Consensus of high-order multi-agent systems with large input and communication delays

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Abstract

We study in this paper the consensus problem for multi-agent systems with agents characterized by high-order linear systems with time delays in both the communication network and inputs. Provided that the open-loop dynamics of the agents is not exponentially unstable, but may be polynomially unstable, and the communication topology contains a directed spanning tree, a truncated predictor feedback approach is established to solve the consensus problem. It is shown that, if the delays are constant and exactly known, the consensus problems can be solved by both full state feedback and observer based output feedback protocols for arbitrarily large yet bounded delays. If it is further assumed that the open-loop dynamics of the agents only contains zero eigenvalues, the delays are allowed to be time-varying and unknown. Numerical examples are worked out to illustrate the effectiveness of the proposed approaches.

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1. Introduction

Consensus refers to a group of agents under appropriate distributed control policies reaching an agreement on certain quantities of interest (Ishii, Tempo, & Bai, 2012; Lv, Chen, & di Bernardo, 2010; Lv, Chen, & Yu, 2011; Ren & Cao, 2011). Consensus is a fundamental problem in cooperative control of multi-agent systems and is closely related with other high-level problems such as flocking (Su, Wang, & Chen, 2009; Yu, Chen, & Cao, 2010a) and formation control (Fax & Murray, 2004), which find many applications in engineering such as sensor networks, spacecraft formation flying and cooperative surveillance (Chen & Duan, 2008; Duan, Chen, & Huang, 2007; Ren, Beard, & Atkins, 2007). Indeed, a group of autonomous agents connected by a communication or sensing network can coordinate with each other to perform some challenging tasks that cannot be accomplished if they operate uncooperatively. As a result, consensus via cooperative control of a group of agents has received considerable attention from various scientific communities in the past several decades (see Duan, Wang, Chen, & Huang, 2008, Hong, Chen, & Bushnell, 2006, Ishii & Tempo, 2010, Li & Jiang, 2009, Li & Xie, 2011, Lv et al., 2011, Qin, Gao, & Zheng, 2011, Ren & Cao, 2011, Su et al., 2009, Su, Wang, & Chen, 2010, You & Xie, 2011b and the references therein).

While the existing literature deals with consensus problem for multi-agent systems captured by low order dynamics such as single integrator, double integrator, and oscillators, consensus for multi-agent systems with high-order dynamics has received little attention, except for multi-agent systems whose agents are modeled by integrator chains of length greater than two (see, for example, Ren, Moore, & Chen, 2006). Recently, the consensus problem for multi-agent systems with agents modeled by high-order linear dynamic systems is considered in Seo, Shim, and Back (2009) by dynamic output feedback. The consensus problems for both continuous-time and discrete-time linear multi-agent systems with directed communication topologies are addressed in Li, Liu, and Lin (2011), where distributed reduced-order observer-based consensus protocols are developed. For single input high-order nonlinear multi-agent systems with unknown dynamics, an adaptive cooperative tracking control scheme is considered in Zhang and Lewis (2012). Synchronization of identical general linear systems on a digraph containing a spanning tree is studied in Zhang, Lewis, and Das (2011) by establishing both state feedback and observer based output feedback protocols. For more
related work on consensus of high-order multi-agent systems, see Li et al. (2011), Ren et al. (2006), Seo et al. (2009), Su and Huang (2012b), You and Xie (2011a) and the references cited there.

Delay effect on the convergence of consensus protocols is an important issue to be considered. One source of time delay in multi-agent systems is the communication from one agent to another, which is named as communication delay. Another source of time delay is related with the processing and connecting time for the packets arriving at each agent, which is called input delay (Tian & Liu, 2009). Consensus of multi-agent systems with communication and/or input delays has been extensively studied in the literature (see Lin, Jia, Du, & Yuan, 2007, Olfati-Saber & Murray, 2004, Tian & Liu, 2009 and the references therein). In most of these studies, the time delays are assumed to be unknown and the main purpose is to find the upper bounds on the time delays such that the consensus can still be achieved in the presence of time delay. For some simple agent dynamics, for example, those of single integrators or single oscillators, establishing the necessary and sufficient conditions on the maximal allowable time delay is possible under a prescribed protocol by analyzing the roots of certain characteristic equations (see, for example, Lin et al., 2007, Olfati-Saber & Murray, 2004, Rudy & Olgac, 2011 and Yu, Chen, & Cao, 2010b, for details). By using the generalized Nyquist criterion, some set-valued conditions are established in Munz, Papachristodoulou, and Allgower (2010) to guarantee the consensus of general single-input–single-output linear multi-agent systems with delays and explicit conditions for determining the convergence rate of single-integrator multi-agent systems are also proposed there. In Munz, Papachristodoulou, and Allgower (2011a), robust static output-feedback controllers are designed to achieve consensus in networks of heterogeneous agents modeled as nonlinear systems of relative degree two in the presence of heterogeneous communication delays. Very recently, the high-order consensus problem for heterogeneous multi-agent systems with unknown communication delays is studied in Tian and Zhang (2012) and a necessary and sufficient condition is given for the existence of a high-order consensus solution to heterogeneous multi-agent systems.

This paper studies the consensus problem for multi-agent systems with agents characterized by high-order linear systems with time delays. The time delays can be in both the communication network and the inputs of the agents. Since the delays are allowed to be arbitrarily large, under the assumption that the communication topology among the agents contains a directed spanning tree, we first employ the well-known predictor feedback (which is also known as the finite spectrum assignment and model reduction approach, Manitiuss & Olbricht, 1979, Zhou, Lin, & Duan, 2012) to design both distributed state feedback and observer based output feedback protocols such that the delays are completely compensated. However, as the predictor feedback protocols require the exact information of the network and the relative input signals among the agents, they may suffer some implementation problems. To overcome this problem, under the additional condition that the open-loop dynamics of the agents is at most polynomially unstable, we show that the consensus problems can also be solved by truncated predictor feedback protocols which need neither the exact information of the network nor the relative input signals among the agents. The allowable delays under the truncated predictor feedback based protocols can also be arbitrarily large, yet bounded. We also show that, if the open-loop dynamics of the agents only contains zero eigenvalues, the time delays are allowed to be time-varying and unknown, namely, for an arbitrarily given positive number $\tau^*$, the established explicit distributed protocols (dependent on $\tau^*$) achieve consensus for arbitrary time delays bounded by $\tau^*$. Numerical examples are worked out to illustrate the effectiveness of the proposed protocols.

We emphasize that, for observer based output feedback consensus, as the protocol is distributed, the observer error dynamics and the dynamics of the agents are coupled. As a result, an intricate Lyapunov analysis of the stability of the closed-loop multi-agent system has to be performed. We also emphasize that, while most of the literature on consensus of multi-agent systems with time delays deals with the analysis of robustness with respect to delays, that is, to estimate the bounds on the time delays under which a pre-designed consensus protocol continues to achieve consensus in the presence of time delays that are generally of small size (see, for example, Lin et al., 2007, Munz et al., 2010, 2011a, Munz, Papachristodoulou, & Allgower, 2012, Olfati-Saber & Murray, 2004, Qin et al., 2011, Rudy & Olgac, 2011 and Yu et al., 2010b), in the present paper, we take time delays into account in our design of consensus protocol and we allow the delays to be arbitrarily large. Particularly, we adjust a design parameter in the protocols in accordance with the size of the delays.

The remaining part of this paper is organized as follows. The problem formulation is given in Section 2. The state feedback consensus and observer based output feedback consensus problems are respectively solved in Sections 3 and 4. Numerical examples are provided in Section 5 to validate the effectiveness of the proposed protocols. Finally, Section 6 concludes the paper.

Notation: The notation used in this paper is fairly standard. For a complex matrix $A$ with appropriate dimensions, $A^T$, $A^\dagger$, $\lambda(A)$, $|A|$ are respectively its transpose, conjugated transpose, eigenvalue set, and norm. For a positive scalar $\tau$, let $\mathbb{R}_{\tau} = \mathbb{R}([-\tau, 0])$, $\mathbb{R}^\tau$ denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^\tau$ with the topology of uniform convergence, and let $x_t \in \mathbb{R}_{\tau}$ denote the restriction of $x(t)$ to the interval $[t - \tau, t]$. Let $\sigma(t)$ be the time-varying delay, $\sigma(t) \in \mathbb{R}([-\tau, 0])$. For two matrices $A$ and $B$, we use $A \otimes B$ to denote their Kronecker product. For two integers $p$ and $q$ with $p \leq q$, the symbol $I[p, q]$ refers to the set $\{p, p + 1, \ldots, q\}$. For a complex number $s$, we use $\Re\{s\}$ and $|s|$ to denote respectively its real part and modulus. Finally, throughout this paper, if not specified, for a series of vectors $x_i, i \in \{1, \ldots, N\}$, $N$ is the number of agents, $\alpha_i \in \mathbb{R}^N$ with appropriate dimensions, we denote $x = [x_1^T, x_2^T, \ldots, x_N^T]^T$. The remaining part of this paper is organized as follows. The problem formulation is given in Section 2. The state feedback consensus and observer based output feedback consensus problems are respectively solved in Sections 3 and 4. Numerical examples are provided in Section 5 to validate the effectiveness of the proposed protocols. Finally, Section 6 concludes the paper.

