Truncated predictor feedback for linear systems with long time-varying input delays

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Abstract

In this paper we study the problem of stabilizing a linear system with a single long time-varying delay in the input. Under the assumption that the open-loop system is stabilizable and not exponentially unstable, a finite dimensional static time-varying linear state feedback controller is obtained by truncating the predictor based controller and by adopting the parametric Lyapunov equation based controller design approach. As long as the time-varying delay is exactly known and bounded, an explicit condition is provided to guarantee the stability of the closed-loop system. It is also shown that the proposed controller achieves semi-global stabilization of the system if its actuator is subject to either magnitude saturation or energy constraints. Numerical examples show the effectiveness of the proposed approach.

1. Introduction

Time delay, which arises frequently in many engineering systems such as long transmission lines in pneumatic systems, rolling mills, nuclear reactors, hydraulic systems, manufacturing processes, digital control systems and systems that are controlled remotely (Hale, 1977; Krstic, 2010a; Niculescu, 2001; Richard, 2003), is generally recognized as a source of performance degradation and even instability of control systems (Hale, 1977). Control problems, especially, the problems of asymptotic stability analysis and stabilization, for time-delay systems have attracted much attention for several decades. Various types of time-delay systems have been investigated and a large number of results have been reported in the literature (see, e.g., Chen, 1995, Hale, 1977, Krstic, 2010b, Mazenc, Malisoff, & Lin, 2008, Richard, 2003, Yakoubi & Chitour, 2007 and Zhang, Zhang, & Xie, 2004 and the references cited therein). There are several categories of methods for handling asymptotic stability analysis and stabilization of time-delay systems. The most efficient methods are probably the Lyapunov–Krasovskii functional based methods (see, e.g., He, Wang, Lin, & Wu, 2007, Pepe, Jiang, & Fridman, 2008, Xie, Fridman, & Shaked, 2001 and Xu, Lam, & Yang, 2001). The Razumikhin theorem based approach also falls into this category (Hale, 1977). The idea is to find a positive-definite functional such that its derivative along the trajectories of the time-delay system is negative. The results obtained by these methods can be easily recast into linear matrix inequalities, which can be efficiently solved numerically. A drawback of these methods is that in general only sufficient conditions can be obtained, which leads to conservatism. Even though such conservatism can be reduced, the resulting conditions are usually very complicated with the associated computational burden dramatically increased (Gu, 2001; Gu, Kharitonov, & Chen, 2003).

Another efficient approach to dealing with time-delay systems is the predictor feedback (Artstein, 1982; Jankovic, 2009; Krstic, 2010a,b; Manitius & Olbrot, 1979; Michiels & Niculescu, 2007). This approach is especially effective for input delay systems, including unstable ones. However, as noticed in Krstic (2010b), almost all the existing results deal with problems where the delay is constant and the time-varying input delay has received very
little attention. Although the predictor feedback for the time-varying input delay system has been introduced by Artstein (1982), the design is not worked out in detail since the case of time-varying delay is considered only for plants that are time-varying, in which case explicit developments are not possible (Krstic, 2010b). Very recently, by constructing a Lyapunov functional using a backstepping transformation with time-varying kernels, and transforming the actuator state into a transport partial differential equation with a convection speed coefficient that varies with both space and time, the exponential asymptotic stability of the feedback system with the predictor controller proposed in Nihtila (1991) for systems with time-varying input delays has been proved in Krstic (2010b).

Predictor based approaches for controlling time-delay systems have received renewed interest in recent years. For example, the exponential stability for linear systems with distributed actuator delays and under predictor based feedback is established by the use of Lyapunov functionals in Bekiaris-Liberis and Krstic (2011b); a predictor feedback controller for a general system in strict-feedback form with delayed integrators, which is an example of a particularly challenging class of exponentially unstable systems with simultaneous input and state delays, is proposed in Bekiaris-Liberis and Krstic (2010b); for the case that the delays existing in the systems are unknown, adaptive predictor based controllers are proposed in Bekiaris-Liberis and Krstic (2010a) and Bresch-Pietri and Krstic (2010) respectively for a two-block sub-class of linear feedforward systems and linear systems with a single long actuator delay; predictor-based state feedback and output feedback controllers are designed in Bekiaris-Liberis and Krstic (2011a) and Karafyllis and Krstic (2000) for general nonlinear systems with time-varying input and state delays, respectively; and an approximated predictor for nonlinear systems with delayed input is established in Karafyllis (2011).

In this paper and its conference version (Zhou, Lin, & Duan, 2011), inspired by the work in Krstic (2010b) and Nihtila (1991), we consider predictor based controllers for linear systems with long time-varying input delays. Differently from those traditional predictor based controllers, which are infinite-dimensional static feedback laws and may cause difficulties in their practical implementation (see, for example, Richard, 2003 and Van Assche, Dambrine, Lafay, & Richard, 1999), we develop a truncated predictor feedback (TPF) which only involves a finite dimensional static state feedback by safely ignoring the distributed terms in the traditional predictor based feedback. It is shown that if the open-loop system is not exponentially unstable and the nominal feedback gain is designed by our recently developed parametric Lyapunov equation based low gain feedback (Zhou, Duan, & Lin, 2008; Zhou, Lin, & Duan, 2010b), the asymptotic stability of the closed-loop system under the TPF can be established with the aid of the Razumikhin stability theorem. An explicit condition is provided for choosing the free parameter in the controller. It is also shown that the proposed TPF also achieves semi-global stabilization of the considered delay system when the actuator is subject to either magnitude saturation or energy constraints. Numerical examples involving time-varying delays considered in Krstic (2010b) are worked out to illustrate the effectiveness of the proposed approach. We point out that the proposed controller reduces to the one proposed in Zhou et al. (2010b) if the delay in the input is constant, exposing an underlying mechanism of the approaches given in Lin and Fang (2007) and Zhou et al. (2010b). We also point out that, although long time-varying input pointed delays, multiple pointed delays, distributed delays, and state delays have been respectively considered in Zhou, Gao, Lin, and Duan (2012), Zhou, Lin, and Duan (2009) and Zhou, Lin, and Duan (2010a), the open-loop systems are only allowed to have unstable zeros at the origin. In the present paper, the unstable poles are allowed to be on the imaginary axis.

The remainder of this paper is organized as follows. The idea of the TPF for linear systems with time-varying input delays is introduced in Section 2. In Section 3 we prove the asymptotic stability of the closed-loop system with state feedback. Output feedback results are given in Section 4. In Section 5, we further show that the TPF also solves some constrained control problems. Numerical examples are presented in Section 6 to validate the effectiveness of the proposed approach. Finally, Section 7 concludes the paper.

Notation: The notation used in this paper is fairly standard. For a vector \( u \in \mathbb{R}^n \), we use \( \|u\|_\infty \) to denote the \( \infty \)-norm of \( u \) and \( \text{sign}(y) \) to denote the sign function which takes value \( +1 \) if \( y > 0 \) and \( -1 \) if \( y < 0 \). The standard saturation function is defined as \( \text{sat}(u) = \text{sign}(u) \min \{|u|, 1\} \). For a matrix \( A \in \mathbb{R}^{n \times n} \), \( A^T \) and \( \text{tr}(A) \) are respectively its transpose and trace. Finally, for a positive scalar \( \tau \), let \( \hat{v}_{s,t} = \hat{v}([-\tau, 0], \mathbb{R}^n) \) denote the Banach space of continuous vector functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}^n \) with the topology of uniform convergence, and let \( x_\tau \in \hat{v}_{s,t} \), denote the restriction of \( x(t) \) to the interval \([t - \tau, t]\) translated to \([-\tau, 0]\), that is, \( x_\tau(t) = x(t + \tau), \; \theta \in [-\tau, 0] \).

2. The TPF approach

Consider the following linear system with input delay

\[
\dot{x}(t) = Ax(t) + Bu (\phi(t)),
\]

where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^n \) are respectively the state and input vectors, and \( \phi(t) : \mathbb{R}^+ \to \mathbb{R} \) is a continuously differentiable function that incorporates the actuator delay. The function \( \phi(t) \) can be defined in a more standard form

\[
\phi(t) = t - d(t),
\]

where \( d(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) is the time-varying delay. However, as pointed out in Krstic (2010b), the formalism involving the function \( \phi(t) \) turns out to be more convenient because the predictor problem we will consider later requires the inverse function of \( \phi(t) \), namely, \( \phi^{-1}(t) \). In this paper, we will proceed with model (1) and assume (2) whenever necessary. Some necessary assumptions on \( \phi(t) \) will be made clear as follows.

