + \lambda E_3 K E_4) Q E_3 B K \varphi \\
& \leq & \left[ (| \varphi |^2 \| \psi \|^2 \| Q \| + 2 \Re \{ \lambda \} (\lambda E_3^T B K^T \varphi)^T Q \lambda E_3^T B K^T \varphi \\
& + \lambda \psi^T Q (F + \lambda E_3 K E_4) Q E_3 B K \varphi \right]
\end{align}
(49)

Choose $V_1 (\varphi, \psi) - V_2 (\psi) - \gamma \psi^T K \psi$. Then we get from (48) and (51) that
\begin{align}
\nabla V (\varphi, \psi) \leq & -\gamma \psi^T P \varphi \\
& - \frac{1}{4} \mu^2 \psi^T Q \psi \\
& - \frac{1}{\rho} \mu \psi^T Q \psi \\
& - \frac{1}{\rho} \psi^T Q \psi.
\end{align}
(50)

As $\rho_1$ is bounded below by some positive number for any $\gamma \in [0, \gamma^*_0]$, and $\lim_{\gamma \to 0} K (\gamma) = 0$ [21], there exists a $\gamma^*_0 = \gamma^*_1 (\lambda) \in (0, \gamma^*_0]$. Consequently, it follows from (52) that
\begin{align}
\nabla V (\varphi, \psi) \leq & -\gamma \psi^T P \varphi \\
& - \frac{1}{\rho} \mu \psi^T Q \psi \\
& - \psi^T Q \psi.
\end{align}
(53)

which ensures the stability of system (34). The proof is complete.

We are now ready to prove Theorem 2.

Proof of Theorem 2: Similarly to (13), the closed-loop dynamics consisting of (1) and (30) can be written as
\begin{align}
\chi^+ = (I_N \otimes \mathcal{A}_0 + L \otimes \mathcal{A}_1) \chi
\end{align}
(55)

where $\mathcal{A}_0 = \mathcal{A}_0$ is related in (14) and $\mathcal{A}_1$ is given by
\begin{align}
\mathcal{A}_1 = \begin{bmatrix}
B K C & 0 \\
H C & 0
\end{bmatrix}.
\end{align}
(56)

Then similarly to the proof of Theorem 1, the consensus is achieved if the matrix
\begin{align}
\mathcal{A}_1' \triangleq \mathcal{A}_0 + \lambda_1 \mathcal{A}_1 = \begin{bmatrix}
A + \lambda_1 B K C & B K_2 \\
\lambda_1 H C & F
\end{bmatrix}
\end{align}
(57)

is Hurwitz (Schur stable for the discrete-time case) for all $i \in [1, N]$. Notice that
\begin{align}
Q_i \mathcal{A}_i Q_i^{-1} = \begin{bmatrix}
A + \lambda_1 B K & B K_2 \\
-\lambda_1 T B K & F - \lambda_1 T B K
\end{bmatrix}
\end{align}
(58)
which is just in the form of (31). Then, by Lemma 2, there exists a \( \gamma^+ = \gamma^+ \left( \{ \lambda_i \}_{i=1}^N \right) \triangleq \min_{\lambda_i \in \mathbb{R}} |\gamma^+|_{\lambda_i} \in (0, 1) \), and a matrix \( K(\gamma) : (0, \gamma^+) \to \mathbb{R}^{n \times n} \), such that the matrix in (58) is Hurwitz (Schur stable for the discrete-time case) for any \( i \in [1, 2, N] \). The remainder of the proof is similar to the proof of Theorem 1 and is omitted. The proof is complete.

In the discrete-time case, under the same condition of Theorem 2, a reduced-order observer based protocol in the form of (28) was established in [10]. In comparison with that result, our reduced-order observer based protocol (30) possesses not only some advantages stated in Remarks 3 and 4, but also the advantage that explicit formulation of the observer gains \( K_i \) and \( a_i \) in (30) can be obtained as explicit solutions to the AREs (32) and (33) can be computed [20], [21].