2. Problem formulation and preliminaries

We consider a continuous-time high-order multi-agent system described by

$$ \dot{x}_i(t) = A x_i(t) + B u_i(t - \tau_{com}) , \quad y_i(t) = C x_i(t) , \quad i \in \{1, \ldots, N\} , $$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^r$ are the state, the control and the output of Agent $i$, respectively, $N \geq 1$ is a given integer denoting the number of agents, $\tau_{com} \geq 0$ is the input delay, and $(A, B, C)$ is a given matrix triple. Let the communication topology among these agents be characterized by a weighted directed graph $G(N, E, A)$, where $N$ is the node set, $E$ is the edge set, and $A = [\alpha_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix. Denote the corresponding Laplacian by $L = [l_{ij}] \in \mathbb{R}^{N \times N}$.

In the full information case, we assume that Agent $i$ collects the delayed state information of its neighboring agents by the rule

$$ z_i(t) = \sum_{j \in N_i} \alpha_{ij} (x_j(t - \tau_{com}) - x_i(t - \tau_{com})) = \sum_{j = 1}^N l_{ij} (t - \tau_{com}) , \quad i \in \{1, \ldots, N\} , $$

where $\tau_{com} \geq 0$ represents the communication delay and $N_i = \{j : \alpha_{ij} \neq 0\}$. In the partial information case, we assume that Agent $i$...
introduces the concept of consensus. For future use, we assume that the initial conditions of the multi-agent system (1) are \( x_i(0) = \hat{x}_i(0) \in \mathbb{R}^n, \) and \( u_i(0) = \hat{u}_i(0) \in \mathbb{R}^m, \) where \( \tau = \tau_{\text{con}} + \tau_{\text{com}}. \) In this paper, we are interested in the design of the state feedback protocol \( u_i(t) = \hat{u}_i(\tau) \). \( i \in [1, N], \) and observer based output feedback protocol \( \hat{u}_i(t) = \hat{u}_i(\tau, \hat{w}_i) \). \( i \in [1, N], \) that achieve consensus of the multi-agent system (1). To this end, we first introduce the concept of consensus.

**Definition 1.** The high-order linear multi-agent system (1) achieves consensus if \( \lim_{t \to \infty} \| \dot{x}_i(t) - \dot{x}_j(t) \| = 0, \) \( \forall i, j \in [1, N]. \)

**Remark 1.** By definition, if the high-order multi-agent system (1) achieves consensus, then there exists a signal \( s(t) \in \mathbb{R}^n, \) which may be unbounded, such that \( \lim_{t \to \infty} \| \dot{x}_i(t) - x_i(t) \| = 0, \) \( \forall i \in [1, N]. \)

The signal \( s(t) \) is referred to as the reference trajectory. The problems we are to solve can then be formally stated as follows.

**Problem 1 (State Feedback Consensus).** Design a state feedback protocol \( u_i(t) = \hat{u}_i(\tau) \). \( i \in [1, N], \) and \( F \in \mathbb{R}^{n \times n} \) is a constant matrix, such that the high-order linear multi-agent system (1) with time delays in both the inputs and the communication network achieves consensus.

**Problem 2 (Observer Based Output Feedback Consensus).** Design a finite dimensional stable observer based output feedback protocol, expressed in the form of

\[
\begin{align*}
\dot{w}_i(t) &= A_p \hat{w}_i(t) + B_p u_i(t), \\
u_i(t) &= C_p \hat{w}_i(t) + D_p w_i(t),
\end{align*}
\]

where \( A_p \in \mathbb{R}^{n \times n} \) is Hurwitz, such that the linear continuous-time high-order multi-agent system (1) with time delays in both the inputs and the communication network achieves consensus and \( \lim_{t \to \infty} \| \hat{w}_i(t) \| = 0, \) \( \forall i \in [1, N]. \)

We will impose some assumptions on the linear continuous-time high-order multi-agent system (1). First, to ensure that the consensus value reached by the agents will not tend to infinity exponentially, namely, the reference trajectory \( s(t) \) defined in Remark 1 is not exponentially diverging, the matrix \( A \) should not contain eigenvalues in the open right-half plane (Li, Duan, Chen, & Huang, 2010; Seo et al., 2009). Hence we should assume that all the eigenvalues of \( A \) are located on the closed left-half plane. In this case, we can perform a state transformation such that the asymptotically stable modes and the modes on the imaginary axis are de-coupled, and, consequently, only the consensus problem for multi-agent systems associated with these modes on the imaginary axis is required to be solved (see Remark 6 in Su and Huang (2012a) for details). Hence, without loss of generality, we will impose the following assumption on the multi-agent system (1).

**Assumption 1.** The matrix pair \( (A, B) \) is controllable, the matrix pair \( (A, C) \) is observable, and all the eigenvalues of \( A \) are on the imaginary axis.

Notice that multi-agent systems with agents characterized by a chain of integrators or a single oscillator (see, for example, Lin et al., 2007 and Yu et al., 2010b) satisfy Assumption 1 automatically. However, Assumption 1 allows \( A \) to have nonzero eigenvalues with algebraic and geometry multiplicities larger than 1, and is thus weaker than the existing assumptions, for example, Assumption 1 in Su and Huang (2012a). We notice that such an assumption is also made in Wang, Saberi, Stoorvogel, Grip, and Yang (2013).

Our second assumption is concerned with the communication topology among the agents.

**Assumption 2.** The communication topology \( g(N, E, A) \) contains a directed spanning tree.

This assumption is necessary for guaranteeing a solution to the consensus problem even in the absence of delay (see, for example, Li et al., 2010 and Seo et al., 2009). Under Assumption 2, there exists a nonsingular matrix \( T \in \mathbb{C}^{n \times N}, \) whose first column is \( T_1 = [1, 1, \ldots, 1]^T \) in \( \mathbb{R}^n, \) such that (Ren & Cao, 2011)

\[
T^{-1}LT = \begin{bmatrix}
0 & \cdots & 0 \\
\lambda_2 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix} \equiv \begin{bmatrix}
0 & 0 \\
0 & D_n \\
0 & 0
\end{bmatrix}
\]

where \( \lambda_i \) are such that \( Re(\lambda_i) > 0, \) \( i \in [1, N], \) and \( \delta_i, \) \( i \in [2, N - 1], \) equals either 1 or 0. For future use, we define \( \delta_0 = 0. \)

At the end of this section, we introduce the following lemma which plays the central role in the stability analysis in the sequel. The detailed proof is provided in Appendix A.1 for clarity.

**Lemma 1.** Assume that \( (A, B, C) \) satisfies Assumption 1 and \( \lambda \in \mathbb{C} \) is a given scalar such that \( Re(\lambda) > 0. \) Let \( F = -\mu B^T P \) \( \gamma \) where \( P \) \( \gamma \) is the unique positive definite solution to the following parametric algebraic Riccati equation (ARE)

\[
A^TP + PA - PBB^TP = -\gamma P.
\]

Let \( H \) be such that \( A + HC \) is Hurwitz, \( (\theta, \theta) \) be a pair of constant real numbers and \( \mu^* = \frac{1}{Re(\lambda)}. \)

(1) If \( \tau \) is constant yet can be arbitrarily large, then for any \( \mu \geq \mu^*, \) there exists a scalar \( \gamma^* > \gamma(\mu, \tau, |\lambda|, H) > 0 \) such that the linear time delay system

\[
\left\{ \begin{array}{l}
\dot{\psi}(t) = A\psi(t) + \lambda B \mathcal{F} \psi(t - \tau) + \theta AB \mathcal{F} \psi(t - \tau), \\
\dot{e}(t) = (A + HC) e(t) - \theta \lambda B \mathcal{F} \psi(t - \tau) - e(t - \tau)
\end{array} \right.
\]

is asymptotically stable for all \( \gamma \in (0, \gamma^*], \) where \( \mathcal{F} = Fe^{\beta t}. \)

(2) If, in addition, all the eigenvalues of \( A \) are zero and the delay \( \tau \) is possibly unknown, time varying yet bounded uniformly by \( \tau_e, \) then for any \( \mu \geq \mu^*, \) there exists a scalar \( \gamma^* = \gamma(\mu, \tau_e, |\lambda|, H) > 0 \) such that the linear time-delay system

\[
\left\{ \begin{array}{l}
\dot{\psi}(t) = A\psi(t) + \lambda B \mathcal{F} \psi(t - \tau) + \theta AB \mathcal{F} \psi(t - \tau), \\
\dot{e}(t) = (A + HC) e(t) - \theta \lambda B \mathcal{F} \psi(t - \tau) - e(t - \tau)
\end{array} \right.
\]

is asymptotically stable for all \( \gamma \in (0, \gamma^*]. \)

**3. Consensus by the state feedback protocol**

In this section we present solutions to Problem 1 by developing TPF based state feedback protocols. Both delay-dependent and delay-independent protocols will be established. To deduce our main results, we first introduce the following lemma.
Lemma 2. The multi-agent system in (1) achieves consensus by the following state feedback protocol

\[ u_i(t) = F \left( e^{At} z_i(t) + \sum_{j=1}^{N} I_j \int_{-\tau}^{0} e^{-A} B_j (t + s) \, ds \right), \tag{9} \]

where \( F \in \mathbb{R}^{m \times n} \) is such that \( A + \lambda_j B_i, \, i \in [1, N] \), are all Hurwitz.