Assumption 1. The function \( \phi : \mathbb{R}^+ \to \mathbb{R} \) is a continuously differentiable, invertible and exactly known function such that \( \frac{d}{dt}\phi(t) > 0 \), \( \forall t \in [0, \infty) \), and the delay \( D(t) \) is bounded, namely, there exists a finite, yet arbitrarily large, number \( D > 0 \) such that

\[
0 \leq d(t) \leq D, \; \forall t \in [0, \infty).
\]

The main idea of the predictor feedback is to design the feedback controller

\[
u(\phi(t)) = Fx(t), \quad \forall \phi(t) \geq 0,
\]

such that the closed-loop system consisting of (1) and (4) is

\[
\dot{x}(t) = (A + BF)x(t), \quad \forall \phi(t) \geq 0,
\]

where \( F \) is such that \( A + BF \) is asymptotically stable. The controller (4) can also be written as

\[
u(t) = Fx(\phi^{-1}(t)), \quad t \geq 0.
\]

However, as \( \phi^{-1}(t) \geq t \), \( \forall t \geq 0 \), the above controller is impossible to implement in practice. To overcome this problem, \( x(\phi^{-1}(t)) \) should be predicted based on the current state. By using the system model (1) and the variation of constants formula (Hale, 1977;
Krstic, 2010b), it can be obtained that
\[ x\left(\phi^{-1}(t)\right) = e^{A(\phi^{-1}(t)-t)}x(t) \]
\[ + \int_{t}^{\phi^{-1}(t)} e^{A(\phi^{-1}(t))-s}Bu_s(s) \, ds. \]  
(7)
Substituting the above relation into (6) gives the following predictor feedback
\[ u(t) = F\left(e^{A(\phi^{-1}(t)-t)}x(t) \right) \]
\[ + \int_{t}^{\phi^{-1}(t)} e^{A(\phi^{-1}(t))-s}Bu_s(s) \, ds. \]  
(8)
For easy reference, the first term
\[ u_1(t) = Fe^{A(\phi^{-1}(t)-t)}x(t), \]  
(9)
is referred to as the finite dimensional predictor while the second term
\[ u_2(t) = F \int_{t}^{\phi^{-1}(t)} e^{A(\phi^{-1}(t))-s}Bu_s(s) \, ds, \]  
(10)
is referred to as the infinite dimensional predictor.

The predictor based controller (8) is conceptually appealing as it ensures that the closed-loop system (5) has finite spectrum (this is why this method is also referred to as the finite spectrum assignment method). However, the controller in (8) is implicit since \( u \) is present on both sides of Eq. (8) and under an integral sign, which makes the implementation hard even when the delay \( d(t) \) is a constant one (Richard, 2003; Zhong, 2004). As explained in Mirkin and Olbrot (1979), obtaining this integral term as the solution to a differential equation must be discarded because it involves unstable pole–zero cancellations when \( A \) is unstable. An alternative is to approximate the integral term with a sum of point-wise delays by using a numerical quadrature rule such as rectangular, trapezoidal and Simpson’s rules. During the past several decades, the effect of such a semi-discretization on the asymptotic stability of the closed-loop system has been examined thoroughly. It is demonstrated in Mirkin (2004) and Van Assche et al. (1999) with a scalar example that for some prescribed system parameters, the control law approximated by numerical quadrature methods such as Simpson’s rule may lose its ability to stabilize the delay system no matter how precise the approximation is. It is then analyzed theoretically in Mirkin (2004) that such a loss of stability is caused by an undesirable combination of poor high-frequency accuracy of the approximation method and excessive sensitivity to high-frequency additive plant uncertainties. Consequently, a safe implementation can be achieved by eliminating either of these two factors (Mirkin, 2004). Considerable attention has been paid to overcoming this problem in different aspects in the past several decades (see Mirkin, 2004, Richard, 2003 and Zhong, 2004 and the references therein).

In this paper, we will show that the distributed term in the predictor based controller (8) is not required if some additional requirements are imposed on the system. Consequently, the implementation problem for such type of controllers is avoided entirely. To this end, we first notice that, since \( d(t) \) is bounded, the function \( \phi^{-1}(t) - t \), which was referred to as the prediction time, is also bounded. In fact,
\[ 0 \leq \phi^{-1}(t) - t \leq D. \]  
(11)
Let the nominal feedback gain \( F \) be parameterized as \( F = F(\gamma) \), \( \gamma \in (0, 1) \). If \( F(\gamma) \) is of order 1 with respect to \( \gamma \), namely,
\[ \lim_{\gamma \to 0} \frac{1}{\gamma} \| F(\gamma) \| < \infty, \]  
(12)
then the finite dimensional predictor term \( u_1(t) \) in the predictor feedback law (8) is also “of order 1” with respect to \( \gamma \) in view of (11). Consequently, control \( u \) itself is “of order 1” with respect to \( \gamma \), namely,
\[ \lim_{\gamma \to 0} \frac{1}{\gamma} \| u(\gamma) \| < \infty, \]  
(13)
As a result, by virtue of (11) and defining \( e^{i(\phi^{-1}(t)-t)} \) in view of (11), and \( B = T(t, s) \),
\[ \lim_{\gamma \to 0} \frac{1}{\gamma} \| F(\gamma) \| \int_{t}^{\phi^{-1}(t)} \| T(t, s) \| \frac{1}{\gamma} \| u(\gamma) \| \, ds, \]  
(14)
namely, the infinite dimensional predictor term \( u_2(t) \) is at least “of order 2” with respect to \( \gamma \). This indicates that, no matter how large the value of \( D \) is, the infinite dimensional predictor term \( u_2(t) \) in (10) is dominated by the finite dimensional predictor term \( u_1(t) \) in (9) and thus might be safely neglected in \( u(t) \) when \( \gamma \) is sufficiently small. As a result, the predictor feedback law (8) can be truncated as
\[ u(t) = u_1(t) = F(\gamma) e^{A(\phi^{-1}(t)-t)}x(t), \]  
(15)
which we refer to as the “truncated predictor feedback (TPF)”. The main advantage of the TPF (15) over the predictor feedback (8) is that the numerical problems encountered in the implementation of the integral predictor (distributed) term (10) are entirely avoided.

However, to ensure that the TPF is indeed feasible, two problems should be solved. On the one hand, we need to identify what type of systems can be stabilized by a parameterized feedback gain \( F = F(\gamma) \), \( \gamma \in (0, 1) \) such that (12) is satisfied. On the other hand, we need to verify that the TPF (15) can indeed stabilize the time-varying delay system (1). For the first problem, it is well-known that such a parameterized feedback gain exists if and only if \( (A, B) \) is stabilizable and all the eigenvalues of \( A \) are on the closed left-half plane (see, for example, Lin, 1998 and Zhou et al., 2008). Since the stable eigenvalues of \( A \) do not affect the stabilizability of the system, for simplicity, we impose the following assumption on the system.

Assumption 2. The matrix pair \((A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\) is controllable with all the eigenvalues of \( A \) being on the imaginary axis.

The main purpose of this paper is to give a positive answer to the second problem mentioned above.

Remark 1. When the delay in the time-delay system (1) is constant, say, \( \phi(t) = t - D \), where \( D \) is a positive constant, then \( \phi^{-1}(t) = t + D \), and the TPF (15) becomes
\[ u(t) = F(\gamma) e^{Ao}x(t). \]  
(16)
In this case, it has been proven in Lin and Fang (2007) and Zhou et al. (2010b) that, if \( F(\gamma) \) is properly designed, such a controller can indeed globally stabilize the time-delay system (1). This also explains why we have designed such a controller in Lin and Fang (2007) and Zhou et al. (2010b).