Remark 6: It follows from the proofs of Theorems 1 and 2 that the convergence rate (denoted by \( \rho(\gamma) \)) of the agents reaching consensus under the truncated reduced-order observer based protocol (30) is inversely proportional to the maximal spectral abscissa (spectral radius for the discrete-time case) of \( \mathcal{A}_i \) for all \( i \in [2, N] \). Hence, for simplicity, we can denote

\[
\rho(\gamma) = \max_{i \in [2, N]} \max_{\lambda_i \in \mathbb{R}} |\lambda_i| 
\]

for the continuous-time case and

\[
\frac{1}{\rho(\gamma)} = \max_{i \in [2, N]} \max_{\lambda_i \in \mathbb{R}} |\lambda_i| 
\]

for the discrete-time case. Then, to guarantee a satisfactory convergence rate, we need to find the optimal \( \gamma \) (denoted by \( \gamma_{\text{opt}} \)) such that \( \rho(\gamma) \) is maximized. Such an optimal value can be easily found by calculating the function \( \rho(\gamma) \) at discrete values \( \gamma_k = k \Delta \gamma, k = 1, 2, \ldots \), where \( \Delta \gamma > 0 \) is a sufficiently small number. The design process will be illustrated in the next section.

IV. A NUMERICAL EXAMPLE

In this section we use a numerical example to demonstrate the effectiveness of the proposed approaches. For space limitation and the purpose of comparison, we only illustrate Theorem 2 for the continuous-time case by borrowing the example from [10]. Assume that \( N = 6 \) and the coefficient matrices in (1) are

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

which corresponds to a double integrator system. Let the Laplacian of the communication network be given by

\[
L = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.
\]

It follows that \( \lambda(A) = \{0, 1, 1.3376 \pm 0.5623 i, 2, 3.3247 \} \). Hence the network contains a directed spanning tree and Assumption 1 is fulfilled.

As \( \gamma = 1 \), a truncated reduced-order observer of order 1 in the form of (30) can be constructed to achieve consensus. Let

\[
F = -7, \quad H = 1, \quad T = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}
\]

which is such that \( T \) satisfies (6) and \( W \) is invertible. Hence, according to Lemma 2 and Theorem 2, we can choose \( \mu = 1 \), and solve the ARE (32) to give

\[
K(\gamma) = \mu \left[ -\gamma^2 - 2\gamma \right], \quad \gamma > 0.
\]

To determine the optimal value of \( \gamma \) such that the convergence rate \( \rho(\gamma) \) is maximized, the function \( \rho(\gamma) \) is plotted in Fig. 1. It follows that the consensus is achieved if and only if \( \gamma \in (0, 5.108) \) and \( \rho(\gamma) \) is maximized with \( \gamma = \gamma_{\text{opt}} = 2.5 \), which will be used in our simulation. The corresponding maximal convergence rate is \( \rho(\gamma_{\text{opt}}) = 1.21 \). Moreover, the convergence rate associated with (28), where the parameters are chosen as the same as in (30), can be computed as \( \rho = 1.3615 \), which is only a little higher than \( \rho(\gamma_{\text{opt}}) \).

For simulation purpose, we choose the initial conditions for the agents as \( x_1(0) = [1, -2]^T, x_2(0) = [2, 3]^T, x_3(0) = [1, -1]^T, x_4(0) = [3, 1]^T, x_5(0) = [1, -2]^T, x_6(0) = [2, 1]^T \), and the initial conditions for the observers as \( \xi_1(0) = -1, \xi_2(0) = -2, \xi_3(0) = -1, \xi_4(0) = -2, \xi_5(0) = 2 \). Then the states of the agents and the observers associated with the protocols (30) and (28) are respectively plotted in Figs. 2 and 3. From these two figures we clearly see that the consensus of the MAS is achieved. Moreover, the
convergence performance of the truncated reduced-order observer based protocol (30) is comparable to that of the protocol (28), yet the protocol (30) possesses the advantages that it only requires the relative outputs information and the observer dynamics themselves are asymptotically stable.

V. CONCLUDING REMARKS

This technical note studies output feedback consensus of MASs characterized by high-order linear systems with directed communication topologies. A distributed reduced-order observer based protocol is established based on the relative outputs and inputs of neighboring agents. It is shown that the consensus of the MAS can be achieved by the proposed reduced-order observer based protocol if it can be achieved by state feedback protocols. Provided that the open-loop dynamics of the MASs is not exponentially unstable, a truncated reduced-order observer based protocol is also established, which only requires the relative outputs of neighboring agents. All the results are established for both continuous-time and discrete-time MASs. Corresponding results based on full-order observers have been developed and will be reported elsewhere. In fact, in the full-order observer case, we need only to replace $L = I$, $F = A - HC$, $K_1 = 0$ and $K_2 = K$ in (5), (6) and (30), where $K$ is determined in Theorem 1 for the distributed observer and is determined in Theorem 2 for the truncated observer.

REFERENCES