Proof. We consider an artificial protocol as

\[ u_i(t) = \dot{z}_i(t) + \epsilon_i(t), \quad i \in [1, N]. \tag{10} \]

Then the closed-loop dynamics of the multi-agent system (1) and the protocol (10) can be represented as, for all \( t \geq \tau \),

\[ \dot{x}(t) = (T \otimes I_n) (L_n \otimes A) (T^{-1} \otimes I_n) x(t) + (T \otimes I_n) (L_n \otimes B) (T^{-1} \otimes I_n) x(t), \tag{11} \]

where we have used (5). The above equation can be rewritten as

\[ \dot{x}(t) = (L_n \otimes A) \chi(t) + (L_n \otimes B) \chi(t). \tag{12} \]

in which

\[ \chi = [X_1^T \quad X_2^T \quad \cdots \quad X_N^T] \triangleq (T^{-1} \otimes I_n) x. \tag{13} \]

Since the first column of \( T \) is \( 1_N \), if \( \lim_{t \to -\infty} \| \chi(t) \| = 0, \forall i \in [1, N] \), then it follows from (11) that \( \chi(t) \to \chi_i(t) \triangleq s(t), \forall i \in [1, N] \), as \( t \) approaches infinity and the consensus is achieved. On the other hand, in view of (12) and (5), the dynamics of \( \chi_i(t), \, i \in [1, N] \), obey the following equations

\[ \dot{\chi}_i(t) = A \chi_i(t) + \lambda_i B_i \chi_i(t) + \delta_i B_i \chi_{i+1}(t), \tag{14} \]

where \( i \in [1, N] \) and \( \chi_{N+1}(t) = 0 \). Hence the problem of consensus is solved by the protocol (10) as \( A + \lambda_i B_i, \, i \in [1, N] \), are all Hurwitz.

We next show that the artificial protocol (10) is equivalent to (9). In view of (2), we can write

\[ u_i(t) = F \left( \sum_{j=1}^{N} I_j \dot{x}_j(t + \tau_{\text{con}}) \right), \quad i \in [1, N]. \tag{15} \]

On the other hand, by using the system equation in (1), we can predict \( \dot{x}_j(t + \tau_{\text{con}}) \) from \( \dot{x}_j(t - \tau_{\text{con}}) \) as

\[ \dot{x}_j(t + \tau_{\text{con}}) = e^{At} \dot{x}_j(t - \tau_{\text{con}}) + \int_{-\tau}^{0} e^{-A} B_j (t + s) \, ds, \tag{16} \]

by which and (2) the artificial protocol (15) can be further expressed as (9). The proof is completed. \( \blacksquare \)

Remark 3. We would like to point out that the predictor feedback based protocol in (9) (as well as the observer based protocol given in Lemma 3 later) does not require the matrix \( A \) to satisfy Assumption 1, namely, \( A \) can be any square matrix. Moreover, if \( L \) is exactly known and the relative input \( u_i - u_j, \, i, j \in [1, N] \), is accessible (we would like to point out that the accessibility of the relative input is indeed assumed in some existing literature, for example, You & Xie, 2011a), this protocol is implementable, though computationally expensive.

We now develop a truncated version of the predictor feedback based protocol in (9) under Assumption 1. Under this assumption, there exists a parameterized feedback gain \( F = F(\gamma) : \mathbb{R}^{m} \to \mathbb{R}^{m \times n} \), which is of order 1 with respect to \( \gamma \), namely (Zhou, Lin, & Duan, 2011),

\[ \lim_{\gamma \to 0^+} \frac{1}{\gamma} \| F(\gamma) \| < \infty. \tag{18} \]

Then the predictor feedback based protocol \( u_i(t), \, i \in [1, N] \), itself is “of order 1” with respect to \( \gamma \), namely,

\[ \lim_{\gamma \to 0^+} \frac{1}{\gamma} \| u_i(t) \| = 0, \quad \lim_{\gamma \to 0^+} \frac{1}{\gamma} \| u_i(t) \| < \infty, \quad \forall \tau > 0, \, i \in [1, N]. \tag{19} \]

As a result, by virtue of (18) and denoting \( v_i(t) = F \sum_{j=1}^{N} I_j \int_{-\tau}^{0} e^{-A} B_j (t + s) \, ds \), we have

\[ \lim_{\gamma \to 0^+} \frac{1}{\gamma^2} \| v_i(t) \| \leq \lim_{\gamma \to 0^+} \frac{1}{\gamma} \| F(\gamma) \| \sum_{j=1}^{N} \int_{-\tau}^{0} \| e^{-A} B_j \| \, ds \times \lim_{\gamma \to 0^+} \frac{1}{\gamma} \| u_j(t + s) \| < \infty, \tag{20} \]

namely, the second term \( \sum_{j=1}^{N} \int_{-\tau}^{0} e^{-A} B_j (t + s) \) in (9) is at least "of order 2" with respect to \( \gamma \). This indicates that, no matter how large the value of \( \tau \) is, the distributed term \( \sum_{j=1}^{N} \int_{-\tau}^{0} e^{-A} B_j (t + s) \) in (9) is dominated by the term \( F e^{At} z_i(t) \) in (9) and thus might be safely neglected in \( u_i(t) \) when \( \gamma \) is sufficiently small (Zhou et al., 2012). As a result, the predictor feedback based protocol in (9) can be truncated as

\[ u_i(t) = F e^{At} z_i(t), \quad i \in [1, N], \tag{21} \]

which we refer to as the TPF based protocol.

Remark 4. Compared with the predictor feedback protocol in (9), the TPF based protocol \( u_i(t) \) in (21) possesses the following advantages:

(1) The TPF based protocol \( u_i(t) \) in (21) is independent of the signals \( u_j, \, j \in [1, N], j \neq i \), which is not the case for the predictor feedback based protocol in (9).

(2) The TPF based protocol \( u_i(t) \) in (21) is static, while the predictor feedback based protocol \( u_i(t) \) in (9) is dynamic as it involves the history information of \( u_i(t), \, j \in [1, N] \) in the interval \([t - \tau, t]\).  

(3) The TPF based protocol \( u_i(t) \) in (21) does not require the exact information of \( L \), while the predictor feedback based protocol \( u_i(t) \) in (9) does.

The following theorem shows that the TPF based protocol \( u_i(t) \) in (21) indeed solves Problem 1.
Theorem 1. Let Assumptions 1 and 2 be satisfied and $P(\gamma)$ be the unique positive definite solution to the parametric ARE (6). Then, for any bounded delays $\tau_{con}$ and $\tau_{com}$ that are arbitrarily large and exactly known, and for any $\mu \geq \mu^*(L)$, there exists a number $\gamma^* = \gamma^*(\mu, \tau, [\lambda_i]_{i=0}^{N}) > 0$ such that Problem 1 is solved by the TPF based protocol $u_i(t)$ in (21) where

$$F = -\mu B^T P(\gamma), \quad \forall \mu \in [\mu^*(L), \infty), \forall \gamma^* \in (0, \gamma^*].$$

Proof. Similarly to the proof of Lemma 2, with the TPF based protocol $u_i(t)$ in (21), we can show that the consensus problem is solved if the following series of time-delay systems are asymptotically stable

$$\dot{x}_i(t) = A_i x_i(t) + \lambda_i B e^{\tau_i} x_i(t - \tau),$$

where $x_{i+1}(t) \equiv 0$. In addition, all the states $x_i(t)$, $i \in [1, N]$, converge to $x_i(t)$, which satisfies $\dot{x}_i(t) = A_i x_i(t)$. Clearly, stability of (23) is equivalent to the stability of (24). However, the stability of the series of time-delay systems in (24) follows from Item 1 of Lemma 1 with $(\theta, \vartheta) = (0, 0)$.

Remark 5. It follows from the proof of Lemma 1 that the norm of the gain $F = -\mu B^T P(\gamma)$ decreases as the delay gets larger and $\gamma^*(\mu, \tau, [\lambda_i]_{i=0}^{N})$ decreases. Hence the norm of the gain $Fe^{\tau_i}$ in the protocol (21) decreases as the delay gets larger if A is Lyapunov stable (namely, all the eigenvalues on the imaginary axis are simple) since $\|e^{\tau_i}\| = 1$. If A contains at least one pair of imaginary eigenvalues whose algebraic multiplicity is greater than 1, numerical simulation also shows that the norm of the gain $Fe^{\tau_i}$ in the protocol (21) decreases as the delay gets larger. Hence, a large delay will not lead to large control effort. On the other hand, as the term $e^{\tau_i}$ is very close to the identity matrix for very small delays, the gain $Fe^{\tau_i}$ is very close to $F$, which is frequently used in the literature. In this case, our method is comparable to the existing approaches.

In Theorem 1, the delays $\tau_{con}$ and $\tau_{com}$ should be known exactly, which may not be the case all the time. In the following we present an alternative delay-independent protocol which does not require the exact information of $\tau_{con}$ and $\tau_{com}$.