3. Stability analysis of the closed-loop system

In this section, we first prove that the TPF (15) can indeed stabilize the time-varying delay system (1) under Assumptions 1 and 2.
Theorem 1. Consider the linear system (1) with time-varying delay. Let Assumptions 1 and 2 be satisfied and $n \geq 2$. Then the TFP

$$u(t) = -B^TP(\gamma)e^h(\phi^{-1}(t)-1)x(t), \quad \forall t \geq 0,$$

(17)

(globally) asymptotically stabilizes system (1), where $u(t), \forall t \in [\phi(0), 0)$, is a function that is bounded, the matrix $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation (ARE)

$$A^TP + PA - BB^TP = -\gamma P,$$

(18)

and the parameter $\gamma$ satisfies

$$\gamma \in (0, \gamma^*), \quad \gamma^* = \frac{\delta^*}{D\omega},$$

(19)

with $\omega = n - 1$ and $\delta^*$ being the unique positive root of the following equation

$$\frac{\omega^2}{n^2} = \delta e^{\delta} (e^{\delta} - 1).$$

(20)

Proof. For simplicity, we denote $F = F(\gamma) = -B^TP(\gamma)$ and $P = P(\gamma)$. Consider an arbitrary initial condition of the time-varying delay system (1) as

$$x(t) = \phi(t), \quad \forall t \in [\phi(0), 0].$$

(21)

From the TFP (17) we can write

$$u(\phi(t)) = F e^{A(t-\phi(t))}x(\phi(t)), \quad \forall t \geq \phi^{-1}(0).$$

(22)

The closed-loop system can thus be written as

$$\dot{x}(t) = Ax(t) + BF e^{A(t-\phi(t))}x(\phi(t)), \quad \forall t \geq \phi^{-1}(0).$$

(23)

Since $u(t), \forall t \in [\phi(0), 0]$ is a bounded function, the solution in the interval $[0, \phi^{-1}(0)]$ to the closed-loop system is simply given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds,$$

(24)

$$\forall t \in [0, \phi^{-1}(0)],$$

which is also bounded for any bounded initial condition $\phi(t), \forall t \in [\phi(0), 0]$ and $u(t), \forall t \in [\phi(0), 0)$. Hence, we need only to consider the asymptotic stability of the closed-loop system with $t \geq \phi^{-1}(0)$, say, the asymptotic stability of system (23). However, the solution to system (23) in the interval $[\phi^{-1}(0), \phi^{-1}(\phi^{-1}(0))]$ is given by

$$x(t) = e^{A(t-\phi^{-1}(0))}x(\phi^{-1}(0)) + \int_{\phi^{-1}(0)}^t e^{A(t-s)}BF e^{A(s-\phi(s))}x(\phi(s))ds,$$

(25)

which is again bounded for any bounded initial condition $\phi(t), \forall t \in [\phi(0), 0]$ and $u(t), \forall t \in [\phi(0), 0)$ in view of (24). Therefore, we need only to consider the asymptotic stability of system (23) with $t \geq \phi^{-1}(\phi^{-1}(0))$.

Notice that with the help of the above model (23) and the variation of constants formula (Hale, 1977; Krstic, 2010b), we can compute, for all $t \geq \phi^{-1}(\phi^{-1}(0))$,

$$x(t) = e^{A(t-\phi(t))}x(\phi(t)) + \int_{\phi(t)}^t e^{A(t-s)}BF e^{A(s-\phi(s))}x(\phi(s))ds.$$

(26)

Then the closed-loop system (23) can be rewritten as

$$\dot{x}(t) = (A + BF)x(t) - BF \int_0^t e^{A(t-s)}BF e^{A(s-\phi(s))}x(\phi(s))ds \triangleq (A + BF)x(t) - BF\lambda(t).$$

(27)

By virtue of (18) and Lemma 1 in the Appendix, the time derivative of

$$V(x(t)) = x^T(t)Px(t)$$

(28)

along the trajectories of the system in (27) satisfies

$$\dot{V}(x(t)) = x^T(t)\left((A + BF)^TP + P(A + BF)\right)x(t) - 2\lambda^T(t)PBFx(t) - 2\lambda(t)^TPBFx(t) \leq -\gamma V(x(t)) - \lambda^T(t)PBFx(t) + \lambda(t)^TPBFx(t) + \lambda(t)^TPP\lambda(t) = -\gamma V(x(t)) + \lambda^T(t)PBFx(t).$$

(29)

We next simplify the term $\lambda^T(t)P\lambda(t)$. By using the Jensen inequality in Lemma 2 in the Appendix, we get

$$\lambda^T(t)P\lambda(t) \leq (t - \phi(t)) \int_{\phi(t)}^t x^T(s)P(x(s))ds = d(t) \int_{\phi(t)}^t x^T(s)P(x(s))ds \leq D \int_{\phi(t)}^t x^T(s)P(x(s))ds,$$

(30)

where we have used the boundedness assumption on $d(t)$, and

$$L(t, s) = e^{\psi(t-s)}PBB^T e^{A(t-s)}BPBB^Te^{\psi(s-t)}.$$
Notice that
\[ \phi_t(t) = t - d(t) - d((t - d(t))) \quad \text{at } t - d'(t). \]  
Clearly, we have \(|d'(t)| \leq 2D\). Hence, under the condition that
\[ V(t + \theta) < \eta V(t), \quad \forall \theta \in [-2D, 0], \]  
where \( t \geq \phi^{-1}(\phi^{-1}(0)) \) and \( \eta > 0 \) is any given scalar, the inequality in (34) can be continued as
\[ \dot{V}(t) \leq -\chi V(t), \quad \text{where, by denoting } \delta = \omega y D, \]  
\[ \chi = \gamma \left( 1 - e^{\omega y D} n^3 \gamma^2 D \int_{-D}^{t-D} e^{\omega(t-s)} ds \right) \]  
\[ = \gamma \left( 1 - \eta n^2 \gamma^2 D e^{\omega y D} \frac{1}{\omega y D} (e^{\omega y D} - 1) \right) \]  
\[ = \gamma \frac{n^3}{n^3} \left( \omega^2 - \eta \delta e^\delta (e^\delta - 1) \right) \]  
\[ = \gamma \frac{n^3}{D} \omega^3 \delta \left( \omega^2 - \eta \delta e^\delta (e^\delta - 1) \right). \]  
\[ \text{Note that} \]  
\[ f(\delta) = \delta e^\delta (e^\delta - 1), \quad \forall \delta \geq 0, \]  
is a strictly increasing function. Therefore we deduce from Eq. (20) that
\[ \frac{\omega^2}{n^3} - \delta e^\delta (e^\delta - 1) > 0, \quad \forall \delta \in (0, \delta^*), \]  
and, consequently, there exist a number \( \eta > 1 \) and a sufficiently small number \( \epsilon > 0 \) such that
\[ \frac{1}{n^3} D \omega^3 \delta \left( \omega^2 - \eta \delta e^\delta (e^\delta - 1) \right) > \epsilon, \quad \forall \delta \in (0, \delta^*). \]  
Therefore, there exist a constant \( \gamma \in (0, \frac{\epsilon}{\omega y D}) \). The closed-loop system (23) is then asymptotically stable by virtue of the Razumikhin stability theorem (Theorem 5 in the Appendix). The proof is completed. \( \square \)

Remark 2. Above we have assumed that \( n \geq 2 \). If \( n = 1 \), say, the system (1) is of the form \( x(t) = -u(\phi(t)) \), then we get from (39) that \( \chi = 1 - \eta y^2 D^2 \). Consequently, inequality (38) reads
\[ \dot{V}(x(t)) \leq -\chi (1 - \eta y^2 D^2) V(x(t)). \]  
Therefore the asymptotic stability of the closed-loop system is guaranteed provided that \( \gamma \in (0, \frac{\epsilon}{\omega y D}) \).

Regarding the convergence rate of the closed-loop system, we have the following result.