Theorem 2. Let Assumptions 1 and 2 be satisfied and, in addition, all the eigenvalues of $A$ be zero. Let $P(\gamma)$ be the unique positive definite solution to the parametric ARE (6). Assume that the delays $\tau_{con}$ and $\tau_{com}$ are time-varying, unknown and arbitrarily large, yet bounded, namely, there exists a scalar $\tau^* > 0$ such that

$$\sup_{t \in \mathbb{R}} |\tau_{con}(t)| + \sup_{t \in \mathbb{R}} |\tau_{com}(t)| \leq \tau^*.$$  

Then, for any $\mu \geq \mu^*(L)$, there exists a number $\gamma^* = \gamma^*(\mu, \tau^*, [\lambda_i]_{i=0}^{N})$ such that Problem 1 is solved by

$$u_i(t) = F z_i(t), \quad i \in [1, N],$$

where

$$F = -\mu B^T P(\gamma), \quad \forall \mu \in [\mu^*(L), \infty), \forall \gamma^* \in (0, \gamma^*].$$

Proof. The proof is similar to that of Theorem 1 except for the use of Item 2 of Lemma 1 with $(\theta, \vartheta) = (0, 0)$ and $\tau = \tau_i(t) = \tau_{con}(t) + \tau_{com}(t) - \tau_{con}(t)$. The details are omitted for brevity.

A couple of remarks regarding Theorems 1 and 2 are given in order.

Remark 6. In the TPF based protocols proposed in Theorems 1 and 2, the graph $g(N, E, A)$ is not necessarily known exactly. In fact, only the information of the bound on the eigenvalues of $L$ is required. For example, if it is known that the $i$th eigenvalue of $L$ lies in a compact bounded set $\Omega_i$, $i \in [2, N]$, then the scalar $\mu^*(L)$ in Theorems 1 and 2 can be replaced by

$$\mu^*(\Omega) = \sup_{\lambda_i \in \Omega_i, i \in [2, N]} \left\{ \frac{1}{\Re(\lambda_i)} \right\}.$$  

Remark 7. In the existing literature (see, for example, Lin et al., 2007, Olfati-Saber & Murray, 2004, Qin et al., 2011, Tian & Liu, 2009 and Yu et al., 2010b), the problem of finding the maximal allowable delay with a prescribed protocol has been investigated. In this literature, even for multi-agent systems with agents characterized by a single or double integrator, the allowable delay cannot be arbitrarily large (see, for example, Olfati-Saber & Murray, 2004 and Yu et al., 2010b). Here we have shown in Theorems 1 and 2 that arbitrarily large bounded delays are allowable as long as the gain in the protocol is adjusted low enough.

Remark 8. It is shown in Munz, Papachristodoulou, and Allgower (2011b) that consensus in single-integrator multi-agent systems can be guaranteed for arbitrary large delays. The same system and a similar result are also reported in Liu, Lu, and Chen (2010), where some mild conditions are established to guarantee the consensus. Extension of the results in Munz et al. (2011b) to a more general case that the agents are characterized by non-identical single-input–single-output systems having only unstable poles at zero is given in Munz et al. (2012). Theorem 2 can be considered as a generalization of these results since the systems considered in Theorem 2 include the single-integrator multi-agent systems and/or single-input–single-output systems as very special cases. On the other hand, under the same assumptions made in this paper, delay-dependent protocols are established in Wang et al. (2013) for the consensus of high-order multi-agent systems with constant communication delays. However, in that paper the delays are not allowed to be arbitrarily large in the case that $A$ has nonzero eigenvalues and only constant delays are allowed in the case that $A$ has only zero eigenvalues.

Remark 9. As pointed out in Munz et al. (2011b), the convergence rate tends to decrease as the delays increase, in other words, consensus is reached slower for larger delays. This is also the case for the other theorems in this paper and should be considered as the inherent trade-off between the size of the allowable delay and the convergence rate of the consensus.

4. Consensus by an observer based output feedback protocol

In this section, we give solutions to Problem 2 by proposing two kinds of observer based output feedback protocols with the help of the idea of TPF. To this end, we first introduce a lemma which is a generalization of Lemma 2 to the observer based output feedback case.

Lemma 3. The multi-agent system in (1) achieves consensus by the following observer based output feedback protocol

$$\dot{\omega}_i(t) = (A + HC)\omega_i(t) + B \sum_{j=1}^{N} l_{ij} u_j(t - \tau) - H w_i(t),$$

$$u_i(t) = Fe^{\tau_i} \omega_i(t) + F \sum_{j=1}^{N} \int_{t-\tau}^{t} e^{\tau_i - s} B l_{ij} u_j(t + s) ds.$$  

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where  \( i \in I [1,N] \), \( F \in \mathbb{R}^{m \times n} \) is such that \( A + \lambda_i BF \), \( i \in I [2,N] \), are all Hurwitz and \( H \in \mathbb{R}^{q \times q} \) is such that \( A + HC \) is Hurwitz. In addition, the observer states \( o_i(t) \) satisfy \( \lim_{t \to \infty} \| o_i(t) \| = 0 \), \( i \in I [1,N] \).

**Proof.** Consider the following new state variable

\[
\tilde{r}_i(t) = \sum_{j=1}^{N} l_j x_j (t - \tau_{com}), \quad \forall t \geq \tau, \quad i \in I [1,N],
\]

on which the dynamics of the multi-agent system (1) can be rewritten as, for \( t \geq \tau \),

\[
\dot{\tilde{r}}_i(t) = A \tilde{r}_i(t) + B \sum_{j=1}^{N} l_j u_j (t - \tau), \quad i \in I [1,N].
\]

In addition, the information for the feedback given in (3) reduces to

\[
u_i(t) = Cr_i(t), \quad i \in I [1,N], \quad \forall t \geq \tau.
\]

The relation in (30) can be written in compact form

\[
r(t) = (L \otimes I) x (t - \tau_{com})
= (T \otimes I_a) (L_j \otimes I_a) (L^{-1} \otimes I_a) x(t - \tau_{com}),
\]

which, by denoting

\[
\rho = (T^{-1} \otimes I_a) r, \quad \chi = (T^{-1} \otimes I_a) x,
\]

can be further rewritten as

\[
\rho(t) = J \chi (t - \tau_{com}), \quad \forall t \geq \tau.
\]

It follows from (35) and (5) that \( \rho_i(t) \equiv 0, \forall t \geq \tau, \) and

\[
\rho_i(t) = \lambda_i \chi_i(t - \tau_{com}) + \delta_i \chi_{i+1}(t - \tau_{com}),
\]

where \( t \geq \tau, \) \( i \in I [2,N], \) and \( \chi_N(t+1) \equiv 0. \) Now, by denoting \( \varepsilon_i(t) = r_i(t) - o_i(t), \) \( i \in I [1,N], \) we get from (29) and (31) that

\[
\varepsilon_i(t) = (A + HC) o_i(t), \quad i \in I [1,N], \quad \forall t \geq \tau.
\]

which implies that \( \lim_{t \to \infty} \| \varepsilon_i(t) \| = 0, \) \( i \in I [1,N]. \)

Now consider the following artificial protocol

\[
u_i(t) = Fr_i(t + \tau) - Fe_i(t), \quad \forall t \geq 0,
\]

by which the equation in (31) can be written as

\[
\dot{\varepsilon}_i(t) = Fr_i(t + \tau) - e_i(t), \quad t \geq \tau,
\]

where \( i \in I [2,N] \), \( \forall t \geq \tau. \) The above equation can be expressed in compact form

\[
\dot{\varepsilon}(t) = (L \otimes A) \varepsilon(t) + (L \otimes BF) \varepsilon(t - \tau) - Fe_i(t - \tau), \quad \forall t \geq \tau,
\]

which, via the transformation in (34), is equivalent to

\[
\left\{
\begin{array}{l}
\dot{\rho}_i(t) = A \rho_i(t), \\
\dot{\rho}_i(t) = (A + \lambda_i BF) \rho_i(t) + \delta_i \rho_{i+1}(t), \\
- \lambda_i BF e_i(t - \tau) - \delta_i BF e_{i+1}(t - \tau),
\end{array}
\right.
\]

where \( i \in I [1,N], \) \( e_i(t) = (T^{-1} \otimes I_a) e_i(t), \) \( \rho_{i+1}(t) \equiv 0 \) and \( e_{N+1}(t) \equiv 0. \) As \( \lim_{t \to \infty} \| e_i(t) \| = 0, \) \( \rho_i(t) \equiv 0, \) and \( A + \lambda_i BF \), \( i \in I [2,N], \) are all Hurwitz, it follows from (41) that \( \lim_{t \to \infty} \| \rho_i(t) \| = 0, \) \( i \in I [1,N]. \) Thus, we have from (36) that \( \lim_{t \to \infty} \| \chi_i(t) \| = 0, \) \( i \in I [2,N]. \) Consequently, by the property of \( T, \) we get from (34) that \( \chi_i(t) \to \chi_i(t), \) \( \forall i \in I [1,N] \) as \( t \) approaches infinity, namely, the consensus is achieved. Moreover, it follows from \( \lim_{t \to \infty} \| \rho_i(t) \| = 0, \) \( i \in I [1,N], \) that \( \lim_{t \to \infty} \| r_i(t) \| = 0, \) \( i \in I [1,N], \) which together with \( \lim_{t \to \infty} \| e_i(t) \| = 0, \) \( i \in I [1,N], \) indicate that \( \lim_{t \to \infty} \| o_i(t) \| = \lim_{t \to \infty} \| r_i(t) - e_i(t) \| = 0, \) \( i \in I [1,N]. \)

The proof is finished by noting that the artificial protocol \( u_i(t) \) defined in (38) is equivalent to the protocol \( u_i(t) \) defined in (29) with the help of Eq. (31).