Proposition 1. For any \( \gamma \in (0, \gamma^* ) \), let \( \mu = \mu(\gamma) \) be the unique positive root of the nonlinear equation
\[ \mu - \delta \frac{D}{\omega} + \frac{n^3}{D} e^\delta (e^\delta - 1) e^{2\mu} = 0, \tag{46} \]  
where \( \delta = \omega y D \) and \( \omega = n - 1 \). Then there exists a constant \( c(\gamma) > 0 \) such that the state \( x(t) \), \( \forall t > 0 \) of the closed-loop system (23) satisfies
\[ \|x(t)\| \leq c(\gamma) \left( \sup_{\theta \in [0, \phi(0)]} \left\| \begin{array}{c} x(\theta) \\ u(\theta) \end{array} \right\| \right) e^{-\frac{\gamma t}{2}}, \]  
\[ \text{Proof.} \]  
Define a function
\[ g(\mu) = \mu - \delta \frac{D}{\omega} + \frac{n^3}{D} e^\delta (e^\delta - 1) e^{2\mu}. \tag{48} \]  
Then it follows from (42) that
\[ g(0) = -\delta \frac{D}{\omega} + \frac{n^3}{D} e^\delta (e^\delta - 1) = -\frac{1}{n^3} \left( \frac{\omega^2}{n^3} - \delta e^\delta (e^\delta - 1) \right) \]  
\[ < 0, \quad \forall \delta \in (0, \delta^*). \]  
Moreover, we clearly have \( \frac{\partial}{\partial \mu} g(\mu) > 0, \forall \delta \geq 0 \). Hence, by continuity, the nonlinear equation (46) has a unique positive root for all \( \gamma \in (0, \gamma^* ) \).

With \( h(V) \) defined in (35), we obtain
\[ h(V(t)) = (n \gamma)^3 e^{\omega y D} \int_{\phi(t)}^{t} e^{\omega(t-s)} V(x(s)) ds \leq (n \gamma)^3 e^{\omega y D} \int_{\phi(t)}^{t} e^{\omega(t-s)} ds \max_{s \in [t-2D, t]} \{ V(x(s)) \} \]  
\[ = (n \gamma)^3 e^{\omega y D} \frac{1}{\omega y} \max_{s \in [t-2D, t]} \{ V(x(s)) \} \]  
\[ = \frac{n^3}{D} \omega^3 \delta e^\delta (e^\delta - 1) \max_{s \in [t-2D, t]} \{ V(x(s)) \}. \]  
Therefore, we have from (34) that, for all \( t \geq t_0 = \phi^{-1}(\phi^{-1}(0)) \),
\[ \dot{V}(x(t)) \leq -\delta \frac{D}{\omega} V(x(t)) \]  
\[ + \frac{n^3}{D} \omega^3 \delta e^\delta (e^\delta - 1) \max_{s \in [t-2D, t]} \{ V(x(s)) \}. \]  
Notice that the above inequality is in the form of (138). Then, by the Halanay lemma (Lemma 3 in the Appendix), we have
\[ V(x(t)) \leq \sup_{s \in [t-20D, t_0]} \{ V(x(s)) \} e^{-\frac{\gamma t}{2}}, \quad \forall t \geq t_0. \]  
Let \( \lambda_{\min}(P) \) and \( \lambda_{\max}(P) \) denote respectively the minimal and maximal eigenvalues of \( P \).

\[ \lambda_{\min}(P) \|x(t)\|^2 \leq V(x(t)) \leq \lambda_{\max}(P) \|x(t)\|^2. \]  
With this, it follows from (52) that, for all \( t \geq t_0 \),
\[ \|x(t)\| \leq \left( \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right) \sup_{s \in [t_0-2D, t_0]} \{ \|x(s)\| \} e^{-\frac{\gamma t_0}{2}}. \]  
On the other hand, as \( t_0 \) is finite, it follows from (24)–(26) that there exists a positive constant \( c'(\gamma) \) such that
\[ \|x(t)\| \leq c'(\gamma) \left( \sup_{\theta \in [0, \phi(0)]} \left\| \begin{array}{c} x(\theta) \\ u(\theta) \end{array} \right\| \right), \quad \forall t \in [0, t_0]. \]  
Combining (54) and (55) gives the inequality (47) where \( c'(\gamma) \) is defined in an obvious way. The proof is finished. \( \square \)

Remark 3. It follows from (46), (49) and (20) that
\[ \lim_{\gamma \to \gamma^*} \mu(\gamma) = \lim_{\gamma \to \gamma^*} \mu(\gamma) = 0, \]  
which indicates that there exists at least one number \( \gamma^* \) such that \( \mu(\gamma) \) is minimized, namely, the estimation of the convergence rate is maximized. The optimal value \( \gamma^* \) can be obtained by using a bisection method.
Remark 3 implies that, for a given delay bound $D$, a larger value of $\gamma$ (or $\delta$) does not necessarily lead to a higher convergence rate of the closed-loop system. In practice, we may want to choose the optimal value of $\gamma$ (denoted by $\gamma_{\text{opt}}$) such that the convergence rate of the closed-loop system is maximized. Since the convergence rate given in Proposition 1 is only an estimation, the optimal value $\gamma_{\text{opt}}$ may not be simply determined by $\gamma_{\text{opt}} = \frac{\delta}{2D}$ according to Remark 3. We notice that determining $\gamma_{\text{opt}}$ is generally a hard problem if the delay $d(t)$ is time-varying. However, if the delay is a constant, we can propose a numerical method to compute $\gamma_{\text{opt}}$.

When $d(t)$ is a constant, the characteristic quasi-polynomial of the closed-loop system (23) is given by

$$\beta(s, \gamma) = \det(sI - A - BF(\gamma)e^{\lambda_0}e^{-\delta D}).$$

(57)

It is well-known that, for any fixed $\gamma$ and any prescribed number $l$, equation $\beta(s, \gamma) = 0$ has only a finite number of zeros on $\{s : \text{Re}[s] \geq l\}$ (Hale, 1977). The right-most zeros of equation $\beta(s, \gamma) = 0$ can be computed by the efficient software package DDE-BIFTOOL (Engelborghs, Luzyanina, & Samaey, 2001).

For a given $\gamma \geq 0$, denote the real part of the right-most roots of the characteristic quasi-polynomial equation $\beta(s, \gamma) = 0$ by $\lambda_{\text{max}}(\gamma) = \max\{\text{Re}[s] : \beta(s, \gamma) = 0\}.$

(58)

It follows that the closed-loop system (23) is asymptotically stable if and only if $\lambda_{\text{max}}(\gamma) < 0$. Moreover, it is well-known that the convergence rate of the closed-loop system (23) is completely determined by $\lambda_{\text{max}}(\gamma)$, namely, the smaller the value of $\lambda_{\text{max}}(\gamma)$ is, the faster the state converges to the origin (Hale, 1977).

According to Theorem 1, there is an interval $\gamma_{\text{opt}} \subset (0, \infty)$ such that $\lambda_{\text{max}}(\gamma_{\text{opt}}) < 0, \forall \gamma \in \gamma_{\text{opt}}$. Let $\gamma_{\text{opt}} = \sup_{\gamma \in \gamma_{\text{opt}}} \gamma$. By definition, we have $\lambda_{\text{max}}(0) = \lambda_{\text{max}}(\gamma_{\text{opt}}) = 0$, namely, the closed-loop system (23) is marginally unstable with $\gamma = 0$ and $\gamma = \gamma_{\text{opt}}$. Moreover, we have $\lambda_{\text{max}}(\gamma) < 0, \forall \gamma \in (0, \gamma_{\text{opt}})$, i.e., the closed-loop system is asymptotically stable with $\gamma \in (0, \gamma_{\text{opt}})$. Therefore, by continuity of zeros of quasi-polynomials (Ruan & Wei, 2003), there exists a value $\gamma_{\text{opt}} \in (0, \gamma_{\text{opt}})$ such that $\lambda_{\text{max}}(\gamma_{\text{opt}})$ is minimized at $\gamma = \gamma_{\text{opt}}$. Denote such minimal value by $\lambda_{\text{max}}(\gamma_{\text{opt}})$, namely,

$$\lambda_{\text{max}}(\gamma_{\text{opt}}) = \min_{\gamma \in (0, \gamma_{\text{opt}})} \lambda_{\text{max}}(\gamma).$$

(59)

Then, $\lambda_{\text{max}}(\gamma_{\text{opt}})$ is the maximal convergence rate that the TPF (17) can achieve.

In practice, we can compute the function $\lambda_{\text{max}}(\gamma)$ defined in (58) with the software package DDE-BIFTOOL (Engelborghs et al., 2001) by choosing

$$\gamma = k\Delta t, \quad k = 0, 1, \ldots, N,$$

(60)

where $\Delta t$ is a sufficiently small number denoting the step size. Here, $N$ is chosen as the minimal number such that $\lambda_{\text{max}}(N\Delta t) = 0$. According to the computational results of $\lambda_{\text{max}}(\gamma)$, the optimal value $\lambda_{\text{opt}}$ and the maximal convergence rate $\lambda_{\text{max}}$ can be obtained accordingly. The process will be illustrated via numerical examples in Section 6.