**Remark 10.** The introduction of the new state variable \( r_i(t) \) defined in (30) is to put the term \( B \sum_{j=1}^{N} l_j u_j (t - \tau) \) into the dynamic equation (31) so that an observer in the form of (29) can be constructed to obtain the clean error system (37). On the other hand, we can see that the protocol \( u_i(t) \) defined in (29) is also based on the predictor feedback, which is similar to the state feedback case. For this reason, we call (29) an observer based output predictor feedback protocol.

We next establish a truncated version of the observer based output predictor feedback protocol in (29). Similarly to the state feedback case, if \( F (\gamma) \) is such that (18) is satisfied, then the term

\[
F \sum_{j=1}^{N} \int_{-\tau}^{0} e^{-\Delta \gamma} B_l u_l (t + s) \, ds,
\]

in (29) is at least “of order 2” with respect to \( \gamma \) and can thus be neglected provided \( \gamma \) is sufficiently small. On the other hand, as the term \( B \sum_{j=1}^{N} l_j u_j (t - \tau) \) is at least “of order 1” with respect to \( \gamma \) and the other three terms in the \( o_i \) dynamics in (29) are independent of \( \gamma \), it can also be neglected if \( \gamma \) is small enough. Therefore, the observer based predictor feedback protocol in (29) can be truncated as

\[
\left\{\begin{array}{l}
o_i(t) = (A + HC) o_i(t) - H u_i(t), \\
u_i(t) = F e_i(t), \quad i \in I [1,N].
\end{array}\right.
\]

which will be referred to as the observer based TPF protocol. Compared with the observer based predictor feedback protocol (29), the observer based TPF protocol (43) also possesses the three advantages listed in Remark 4.

Similarly to the state feedback case, the following theorem shows that the observer based TPF protocol (43) indeed solves Problem 2 for arbitrarily large yet bounded delay \( \tau \).

**Theorem 3.** Let Assumptions 1 and 2 be satisfied. \( H \) be such that \( A + HC \) is Hurwitz, and \( P (\gamma) > 0 \) be the unique positive definite solution to the parametric ARE (6). Then for any delays \( \tau \) and \( \tau_{com} \) that are exactly known and can be arbitrarily large yet bounded and for any \( \mu \geq \mu^* (T), \) there exists a number \( \gamma^* = \gamma^* (\mu, \tau, \| L_j \|_{i=2}^{N}, H) \) such that Problem 2 is solved by (43), where \( o_0(t) \in \mathbb{R}^q \) and

\[
F = -\mu^B P (\gamma^*), \quad \forall \mu \in [\mu^* (T), \infty), \forall \gamma \in (0, \gamma^*].
\]

**Proof.** By denoting \( \chi(t) = [x_1(t), x_N^T (t)]^T, \) we can write the closed-loop multi-agent system consisting of (1) and (43) as

\[
\dot{x}(t) = (L \otimes A) x(t) + (L_N \otimes A_2) x(t - \tau_{com}) - (L \otimes A_3) x(t - \tau_{com}),
\]

where

\[
A_1 \triangleq \begin{bmatrix} B & 0 \\ 0 & A + HC \end{bmatrix}, \quad A_2 \triangleq \begin{bmatrix} 0 & BF e_i(t) \\ 0 & 0 \end{bmatrix}, \quad A_3 \triangleq \begin{bmatrix} 0 & 0 \\ 0 & HC \end{bmatrix}.
\]

By denoting \( \chi(t) = (T^{-1} \otimes I_{2m}) x(t), \) we can express the dynamics in (45) as

\[
\dot{\chi}(t) = (L_N \otimes A_1) \chi(t) + (L_N \otimes A_2) \chi(t - \tau_{com}) - (L \otimes A_3) \chi(t - \tau_{com}),
\]

(47)
which, in view of the structures of $A_1$, $A_2$ and $A_3$, is equivalent to
\begin{align}
\dot{x}_1(t) &= A_1x_1(t) + A_2x_1(t - \tau), \\
\dot{x}_2(t) &= A_3x_1(t) - \lambda x_3(t - \tau) - \lambda_3A_3x_1(t - \tau),
\end{align}
(48)
where $A_3x_1(t) \equiv 0$. If $\lim_{t \to \infty} \|x_1(t)\| \to 0$, $\forall i \in I[2, N]$, then, by the property of $T$, we see that
\[\lim_{t \to \infty} \|x_i(t) - x_1(t)\| = 0, \quad \forall i \in I[1, N].\]
(49)
Notice that the first equation in (48) can be written as
\begin{align}
\dot{x}_{11}(t) &= A_{11}x_{11}(t) + BFe^{\delta t}x_{12}(t - \tau), \\
\dot{x}_{12}(t) &= (A + HC)x_{12}(t),
\end{align}
(50)
where $x_{11}(t) = \left[ x_{11}^T(t), x_{12}^T(t) \right]^T$. Since the second subsystem in (50) is exponentially stable, it follows from (49) that, for all $i \in I[1, N], \lim_{t \to \infty} \|x_i(t) - x_{11}(t)\| = 0, \quad \lim_{t \to \infty} \|x_i(t) - x_{12}(t)\| = 0, \quad \forall i \in I[1, N].$
which implies that the consensus is achieved and the observer is asymptotically stable. Therefore, in the remaining of the proof, we need only to prove the stability of the second systems in (48), or equivalently, the stability of
\[\dot{x}_i(t) = A_1x_i(t) + \lambda A_3x_i(t - \tau), \quad \forall i \in I[2, N].\]
(52)
where $i \in I[2, N].$
By denoting $\phi_i(t) = \left[ \phi_{i1}^T(t), \phi_{i2}^T(t) \right]^T, \forall i \in I[2, N],$ we can write the systems in (52) as
\begin{align}
\dot{\phi}_{i1}(t) &= A_{i1}\phi_{i1}(t) + BF e^{\delta t}\phi_{i2}(t - \tau), \\
\dot{\phi}_{i2}(t) &= (A + HC)\phi_{i2}(t) - \lambda HC\phi_{i2}(t - \tau),
\end{align}
(53)
where $i \in I[2, N].$ Let $\phi_i(t) = \lambda_i^{-\gamma}\phi_{i1}(t)$ and
\[\psi_i(t) = \phi_{i1}(t) - \tau, \quad i \in I[2, N].\]
(54)
Then system (53) simplifies to, for all $i \in I[2, N],$
\begin{align}
\dot{\phi}_i(t) &= A_{i1}\phi_i(t) + \lambda_i BF e^{\delta t}\phi_i(t - \tau), \\
\dot{\psi}_i(t) &= (A + HC)\phi_i(t) - \tau, \\
&\quad \times \phi_i(t) - \tau, \quad i \in I[2, N].
\end{align}
(55)
Stability of (56) then follows from Item 1 of Lemma 1 where
\[(\theta, \vartheta) = (-1, -1).\] The proof is finished. $
\square$
If the delays in the inputs and communication network are unknown and/or time-varying, we can obtain the following result, which can be regarded as the output feedback version of Theorem 2.

**Theorem 4.** Let Assumptions 1 and 2 be satisfied, $H$ be such that $A + HC$ is Hurwitz, and, in addition, all the eigenvalues of $A$ are zero. Let $P(\gamma) > 0$ be the unique positive definite solution to the parametric ARE (6). Assume that $\tau_{com}$ and $\tau_{con}$ are unknown and arbitrarily large, yet bounded, $\tau_{con}$ is possibly time-varying and $\tau_{com}$ is constant. Let
\[\sup_{r \geq 0} \|\tau_{com}(r)\| + \tau_{con} \leq \tau^* < \infty.\]
(57)
Then for any $\mu \geq \mu^*(L)$, there exists a number $\gamma^* = \gamma^*(\mu, \tau^*, [\lambda_i]_{i=2}^{N}, H)$ such that Problem 2 is solved by the following observer based output feedback protocol
\[
\begin{align}
\dot{\omega}_i(t) &= (A + HC)\omega_i(t) - Hw_i(t), \\
u_i(t) &= F\omega_i(t), \quad i \in I[1, N],
\end{align}
(58)
where $\omega_i(0) \in \mathbb{R}^n$ and
\[F = -\mu B^T(\gamma), \quad \forall \mu \in [\mu^*(L), \infty), \forall \gamma \in (0, \gamma^*].\]
(59)
**Proof.** The proof is quite similar to the proof of Theorem 3 and is omitted for brevity. The only difference is that $\tau_{com}$ should be assumed to be constant so that the transformation (54) (or the transformation (30)) is valid. $
\square$

**Remark 11.** Differently from the observer based output predictor feedback protocol (29), which makes the error dynamics and the dynamics of the original systems decoupled, as indicated by (37) and (39), the error dynamics and the dynamics of the original systems under the observer based TPF protocol (43) are coupled, as indicated by (56). Hence an intricate Lyapunov analysis for the stability of the closed-loop multi-agent system has to be developed in the proof of Lemma 1.

**Remark 12.** We point out that we can obtain an explicit formulation of the functions $s(t)$ associated with Theorems 1-4. We however have not presented it here for the sake of brevity. In addition, as all the eigenvalues of $A$ are on the imaginary axis, we can show that there exists a number $k > 0$ such that
\[\|s(t)\| \leq k \left( 1 + t^{N^* - 1} \right) \sup_{|t| \leq r} \left\{ \|(\theta)\| + \|\omega(0)\| \right\},\]
(60)
where $N^*$ is the maximal algebraic multiplicity of the eigenvalues of $A$.

In the case that $\tau = \tau_{com} = \tau_{con} = 0$, namely, the dynamics of the multi-agent system (1) becomes
\[
\begin{align}
\dot{x}_i(t) &= A_{i1}x_i(t) + Bu_i(t), \\
y_i(t) &= Cx_i(t), \quad i \in I[1, N],
\end{align}
(61)
and the partial information for feedback given in (3) becomes
\[w_i(t) = \sum_{j=1}^N l_jy_j(t), \quad i \in I[1, N],\]
(62)
we obtain the following corollary of Theorem 3.

**Corollary 1.** Let Assumptions 1 and 2 be satisfied, $H \in \mathbb{R}^{p \times p}$ be such that $A + HC$ is Hurwitz, $\mu$ be any positive number such that (17) is satisfied, and $P(\gamma) > 0$ be the unique positive definite solution to the parametric ARE (6). Then for any $\mu \geq \mu^*(L)$, there exists a number $\gamma^* = \gamma^*(\mu, [\lambda_i]_{i=2}^{N}, H)$ such that the consensus of the multi-agent system in (61)-(62) is achieved by the following observer based output feedback protocol
\[
\begin{align}
\dot{\omega}_i(t) &= (A + HC)\omega_i(t) - Hw_i(t), \\
u_i(t) &= F\omega_i(t), \quad i \in I[1, N],
\end{align}
(63)
where $\omega_i(0) \in \mathbb{R}^n$ and
\[F = -\mu B^T(\gamma), \quad \forall \mu \in [\mu^*(L), \infty), \forall \gamma \in (0, \gamma^*].\]
(64)
**Remark 13.** Consensus of the multi-agent system in (61)-(62) by dynamic output feedback was solved in Seo et al. (2009) where a dynamic output feedback protocol in the form of
\[
\begin{align}
\dot{\omega}_i(t) &= (A + HC + BF)\omega_i(t) - Hw_i(t), \\
u_i(t) &= F\omega_i(t), \quad i \in I[1, N],
\end{align}
(65)
is proposed, where $F = -B^TP$ with $P (\varepsilon) > 0$ being the solution to $A^TP + PA - \tau_0PB^TP + \varepsilon I_n = 0$, $\varepsilon \in (0, 1]$.

$$
(66)
$$

with $\tau_0 = \min_{\lambda \in \mathbb{R}^N} |\text{Re} \{\lambda_i\}|$. Our dynamic protocol (63) in Corollary 1 possesses a simpler structure than the dynamic protocol (65) as we do not require that the term corresponds to the term $BF\omega_i(t)$ in (65). More importantly, as seen in the Appendix and Example 1, the analytical expression of the gains in (63) can be obtained by solving a linear Lyapunov equation, whereas the ARE in (66) has to be solved numerically for each given value of $\varepsilon$.

The following result is a corollary of Lemma 3.

**Corollary 2.** Let Assumption 2 be satisfied, $(A, B)$ be controllable and $(A, C)$ be observable. Then the consensus of the multi-agent system in (61)-(62) is achieved by the following observer based output feedback protocol

$$
\begin{align*}
\dot{\omega}_i(t) &= (A + HC) \omega_i(t) + \sum_{j=1}^{N} l_{ij}BF\omega_j(t) - Hw_i(t), \\
\dot{u}_i(t) &= F\omega_i(t),
\end{align*}
$$

(67)

where $F \in \mathbb{R}^{m \times n}$ is such that $A + \omega_i BF, i \in I[2, N]$, are all Hurwitz and $H \in \mathbb{R}^{m \times p}$ is such that $A + HC$ is Hurwitz. In addition, the observer states $\omega_i(t)$ satisfy $\lim_{t \to \infty} \|\omega_i(t)\| = 0$, $i \in I[1, N]$.

**Remark 14.** In Li et al. (2010) the following observer based output feedback protocol

$$
\begin{align*}
\dot{\omega}_i(t) &= (A + BF) \omega_i(t) + \sum_{j=1}^{N} l_{ij}HC\omega_j(t) - Hw_i(t), \\
\dot{u}_i(t) &= F\omega_i(t),
\end{align*}
$$

(68)

is proposed to solve the consensus problem for the multi-agent system (61)-(62), where $F \in \mathbb{R}^{m \times n}$ is such that $A + BF$ is Hurwitz and $H \in \mathbb{R}^{m \times p}$ is such that $A + \omega_i HC, i \in I[2, N]$, are all Hurwitz. It is very interesting to notice that our observer based output feedback protocol (67) in Corollary 2 can be regarded as a dual form of (68).

5. Two numerical examples

**Example 1.** We consider four identical agents whose dynamics is given by

$$
\dot{x}_i(t) = \begin{bmatrix}
0 & \omega_0 & 1 & 0 \\
-\omega_0 & 0 & 0 & 1 \\
0 & 0 & 0 & \omega_0 \\
0 & 0 & -\omega_0 & 0
\end{bmatrix} x_i(t) \\
+ \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} u_i(t - \tau_{\text{con}}),
$$

(69)

$$
y_i(t) = [-1 & 0 & 0 & 0] x_i(t), \quad i \in I[1, 4],
$$

(70)

where $\omega_0 > 0$ is a constant. It follows that Assumption 1 is fulfilled since all the eigenvalues of $A$ are at $\pm \omega_0$ and whose algebraic multiplicity is 2. $(A, B)$ is controllable and $(A, C)$ is observable. Assume that $\tau_{\text{con}} = \frac{\pi}{2}$ and $\tau_{\text{con}} = \frac{3\pi}{2}$, and consequently, $\tau = 2\pi$. Let the communication network be given in Fig. 1, which is characterized by the Laplacian

$$
L = \begin{bmatrix}
3 & 0 & -2 & -1 \\
-2 & 0 & 0 & 0 \\
-1 & 1 & 2 & 0 \\
0 & 0 & -3 & 3
\end{bmatrix}.
$$

(71)

The network in Fig. 1 clearly contains a directed spanning tree. In fact, the eigenvalue set of $L$ can be computed as $\lambda(L) = \{0, 3.5698, 3.2151 \pm 1.3071i\}$. We choose $\mu = \max_{i \in I[1, 4]} \left| \frac{\omega_i}{\omega_0} \right|$.

![Fig. 1. Weighted communication network topology.](image)

We construct the observer based output TPF protocol to solve the consensus problem. By solving the parametric ARE (66), the feedback gain defined in (44) can be computed as $Fe^{\Delta r} = -\mu [f_1]$, $i \in I[1, 4]$, where

$$
\begin{align*}
\Phi &= \begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\
\Phi_{15} & \Phi_{16} & \Phi_{17} & \Phi_{18} \\
\Phi_{19} & \Phi_{20} & \Phi_{21} & \Phi_{22} \\
\Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} \\
\Phi_{27} & \Phi_{28} & \Phi_{29} & \Phi_{30}
\end{bmatrix},
\end{align*}
$$

(72)

$$
\begin{align*}
\Phi_{11} &= \frac{2\gamma^3 \cos (\tau \omega_0)}{\omega_0} - \frac{2\gamma^2 - \gamma^4}{2\omega_0} \sin (\tau \omega_0), \\
\Phi_{12} &= \frac{2\gamma^3 \sin (\tau \omega_0)}{\omega_0} + \frac{2\gamma^2 - \gamma^4}{2\omega_0} \cos (\tau \omega_0), \\
\Phi_{13} &= \frac{2\gamma^3 \cos (\tau \omega_0)}{\omega_0} - \frac{2\gamma^2 - \gamma^4}{2\omega_0} \tau \sin (\tau \omega_0), \\
&\quad + \frac{\gamma^4}{2\omega_0} + \frac{4\gamma^2}{\omega_0} \sin (\tau \omega_0) - 4\gamma \tau \sin (\tau \omega_0),
\end{align*}
$$

$$
\begin{align*}
\Phi_{14} &= \frac{2\gamma^3 \tau \cos (\tau \omega_0)}{\omega_0} + \frac{2\gamma^2 - \gamma^4}{2\omega_0} \tau \cos (\tau \omega_0)
\end{align*}
$$

$$
\begin{align*}
&\quad + \frac{\gamma^4}{\omega_0} + \frac{4\gamma^2}{\omega_0} \sin (\tau \omega_0) + 4\gamma \cos (\tau \omega_0).
\end{align*}
$$

Let the observer gain $H$ be chosen as

$$
H = \begin{bmatrix}
11 & 37.2391 & -0.6667 & 29.5892
\end{bmatrix}^T,
$$

(73)

which is such that $\lambda (A + HC) = \{-4, -3, -2 \pm i\}$. The observer based output TPF protocol can then be constructed according to (43).