4. Output feedback stabilization via TPF

In this section, we discuss about the output feedback stabilization of system (1) by a TPF. Let us assume that the time-delay system (1) has an output

$$y(t) = Cx(t), \quad C \in \mathbb{R}^{p \times n},$$

(61)

where $(A, C)$ is detectable. We construct the following observer-based controller

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(\phi(t)) + L(y(t) - Cz(t)), \\
u(t) &= -BB^TP(\gamma)e^{(\lambda_0-\delta D)}z(t), \quad \forall t \geq 0,
\end{align*}$$

(62)

where $P(\gamma)$ is the unique positive definite solution to the parametric ARE (18) and $L \in \mathbb{R}^{p \times p}$ is such that $A - LC$ is asymptotically stable.

Theorem 2. Consider the linear system (1) with time-varying delay. Let Assumptions 1 and 2 be satisfied and $n \geq 2$. Then there exists a $\gamma^* > 0$ such that the observer based TPF (62) (globally) asymptotically stabilizes the system for any $\gamma \in (0, \gamma^*)$.

Proof. Let $e(t) = x(t) - z(t)$ and $F = -BB^TP$. Then the closed-loop system (1) and (62) becomes

$$\begin{align*}
\dot{x}(t) &= Ax(t) + BF e^{A(t-\phi(t))}x(\phi(t)) - e(\phi(t))) \\
nu(t) &= -(A - LC)e(t), \quad t \geq 0.
\end{align*}$$

(63)

As in the proof of the state feedback results, we need only to prove the asymptotic stability of the closed-loop system with $t \geq \phi^{-1}(\phi^{-1}(0))$. In this case, by using the model in the first equation in (63) and the variation of constants formula, we get

$$x(t) = \int_{\phi(t)}^{t} e^{A(t-s)}BF e^{A(s-\phi(s))}x(\phi(s))e^{-e(\phi(t))}ds + e^{A(t-\phi(t))}x(\phi(t)).$$

(64)

Note that the second equation in (63) implies

$$e(\phi(t)) = e^{A(t-\phi(t))}e(\phi(t)), \quad \forall t \geq \phi^{-1}(\phi^{-1}(0)).$$

(65)

Inserting (64) and (65) into the first equation in (63) gives

$$\dot{x}(t) = (A + BF)x(t) + BF(\lambda_2(t) - \lambda_1(t))$$

$$+ BF e^{A(t-\phi(t))}e^{A(t-\phi(t))}e^{-e(\phi(t))}e(t),$$

(66)

where $\lambda_1(t)$ and $\lambda_2(t)$ are respectively defined as

$$\begin{align*}
\lambda_1(t) &= \int_{\phi(t)}^{t} e^{A(t-s)}BF e^{A(s-\phi(s))}x(\phi(s))ds, \\
\lambda_2(t) &= \int_{\phi(t)}^{t} e^{A(t-s)}BF e^{A(s-\phi(s))}e(\phi(s))ds.
\end{align*}$$

(67)

Choose the following Lyapunov function

$$V(\zeta(t)) = x^T(t)Px(t) + e^T(t)Re(t), \quad \zeta(t) = (x(t), e(t)),$$

where $R > 0$ solves the following Lyapunov matrix equation

$$(A - LC)^T R + R (A - LC) = -I_n.$$ 

(69)

Then, with the help of (18) and (69), the time derivative of $V(\zeta(t))$ along the trajectories of system (63) reads

$$\dot{V}(\zeta(t)) = -x^T(t) Px(t) - x^T(t) PBB^TPx(t)$$

$$- \|e(t)\|^2 + E(t) + F(t),$$

(70)

where

$$E(t) = -2x^T(t)PBB^TP e^{A(t-\phi(t))}e^{(A-LC)(t-\phi(t))}e(t)$$

$$= -2x^T(t)PBB^TP e^{A(t-\phi(t))}e^{(A-LC)(t-\phi(t))}e(t),$$

(71)

and

$$F(t) = 2(\lambda_2(t) - \lambda_1(t))^T PBB^TPx(t).$$

(72)

By virtue of Lemmas 1 and 2, we get

$$E(t) \leq \frac{1}{2}x^T(t)PBB^TP(\gamma)x(t) + 2e^T(t)E_1(t)e(t),$$

(73)
where
\[ F(t) = e^{-(A-LC) t} P B B^T P e^{(A-LC) t} \]
\[ \leq n \gamma e^{\omega \gamma D} e^{-(A-LC) t} P B B^T P e^{-(A-LC) t} \]
\[ \leq n \gamma e^{\omega \gamma D} e^{-(A-LC) t} P B B^T P e^{-(A-LC) t} \]
\[ \leq n \gamma e^{\omega \gamma D} e^{-(A-LC) t} P B B^T P e^{-(A-LC) t} \]
\[ \equiv E(t). \quad (74) \]

Using Lemmas 1 and 2 again gives
\[ F(t) \leq 2 (\lambda(t) - \lambda_1(t)) \] \[ + \frac{1}{2} t^2 \gamma^2 (t) P B B^T P (t) \]
\[ \leq 2 n \gamma (\lambda(t) - \lambda_1(t)) P (\lambda(t) - \lambda_1(t)) \]
\[ + \frac{1}{2} t^2 \gamma^2 (t) P B B^T P (t) \]
\[ \leq 4 n \gamma (\lambda(t) \gamma^2 (t) P) \]
\[ + \frac{1}{2} t^2 \gamma^2 (t) \]
\[ \leq 4 n \gamma (\lambda(t) P) \]
\[ + \frac{1}{2} t^2 \gamma^2 (t) \]
\[ \equiv 4 n \gamma (\lambda(t) + \frac{1}{2} t^2 \gamma^2 (t) P B B^T P (t). \quad (75) \]

Substituting (73)–(75) into (70) yields
\[ \dot{V}(x(t), P(t)) \leq -\gamma x^2 (t) P(t) + 4 n \gamma \lambda_1 (t) \]
\[ - e^2 (t) (I_n - 2 E(t)) e(t). \quad (76) \]

We note that \( \lambda(t) \) is in the form of \( \lambda \) defined in the proof of Theorem 1. Then, similar to (33), we can obtain
\[ \lambda(t) P \lambda(t) \leq (n \gamma)^2 D \int_{\phi(t)}^t e^{\omega \gamma (t - \phi(s))} \phi (s) P (s) \phi (s) ds. \quad (77) \]

Moreover, by Lemma 1, it is easy to get that
\[ \lambda(t) P \lambda(t) \leq \int_{\phi(t)}^t e^T (s) E(s) E(s) ds, \quad (78) \]
where
\[ E(s) = e^{A s} P B B^T P e^{A s} \]
\[ \leq (n \gamma)^2 e^{\omega \gamma s} P. \quad (79) \]