For the simulation purpose, we choose $\omega_0 = 2/\sqrt{3}, \gamma = 0.1$, and the initial conditions for the four agents as $x_1(\theta) = [3, -4, 2, 4]^T$, $x_2(\theta) = [-2, 3, -3, 2]^T$, $x_3(\theta) = [4, -2, 2, 3]^T$, $x_4(\theta) = [2, 3, -3, 2]^T$, and $u_i(0) = 0, i \in I[1, 4]$, $\forall \theta \in [-\tau, 0]$. The initial conditions for the observer are respectively assigned as $\omega_1(0) = [6, -3, 4, 1]^T$, $\omega_2(0) = [4, -3, 5, 1]^T$, $\omega_3(0) = [-4, 4, 7, 2]^T$ and $\omega_4(0) = [5, 2, -4, -2]^T$. The differences between the states of Agent 1 and those of the other agents are recorded in Fig. 2. The observer states are also depicted in Fig. 3. In these figures, $x_{ij}, i, j \in I[1, 4]$, denotes the $j$th element of $x_i$ (the state of Agent $i$), and $\omega_0$ denotes the $i$th element of $\omega_0$. From these figures we clearly see that the consensus is achieved by this observer based output TPF protocol.
Example 2. We consider four agents characterized by

\[
\begin{align*}
\dot{x}_i(t) &= 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} x_i(t) + 
\begin{bmatrix}
0 & 0 \\
1 & 1 \\
0 & 1 \\
1 & 1 \\
\end{bmatrix} u_i(t - \tau_{\text{con}}(t)), \\
y_i(t) &= 
\begin{bmatrix}
-1 & 0 & 0 & 1 \\
0 & -1 & 0 & 1 \\
\end{bmatrix} x_i(t), \quad i \in [1, 4].
\end{align*}
\]

(74)

Notice that the associated system matrix \(A\) is in a Jordan canonical form. It follows that \(A\) has only zero eigenvalues whose geometry multiplicity is 2 and maximal algebraic multiplicity is 3. Moreover, we have that \((A, B)\) is controllable and \((A, C)\) is observable. Let the network between these four agents be again characterized by the topology shown in Fig. 1. We assume that the delays in the communication network and inputs are respectively characterized by

\[
\tau_{\text{con}}(t) = \frac{1}{2} + \cos\left(\frac{\sqrt{t}}{2}\right), \quad \tau_{\text{com}} = \frac{1}{2},
\]

(76)

which is such that \(|\tau_{\text{con}}(t)| + \tau_{\text{com}} \leq \tau^* = 2, \forall t \in \mathbb{R}\). We construct the observer based output TPF protocol to solve the consensus problem.

By solving the parametric ARE (6), the feedback gain \(F\) defined in (58) can be computed as in Box 1.

Here we also choose \(\mu = 0.3310\) as in Example 1 and \(\gamma = 0.075\). By applying the function place in Matlab, we get the
observer gain $H$ such that $\lambda (A + HC) = \{-4, -3, -2 \pm i\}$ as

$$H = \begin{bmatrix} 18.0416 & 12.7979 & 5.0134 & 11.1312 \\ -5.0836 & -0.5969 & 3.2795 & -4.6864 \end{bmatrix}^T.$$ (78)

For the simulation purpose, we choose the initial conditions for these four agents as the same as in Example 1. The differences between the states of Agent 1 and those of the other agents and the observer states are respectively recorded in Figs. 4 and 5. From these figures we clearly see that the consensus of these four agents is also achieved by the proposed observer based output TPF protocol.

6. Conclusions

This paper studies the consensus problem for multi-agent systems with agents characterized by high-order linear systems with time delays existing in both the communication network and inputs of the agents. Under the condition that the open-loop dynamics of the agents is at most polynomial unstable and the communication topology for the agents contains a directed spanning tree, the truncated predictor feedback approach is established to solve the consensus problems. It was proven that if the delays are exactly known, the consensus problems can be solved by both full state feedback and observer based output feedback protocol for arbitrarily large yet bounded delays. On the other hand, the communication delays and input delays can be allowed to be time-varying, arbitrarily large yet bounded, and even unknown, if the open-loop dynamics of the agents only contain zero eigenvalues. Numerical examples were worked out to illustrate the effectiveness of the proposed approaches.

Appendix

A.1. Proof of Lemma 1

Proof of Item 1. By the variation of constants formula, we obtain from the first equation in (7) that, for all $t \geq 2\tau$,

$$\dot{\varphi}(t) = e^{\lambda t} \varphi(t - \tau) + \int_{t - \tau}^{t} e^{\lambda(t-s)}(-\lambda \mu BB^T P e^{\lambda s} \varphi(s - \tau) + \theta \lambda \mu BB^T P e^{\lambda s} e(s - \tau) ds,$$ (79)

substitution of which into the first system in (56) gives, for all $t \geq 2\tau$,

$$\dot{\varphi}(t) = (A - \lambda \mu BB^T P) \varphi(t) - \theta \lambda \mu BB^T P e^{\lambda s} e(t - \tau) - (\lambda \mu)^2 BB^T P (\pi_1(t) - \pi_2(t)),$$ (80)

where $\pi_1(t)$ and $\pi_2(t)$ are defined as

$$\begin{cases} 
\pi_1(t) = \int_{t-\tau}^{t} e^{\lambda(t-s)} BB^T P e^{\lambda s} \varphi(s - \tau) ds, \\
\pi_2(t) = \theta \int_{t-\tau}^{t} e^{\lambda(t-s)} BB^T P e^{\lambda s} e(s - \tau) ds.
\end{cases}$$ (81)

The time-derivative of the Lyapunov function $V_1(\varphi(t)) = \varphi^T(t) P \varphi(t)$ along the trajectories of (80) satisfies

$$\dot{V}_1(\varphi(t)) \leq -\gamma \varphi^H(t) P \varphi(t) + (1 - 2\mu \text{Re} \lambda) + 3\kappa |\lambda|^2 |\mu|^2 \varphi^H(t) BB^T P \varphi(t)$$

$$+ \frac{n\gamma}{K} (\varphi^H(t) P \pi_1(t) + \pi^H_2(t) P \pi_2(t))$$

$$+ \theta^2 e^{\lambda s} e(t - \tau) e^{\lambda s} e(t - \tau),$$ (82)
where we have used Lemma 4 in Appendix A.2. On the other hand, by the Jensen inequality (Gu, 2000) and Lemma 4 in Appendix A.2, we can compute

\[ \pi_1^H P \pi_1 \leq (ny)^2 \tau e^{2(\gamma-1)\tau} \int_{t-\tau}^{t} \psi^H(s) P \psi(s) \, ds. \]  

\[ \pi_1^H P \pi_2 \leq \theta^2 (ny)^2 \tau e^{2(\gamma-1)\tau} \int_{t-\tau}^{t} \psi^H(s) P \psi(s) \, ds. \]  

Hence the inequality in (82) simplifies to

\[
\dot{V}_1(\psi(t)) \leq -\gamma \psi^H(t) P \psi(t) + (1 - 2\mu \text{Re} \left( \lambda \right) + 3\kappa \left| \lambda \right|^4 \mu^4) \times \psi^H(t) P \psi(t)
+ \frac{ny}{\kappa} (ny)^2 \tau e^{2(\gamma-1)\tau}
\times \left( \int_{t-\tau}^{t} \psi^H(s) P \psi(s) \, ds + \theta^2 \int_{t-2\tau}^{t} \psi^H(s) P \psi(s) \, ds \right)
+ \frac{ny \theta^2}{\kappa} e^{(\gamma-1)\tau} e^H(t - \tau) P e(t - \tau). 
\]  

(83)

Let \( V_2(\psi, \epsilon_l) \) and \( V_3(\epsilon_l) \) be respectively defined as

\[ V_2(\psi, \epsilon_l) = \frac{ny}{\kappa} (ny)^2 \tau e^{2(\gamma-1)\tau} \int_{t-\tau}^{t} \left( \psi^H(l) P \psi(l) + \theta^2 e^\psi(l) Pe(l) \right) \, dl, \]  

(86)

\[ V_3(\epsilon_l) = \frac{ny \theta^2}{\kappa} e^{(\gamma-1)\tau} \int_{t-\tau}^{t} e^\psi(s) Pe(s) \, ds, \]  

(87)

from which we can compute

\[ \dot{V}_2(\psi, \epsilon_l) = 2\tau \frac{ny}{\kappa} (ny)^2 \tau e^{2(\gamma-1)\tau} (\psi^H(t) P \psi(t) + \theta^2 e^\psi(t) Pe(t)) - \frac{ny}{\kappa} (ny)^2 \tau e^{2(\gamma-1)\tau}
\times \int_{t-\tau}^{t} (\psi^H(s) P \psi(s) + \theta^2 e^\psi(s) Pe(s)) \, ds, \]  

(88)

\[ \dot{V}_3(\epsilon_l) = \frac{ny \theta^2}{\kappa} e^{(\gamma-1)\tau} e^\psi(t) Pe(t) \]  

(89)