Let \( \gamma > 0 \) be such that \( P(\gamma) \leq R, \forall \gamma' \in (0, \gamma_1] \). Then it follows from (77)–(79) that
\[ \lambda(t) \leq \lambda_2 (t) P \lambda_2 (t) + \lambda_1 (t) P \lambda_1 (t) \]
\[ \leq (n \gamma)^2 D \int_{\phi(t)}^t e^{\omega \gamma (t - \phi(s))} \phi (s) ds \]
\[ \leq (n \gamma)^2 D \int_{\phi(t)}^t e^{\omega \gamma (t - \phi(s))} \phi (s) ds \]
\[ \leq (n \gamma)^2 D e^{\omega \gamma t} \int_{\phi(t)}^t e^{\omega \gamma (t - s)} ds. \quad (80) \]
Let \( \gamma > 1 \) be any given constant. If
\[ V (\zeta (\theta + t)) < \eta V (\zeta (t)) \]
\[ \forall \theta \in [-2D, 0]. \quad (81) \]
then it follows from (80) that
\[ \lambda(t) \leq \left( (n \gamma)^2 D e^{\omega \gamma t} \int_{-D}^t e^{\omega \gamma (t - s)} ds \right) V (\zeta (t)) \]
\[ = \left( \frac{1}{\omega} \eta^2 \gamma D e^{\omega \gamma (t - 1)} \right) V (\zeta (t)). \quad (82) \]
Therefore, inequality (76) can be continued as
\[ V (\zeta (t)) \leq 4 n \gamma \left( \frac{1}{\omega} \eta^2 \gamma D e^{\omega \gamma (t - 1)} \right) V (\zeta (t)) \]
\[ \leq \gamma x^2 (t) P(t) - e^2 (t) (I_n - 2 E(t)) e(t). \quad (83) \]
Let \( \gamma > 0 \) be such that, for any \( s \in [-D, 0] \), and any \( \gamma' \in (0, \gamma_2] \)
\[ I_n \geq 2 n \gamma e^{\omega \gamma (t - s)} P \]
\[ \therefore \]
\[ V (\zeta (t)) \leq -\gamma V (\zeta (t)) \left( 1 - \frac{4}{\omega} \eta^2 \gamma D e^{\omega \gamma (t - 1)} \right). \quad (85) \]
Let \( \gamma' > 0 \) be such that, for any \( \gamma' \in (0, \gamma_2] \),
\[ 1 - \frac{4}{\omega} \eta^2 \gamma D e^{\omega \gamma (t - 1)} \geq \frac{1}{2}. \quad (86) \]
Then for arbitrary \( \gamma' \in (0, \gamma_2] \) where \( \gamma_2 = \min \left\{ \gamma_2', \gamma_1' \right\} \), we conclude from (85) that
\[ \dot{V} (\zeta (t)) \leq -\frac{1}{2} \lambda V (\zeta (t)) \quad \forall t \geq \phi^{-1} (0). \quad (87) \]
The (global) asymptotic stability of the closed-loop system then follows from Theorem 5 in the Appendix. The proof is completed. \( \square \)

An estimate of the convergence rate of the closed-loop system (63) can also be obtained by using the methods in the proof of Proposition 1. The details are omitted for brevity. Moreover, with respect to the state feedback case, Remark 2 is also applicable to the output feedback case.

5. Semi-global stabilization with constrained controls

In the event that the time-delay system in (1) is also subject to input saturation, the system becomes
\[ \dot{x} (t) = A x (t) + B s(t) \left( u (\phi (t)) \right), \quad (88) \]
where \( \text{sat}(\cdot) \) is the standard vector-valued saturation function defined as
\[ \text{sat}(u) = \begin{bmatrix} \text{sat}(u_1) & \text{sat}(u_2) & \cdots & \text{sat}(u_m) \end{bmatrix}, \quad (89) \]
with \( \text{sat}(u) \) being the standard scalar saturation function. Here, without loss of generality, we have assumed the unity saturation level. The non-unity saturation level can be absorbed by the matrix \( B \) and the feedback gain. Then we have the following result regarding semi-global stabilization of the time-varying delay system (88). Output feedback results can be obtained accordingly and will not be presented here for brevity.

**Theorem 3.** Let Assumptions 1 and 2 be satisfied. Then the TPF (17), where \( u(t), t \in [\phi(0), 0] \), is such that
\[ \sup_{t \in [\phi(0), 0]} \| u(t) \|_\infty \leq 1, \quad (90) \]
and \( P(\gamma) \) is the unique positive definite solution to the parametric ARE (18), semi-globally asymptotically stabilizes system (88), i.e., for
any a priori given bounded set $\Omega \subset \mathbb{R}_{n,D}$, there exists a $\gamma^* > 0$ such that, for an arbitrary $\gamma \in (0, \gamma^*]$, the closed-loop system is asymptotically stable at the origin with $\Omega$ contained in the domain of attraction.

**Proof.** We first consider the closed-loop system without input saturation. Choose a nonnegative functional as

$$W_1(x_t) = \rho (\gamma) \int_0^T e^{\omega y} \int_{t-s}^T V(x(\phi(s))) \, ds \, dt,$$

where $\rho (\gamma) = (ny)^3 D e^{\omega y}$. The derivative of this function is given by

$$\dot{W}_1(x_t) = \rho (\gamma) \int_0^T e^{\omega y} \int_{t-s}^T V(x(\phi(s))) \, ds \, dt,$$

$$= \rho (\gamma) \int_0^T e^{\omega y} \int_{t-s}^T (V(x(\phi(s)) - V(x(\phi(t-s)))) \, ds \, dt,$$

$$= \rho (\gamma) \int_0^T e^{\omega y} V(x(\phi(t))) \, ds,$$

$$= \rho (\gamma) \int_0^T e^{\omega y} V(x(\phi(t))) \, ds,$$

$$= \rho (\gamma) \left( \int_0^T e^{\omega y} ds \right) V(x(\phi(t))),$$

$$= \rho (\gamma) \left( \int_0^T e^{\omega y} ds \right) V(x(\phi(t))),$$

$$= \rho (\gamma) \left( \int_0^T e^{\omega y} ds \right) V(x(\phi(t))).$$

Choose another nonnegative functional as

$$W_2(x_t) = \rho (\gamma) \int_0^T e^{\omega y} \int_t^{\phi^{-1}(t)} V(x(\phi(s))) \, ds,$$

whose derivative is given by

$$\dot{W}_2(x_t) = \rho (\gamma) \left( \int_0^T e^{\omega y} ds \right) \frac{d\phi^{-1}(t)}{dt} V(x(t)),$$

$$= \rho (\gamma) \left( \int_0^T e^{\omega y} ds \right) V(x(\phi(t))),$$

$$= \rho (\gamma) \left( \int_0^T e^{\omega y} ds \right) V(x(\phi(t))).$$

Hence, in view of (34), (92) and (94), the time derivative of the Lyapunov functional

$$W(x_t) = V(x(t)) + W_1(x_t) + W_2(x_t),$$

along the trajectories of the closed-loop system in (23) satisfies

$$\dot{W}(x_t) \leq -\gamma \theta(t) V(x(t)),$$

where

$$\theta(t) = 1 - n^2 \gamma^2 D e^{\omega y} \left( \int_0^T e^{\omega y} ds \right) \frac{d\phi^{-1}(t)}{dt}.$$

Since $\frac{d}{dt} \phi^{-1}(t) > 0$, for all $t \in [0, \infty]$, it follows from

$$\frac{d}{dt} \phi^{-1}(t) = \left( \frac{d}{ds} \phi(s) \bigg|_{s=\phi^{-1}(t)} \right)^{-1},$$

that $\frac{d}{dt} \phi^{-1}(t)$ is bounded for all $t \in [0, \infty]$. Then there exists a scalar $\gamma^*$ such that

$$\theta(t) > \frac{1}{2}, \quad \forall \gamma \in (0, \gamma^*), \quad \forall t \geq t_0 = \phi^{-1}(\phi^{-1}(0)),$$

and consequently

$$\dot{W}(x_t) \leq -\frac{1}{2} \gamma V(x(t)), \quad \forall \gamma \in (0, \gamma^*], \quad \forall t \geq t_0.$$

The closed-loop system is then claimed to be asymptotically stable by virtue of the Lyapunov stability theorem. Moreover, we get from (100) that, for all $t \geq t_0$,

$$W(x_t) \leq W(x_{t_0}), \quad \forall \gamma \in (0, \gamma^*].$$

With this it follows from (17), Lemma 1, (11) and (100) that, for any $\tau \geq t_0$,

$$u^T(t) u(t) = x^T(t) e^{A(t)} P B \left( e^{A(t)} P B \right)^T x(t) \leq n \gamma x^T(t) e^{A(t)} x(t) \leq n \gamma e^{(\alpha - \gamma) t} x^T(t) P x(t) \leq n \gamma e^{(\alpha - \gamma) t} x^T(t) P x(t) \leq n \gamma e^{(\alpha - \gamma) t} V(x(t))$$

$$\leq n \gamma e^{(\alpha - \gamma) t} W(x_t) \leq n \gamma e^{(\alpha - \gamma) t} W(x_{t_0}).$$

On the other hand, as $\Omega$ is bounded, $\lim_{\gamma \downarrow 0} \rho (\gamma) = 0$, and $\lim_{\gamma \downarrow 0} P (\gamma) = 0$, it is readily seen that, for any initial condition $x(\theta) = \psi(\theta) \in \Omega$, $\forall \theta \in [\phi(0), 0]$.

$$\lim_{\gamma \downarrow 0} W(x_{t_0}) = 0,$$

from which and (103) we get

$$\lim_{\gamma \downarrow 0} u^T(t) u(t) = 0, \quad \forall t \geq \phi^{-1}(\phi^{-1}(0)),$$

which implies that there exists a $\gamma^*_1 \in (0, \gamma^*)$ such that

$$\sup_{t \in [\phi^{-1}(\phi^{-1}(0)), \infty]} \| u(t) \| \leq 1, \quad \forall \gamma \in (0, \gamma^*_1],$$

namely, the input saturation can be avoided for all control signals $u(t)$, $t \geq t_0$. We next consider the control signals $u(t)$ with $t \in [\phi^{-1}(0), t_0]$. In this case, it follows from (25) that, for any $t \in [\phi^{-1}(0), t_0]$,

$$u(t) = -B^T P (\gamma) \left( e^{A(t)} x(\phi^{-1}(0)) + \int_{\phi^{-1}(0)}^t e^{A(t-s)} B F e^{A(t-s)} x(\phi(s)) \, ds \right).$$

from which there holds

$$\| u(t) \| \leq \| B^T P \| \left( \| e^{A(t)} x(\phi^{-1}(0)) \| + \int_{\phi^{-1}(0)}^t \| x(\phi(s)) \| \, ds \right).$$

where

$$Y(t, s) = \| e^{A(t-s)} B F e^{A(t-s)} x(\phi(s)) \|.$$
In view of (24), the boundedness of \( \psi \in \Omega \), and \( \lim_{t \to 0} P(\gamma) = 0 \), we conclude that there exists a \( \gamma_*^+ \in (0, \gamma^*_\epsilon) \) such that
\[
\sup_{t \in \phi^{-1}(t_0)} \{ \|u(t)\| \} \leq 1, \quad \forall \gamma \in (0, \gamma_*^+) ,
\]
(110)

namely, the closed-loop system will also not saturate for any \( t \in [\phi^{-1}(0), t_0] \). We finally consider the control signals \( u(t) \) with \( t \in [0, \phi^{-1}(0)] \). In this case, it follows from (24) that, for any \( t \in [0, \phi^{-1}(0)] \),
\[
\|u(t)\| = \|B^TPe^{\phi(t-\epsilon)}e^{\phi(t)}x(0)\| \leq \|B^TPe^{\phi(t-\epsilon)}\| \|x(0)\| .
\]
(111)

where \( x(\theta) = \psi(\theta) \in \Omega, \forall \theta \in [\phi(0), 0] \). Again, as \( \Omega \) is bounded, there exists a \( \gamma_*^+ \in (0, \gamma^*_\epsilon) \) such that
\[
\sup_{t \in [0, \phi^{-1}(0)]} \{ \|u(t)\| \} \leq 1, \quad \forall \gamma \in (0, \gamma_*^+) ,
\]
(112)

which indicates that the closed-loop system will not saturate for any \( t \in [0, \phi^{-1}(0)] \). Combining (112), (110) and (106) completes the proof. □

We can show that the TPF (17) also solves the so-called \( L_2 \) semi-global stabilization problem. To this end, we define
\[
\phi = \left\{ u(t) : [\phi(0), \infty) \to \mathbb{R}^n, \int_0^\infty \|u(s)\|^2 \, ds \leq 1 \right\}
\]
(113)

which denotes the set of functions that have bounded energy.

**Theorem 4.** Let Assumptions 1 and 2 be satisfied. Then the TPF (17), where \( u(t) \), \( t \in [\phi(0), 0] \) is such that
\[
\int_0^\phi \|u(s)\|^2 \, ds \leq 1,
\]
(114)

and \( P(\gamma) \) is the unique positive definite solution to the parametric ARE (18), semi-globally asymptotically stabilizes system (1) in the \( L_2 \) sense, i.e., for any a priori given bounded set \( \Omega \subset \mathbb{C}_{n,D} \), there exists a \( \gamma^* > 0 \) such that, for any \( \gamma \in (0, \gamma^*) \), the control \( u(t) \in \phi \) and the closed-loop system is asymptotically stable at the origin with \( \Omega \) contained in the domain of attraction.

**Proof.** It follows from (102) and (100) that, for any \( \gamma \in (0, \gamma^*) \) and any \( t \geq \phi^{-1}(\phi^{-1}(0)) = t_0 \),
\[
u^T\nu(t) \leq \eta \nu(e^{\phi(t-\epsilon)}D W(x_t) + \nu(e^{\phi(t-\epsilon)}D W(x_t) ,
\]
(115)

where \( W(x_t) \) is defined in (95). Taking integration on both sides of the above inequality from \( t_0 \) to \( \infty \) gives
\[
\int_{t_0}^\infty \|u(s)\|^2 \, ds \leq -2n(e^{-\phi(t-\epsilon)}D W(x_t) + \nu(e^{\phi(t-\epsilon)}D W(x_t) ,
\]
(116)

where we have noticed that \( \lim_{t \to \infty} W(x_t) = 0 \) since the closed-loop system (23) is asymptotically stable. Therefore, as \( \Omega \) is bounded, it follows from (104) and (116) that
\[
\lim_{\gamma \to \infty} \int_{t_0}^\infty \|u(s)\|^2 \, ds = 0.
\]
(117)

On the other hand, from (107) and (24) we have
\[
\lim_{\gamma \to \infty} \int_{t_0}^\phi \|u(s)\|^2 \, ds = \lim_{\gamma \to \infty} \int_{t_0}^\phi \|u(s)\|^2 \, ds + \lim_{\gamma \to \infty} \|u(s)\|^2 \, ds = 0.
\]
(118)

Hence, by combining (114), (117) and (118), we conclude that there exists a number \( \gamma^* \in (0, \gamma^*) \) such that
\[
\int_{t_0}^\infty \|u(s)\|^2 \, ds \leq 1, \quad \forall \gamma \in (0, \gamma^*) .
\]
(119)

The proof is thus completed. □

**6. Numerical examples**

We consider a delayed double oscillator system characterized by (1) in which \( A \) and \( B \) are given by
\[
A = \begin{bmatrix} 0 & \omega & 0 & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
(120)

where \( \omega \) is a positive number. For this system, the unique solution to the parametric ARE can be computed as in Box I and the matrix exponential is given by
\[
\exp(As) = \begin{bmatrix} \cos(\omega s) & \sin(\omega s) & \frac{1}{2} \sin(\omega s) & \frac{1}{2} \cos(\omega s) \\ -\sin(\omega s) & \cos(\omega s) & \frac{1}{2} \cos(\omega s) & \frac{1}{2} \sin(\omega s) \\ 0 & 0 & \cos(\omega s) & \sin(\omega s) \\ 0 & 0 & -\sin(\omega s) & \cos(\omega s) \end{bmatrix}
\]
(122)

in which \( \sigma(s) = \sin(\omega s) - \cos(\omega s) \) and \( \psi(s) = \omega \cos(\omega s) + \sin(\omega s) \).

We consider two cases of delay function \( \phi(t) \) in the system.

- **In the first case,** the delay function is (Example 5.3 in Krstic (2010b))
\[
\phi(t) = t - \frac{t+1}{2t+1}, \quad \forall t \geq 0.
\]
(123)

It follows that \( D = 1 \) and the inverse function of \( \phi(t) \) is
\[
\phi^{-1}(t) = \frac{t + \sqrt{(t + 1)^2 + 1}}{2}.
\]
(124)

The delay function \( d(t) = t - \phi(t) \) is shown in Fig. 1. Hence, according to Theorem 1, the (time-varying) TPF is given as
\[
u(t) = -B^TP(\gamma) \exp\left( A \left( \frac{t + 1}{\sqrt{(t + 1)^2 + 1}} \right) \right) x(t).
\]
(125)

- **In the second case** the delay function is oscillatory and is given by (Example 5.4 in Krstic (2010b)),
\[
\phi(t) = \rho^{-1}(t), \quad \rho(t) = t + 1 + \frac{1}{2} \cos(t).
\]
(126)

For an illustration of this function, see Fig. 2 in Krstic (2010b). The delay function \( d(t) = t - \phi(t) \) is also shown in Fig. 1. For this function, it is readily verified that
\[
\max_{t \in \mathbb{R}} |\phi(t) - t| = \max_{t \in \mathbb{R}} |\phi^{-1}(t) - t| = \frac{3}{2}.
\]
(127)
Given $\gamma$, the control signal $u(\phi(t))$ for the above two different kinds of delay functions are recorded in Fig. 4, from which we see that the peak values in the control signals decrease as $\gamma$ decreases, which indicates semi-global stabilization in the presence of input saturation.

Hence, we obtain $D = \frac{3}{2}$. Again, according to Theorem 1, the (time-varying) TPF is given as

$$ u(t) = -B^T P(\gamma) \exp \left( A \left( 1 + \frac{1}{2} \cos(t) \right) \right) x(t). \quad (128) $$

For these two cases, with a given initial condition

$$ x(\theta) = [-1 \ 2 \ 2 \ -1]^T, \quad \forall \theta \in [-D, 0], \quad (129) $$

and by setting $\omega = 2$ and $\gamma = 0.3$, the state responses and control signals $u(\phi(t))$ are shown in Figs. 2 and 3, respectively. It is clear that the systems are indeed stabilized by these two TPF controllers. Finally, for such a given initial condition, with different values of $\gamma$, the control signals $u(\phi(t))$ for the above two different kinds of delay functions are recorded in Fig. 4, from which we see that the peak values in the control signals decrease as $\gamma$ decreases, which indicates semi-global stabilization in the presence of input saturation.

We next use this example to demonstrate how the parameter $\gamma$ influences the convergence rate of the states of the closed-loop system by assuming that the delay is a constant, i.e., $d(t) = D = 1$. In this case, by using the software package DDE-BIFTOOL (Engelborghs et al., 2001), the function $\lambda_{\max}(\gamma)$ defined in (58) can be computed by the discretization of $\gamma$ according to (60). The results are recorded in Fig. 5, from which we get $\gamma_{\sup} = 0.4227$ and $\gamma_{\opt} = 0.3765$ (the corresponding $\lambda_{\max}^{\min}$ defined in (59) is $\lambda_{\max}^{\min} = -0.1917$). For a series of $\gamma \in (0, \gamma_{\sup})$, with the same initial condition as in (129), the state trajectories and control signals are respectively plotted in Figs. 6 and 7. It is seen in Fig. 6 that when $\gamma$ is increased from 0.1 to $\gamma_{\opt}$, the convergence rate of states of the closed-loop system increases. However, if we choose $\gamma = 0.41 > \gamma_{\opt}$, the corresponding convergence rate decreases,
The function $\lambda_{\text{max}}(\gamma)$ for the closed-loop system when the delay is a constant ($D = 1$).

![Fig. 5](image1)

State evolution of the closed-loop system when the delay is a constant ($D = 1$).

![Fig. 6](image2)

Control signal of the closed-loop system when the delay is a constant ($D = 1$).

![Fig. 7](image3)

indicating that $\gamma = \gamma_{\sup}$ indeed leads to the highest convergence rate of the closed-loop system. Finally, for $\gamma = \gamma_{\sup}$, we clearly observe that the closed-loop system is marginally unstable. In fact, the state converges to some constants since the largest zero of the characteristic quasi-polynomial equation $\beta(s, \gamma_{\sup}) = 0$ is zero.

7. Conclusions

In this paper, we have proposed a new design approach, referred to as the truncated predictor feedback (TPF), for linear systems with long time-varying input delays. By adopting the idea of the predictor based feedback and the recently developed parametric Lyapunov equation based low gain feedback, a finite dimensional static linear time-varying state feedback, which is obtained by neglecting the infinite-dimensional distributed terms in the predictor based feedback, was proposed to stabilize the system as long as the open-loop system is not exponentially unstable and the delay is bounded. An explicit condition on the free parameter in the controller was obtained to guarantee the asymptotic stability of the closed-loop system. It was also shown that the proposed parameterized controller semi-globally stabilizes the system in the presence of actuator magnitude saturation and energy constraints. In comparison with the predictor based controllers which are infinite dimensional state feedback, the proposed new controllers are more convenient to implement. Numerical examples have demonstrated the effectiveness of the proposed approach.

The research in this paper opened up several future research topics. For example, it would be interesting to consider linear systems with long multiple and distributed time-varying delays in the input by combining the TPF approach and those ideas found in our early studies (Zhou et al., 2012, 2010a). However, our initial study indicates that the controllers involve some nonlinear differential equations leading to some technical difficulties. Also, it is expected that the proposed TPF can be adopted to handle input delayed systems that are exponentially unstable. In that case, although it is expected that the delay cannot be arbitrarily long, it is reasonable to anticipate that less conservative results concerning the maximal allowable delay in the system can be obtained than those approaches without any predication term can since the finite dimensional predictor term in the form of (9) can partially compensate the delay effect (Krstic, 2010a). Further study along these lines is now underway.

Appendix

In this appendix, we recall some existing basic results that are needed in establishing the results of this paper. We first recall the following results from Zhou et al. (2008), Zhou et al. (2010b) regarding properties of solutions to the parametric ARE (18).

Lemma 1. Assume that the matrix pair $(A, B) \in (\mathbb{R}^{n\times n}, \mathbb{R}^{n\times m})$ is controllable and all the poles of $A$ are on the imaginary axis. Then the parametric ARE

$$A^T P + P A - P B B^T P = -\gamma P,$$

has a unique positive definite solution $P(\gamma) = W^{-1}(\gamma)$, where $W(\gamma)$ is the unique positive definite solution to the following Lyapunov equation

$$W \left( A + \frac{\gamma}{2} I_n \right)^T + \left( A + \frac{\gamma}{2} I_n \right) W = B B^T. $$

Moreover,

$$\lim_{\gamma \to 0} P(\gamma) = 0, \quad \frac{d}{d\gamma} P(\gamma) > 0, \quad \forall \gamma > 0 \quad \text{tr} (B^T P(\gamma) B) = \eta \gamma, \quad P(\gamma) B B^T P(\gamma) \leq \eta \gamma P(\gamma), \quad e^{\gamma t} P(\gamma) e^{\gamma t} \leq e^{\omega \gamma t} P(\gamma),$$

where $t \geq 0$ and $\omega \geq n - 1$.

The second technical lemma is the so-called Jensen Inequality.
Lemma 3. \( \mu \) is a constant, be a continuous function such that
\[
\forall t \geq 0, \quad \mu(t) > 0, \quad \mu(t) \text{ is strictly increasing and } \mu(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty.
\]
Then the function \( \psi(t) \) satisfies
\[
\psi(t) \leq \max_{s \in [t-t_0, t_0]} \{ \psi(s) e^{-\mu(t-t_0)}, \quad \forall t \geq t_0, \quad (138)
\]
Moreover, \( \mu = 0 \) if only if \( a = b \).

The final technical result is the Razumikhin stability theorem.

Theorem 5 (Hale, 1977), Consider the functional differential equation
\[
\begin{align*}
\dot{x}(t) &= f(t, x_t), \quad x(t) \in \mathbb{R}^n, t \geq t_0, \\
x_0(\theta) &= \psi(\theta), \quad \forall \theta \in [-\tau, 0].
\end{align*}
\]
(141)
The function \( f : \mathbb{R} \times \mathbb{R}^n \) is such that the image by \( f \) of \( \mathbb{R} \times \) (a bounded subset of \( \mathbb{R}^n \)) is a bounded subset of \( \mathbb{R}^n \) and the functions \( u, v, w, p : \mathbb{R} \rightarrow \mathbb{R} \) are continuous, non-decreasing and positive for all \( s > 0 \), \( u(s) = v(s) = 0 \), \( w(s) = 0 \), \( p(s) \) is strictly increasing and \( p(s) > s \).

If there is a continuous function \( V : \mathbb{R} \rightarrow \mathbb{R} \) such that the following conditions hold
\[
\begin{align*}
(1) \quad u(||x||) &\leq V(x) \leq v(||x||) \\
(2) \quad V(x) &\leq -w(||x||), \quad \forall V(x(t+\theta)) < p(V(x(t))), \forall \theta \in [-\tau, 0].
\end{align*}
\]
Then the trivial solution \( x(t) \equiv 0 \) of the differential equation (141) is globally asymptotically stable.

References

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