On the other hand, if we choose \( \kappa = \frac{1}{2} \) and \( \mu \geq \frac{1}{2} |\lambda|^4 \mu^4 \), then

\[ 1 - 2\mu \text{Re} \left( \lambda \right) + 3\kappa \left| \lambda \right|^4 \mu^4 \leq 0. \]  

(90)

Then we get from (85) and (88)–(90) that

\[ \dot{V}_1(\psi(t)) + \dot{V}_2(\psi, \epsilon_l) + \dot{V}_3(\epsilon_l) \leq -\left( \gamma - 2\tau \frac{ny}{\kappa} (ny)^2 \tau e^{2(\gamma-1)\tau} \right) \psi^H(t) P \psi(t)
+ \frac{ny \theta^2}{\kappa} (2\tau (ny)^2 \tau e^{2(\gamma-1)\tau} + e^{(\gamma-1)\tau}) e^\psi Pe(t). \]  

(91)

Now choose another Lyapunov function \( V_4(\epsilon(t)) = e^\psi(t) Pe(t) \) where \( Q > 0 \) satisfies

\[ (A + HC)^T Q + Q (A + HC) = -I_n. \]  

(92)

Then by using the second equation in (7) and Lemma 4 in Appendix A.2, we have

\[ \dot{V}_4(\epsilon(t)) \leq -\frac{1}{2} \| \epsilon(t) \|^2 + 4c (\psi^H(t - \tau) P \psi(t - \tau)
+ e^\psi(t - \tau) Pe(t - \tau)), \]  

(93)

where \( c = c (\gamma) = |\lambda|^2 \mu^2 \| B \|^2 Q^2 B \| ny e^{2(\gamma-1)\tau} \theta^2. \) Choose finally the Lyapunov functional

\[ V_4(\psi, \epsilon_l) = 4c \left( \int_{t-\tau}^{t} \psi^H(s) P \psi(s) \, ds + \int_{t-\tau}^{t} e^\psi(s) Pe(s) \, ds \right), \]  

(94)

and it follows from (93) that

\[ \dot{V}_4(\epsilon(t)) + \dot{V}_5(\psi, \epsilon_l) \leq -\frac{1}{2} \| \epsilon(t) \|^2 + 4c (\psi^H(t) P \psi(t)
+ e^\psi(t) Pe(t)). \]  

(95)
Now choose the total Lyapunov function as
\[
V(\phi_t, e_t) = V_1(\phi(t)) + V_2(\phi_t, e_t) + \ldots
\]
whose time-derivative, in view of (91) and (95), satisfies
\[
\dot{V}(\phi_t, e_t) \leq -\gamma f(\gamma) |\phi|^2 + \frac{1}{2} \gamma g(\gamma) |e(t)|^2,
\]
where \(f(\gamma)\) and \(g(\gamma)\) are respectively related with
\[
f(\gamma) = 1 - 2\tau n \frac{\mu^2}{k} \int \tau e^{2(1-\lambda)/\tau} t
\]
and
\[
g(\gamma) = 1 - 4\tau n \frac{\mu^2}{k} \int \tau e^{2(1-\lambda)/\tau} t \|P\|
\]
where \(\gamma = \gamma^*(\mu, \tau, ||h||_\infty, H)\) such that \(f(\gamma) \geq \frac{1}{2}, g(\gamma) \geq \frac{1}{2}, \forall \gamma \in (0, \gamma^*)\). Therefore it follows from (97) that
\[
\dot{V}(\phi_t, e_t) \leq -\gamma |\phi|^2 + \frac{1}{4} \gamma |e(t)|^2,
\]
\(\forall \mu \in [\mu^*, \infty), \forall \gamma \in (0, \gamma^*), \forall t \geq 2\tau,\)
and the asymptotic stability then follows from the Lyapunov stability theorem. The proof is completed.

**Proof of Item 2.** For any \(t \geq 2\tau\), integrating both sides of (8) from \(t - \tau\) to \(t\) gives \(\phi(t) = \phi(t - \tau)\), where
\[
\delta(t) = \int_{t - \tau}^{t} (A\phi(s) - \lambda BF\phi(s - \tau) + \theta \lambda BF(e(s - \tau))) ds.
\]
Then we can rewritten the first equation in system (8) as
\[
\dot{\phi}(t) = (A - \lambda BF)\phi(t) + \lambda BF(\delta(t) + \theta \epsilon(e(t - \tau))),
\]
along whose trajectories the time-derivative of \(V_\phi(\phi(t)) = |\phi(t)|^2\) can be evaluated as
\[
\dot{V}_\phi(\phi(t)) \leq -|\phi|^2(\phi(t))P|\phi(t)| + \frac{2\gamma}{k} \delta(t)P\delta(t)
\]
where \(\gamma \in [\mu^*, \infty), \forall \mu \in [\mu^*, \infty), \forall t \geq 2\tau,\)
By using Lemma 4 in Appendix A.2, we can compute
\[
\delta(t) = \int_{t - \tau}^{t} (A\phi(s) - \lambda BF\phi(s - \tau) + \theta \lambda BF(e(s - \tau))) ds
\]
Now compute the time-derivative of \(V_\epsilon(e(t)) = e(t)Q\epsilon(t),\) where \(Q\) solves (92), along the trajectories of the second equation in system (8) as
\[
\dot{V}_\epsilon(e(t)) \leq -\frac{1}{2} |\epsilon(t)|^2 + 4\tau^2 \gamma |\lambda|^2 \mu^2 \|B^2Q^2B\|
\]
where we have again used Lemma 4 in Appendix A.2. Let
\[
V(\phi(t), e(t)) = V_\phi(\phi(t)) + \sqrt{P}\|V_\epsilon(e(t))\|
\]
Then under the condition that
\[
V(\phi(t + s), e(t + s)) < \eta V(\phi(t), e(t)), \forall s \in [0, \gamma^*),
\]
where \(\eta > 1\) is any prescribed number, we have
\[
\max \left\{ \sqrt{P}\|V_\epsilon(e(t + s))\|, V(\phi(t + s)) \right\}
\]
\[
\leq \eta V(\phi(t), e(t)), \forall s \in [0, \gamma^*),
\]
It follows from \(P \leq \|P\| I \leq \frac{\|P\|}{\lambda_{\min}(Q)}\) and (109) that \(\dot{V}(\phi(t))\) in (103) and \(V_\epsilon(e(t))\) in (106) can be respectively continued as
\[
\dot{V}(\phi(t), e(t)) \leq -\gamma |\phi|^2 + \gamma h_1(\gamma)\eta V(\phi(t), e(t)),
\]
\[
\dot{V}(e(t)) \leq -\frac{1}{2} |\epsilon(t)|^2 + \gamma h_2(\gamma)\eta V(\phi(t), e(t)),
\]
where \(h_1(\gamma)\) and \(h_2(\gamma)\) are respectively defined as
\[
h_1(\gamma) = \frac{2\gamma}{k} (9\tau^2 n^2 \gamma^2 + 3\tau^2 \lambda^2 \mu^2 n^2 \gamma^2
\]
\[
\times \left( 1 + \frac{\gamma^2}{\lambda_{\min}(Q)} + \frac{\gamma^2}{\lambda_{\min}(Q)} \right),
\]
\[
h_2(\gamma) = 4\tau^2 n |\lambda|^2 \mu^2 \|B^2Q^2B\| \left( 1 + \frac{\gamma}{\lambda_{\min}(Q)} \right).
\]
It follows from (110) to (111) that
\[
\dot{V}(\phi(t), e(t)) \leq -\gamma \left( 1 - \frac{1}{2} h(\gamma) \eta \right) V(\phi(t), e(t))
\]
\[
-\frac{1}{2} |\epsilon(t)|^2 + \gamma h_2(\gamma)\eta V(\phi(t), e(t)),
\]
where \(h(\gamma) = h_1(\gamma) + \sqrt{P}\|h_2(\gamma).\), Hence, it follows from \(\lim_{\gamma \to 0} h(\gamma) = 0\) and (114) that there exists a number \(\gamma^* = \gamma^*(\mu, \tau, ||h||_\infty, H)\) such that
\[
\dot{V}(\phi(t), e(t)) \leq -\frac{1}{2} \gamma V(\phi(t), e(t)),
\]
\(\forall \mu \in [\mu^*, \infty), \forall \gamma \in (0, \gamma^*), \forall t \geq 2\tau.,\)
The stability of (8) then follows from the Razumikhin stability theorem (Hale, 1977). The proof is completed.

A.2. Properties of solutions to a parametric ARE

In this subsection we recall the following results from Zhou, Lin, and Duan (2010), Zhou et al. (2012) regarding properties of solutions to the parametric ARE (6).

**Lemma 4.** Assume that the matrix pair \((A, B) \in (R^{n \times n}, R^{n \times m})\) is controllable and all the poles of \(A\) are on the imaginary axis. Then the parametric ARE (6) has a unique positive definite solution \(P(\gamma) = W^{-1}(\gamma),\) where \(W(\gamma)\) is the unique positive definite solution to the following Lyapunov equation \(W(A + \frac{1}{2} I_n) + (A + \frac{1}{2} I_n)W = BB^T.\) Moreover, \(\lim_{\gamma \to 0} P(\gamma) = 0, \frac{\partial P(\gamma)}{\partial \gamma} > 0, \forall \gamma > 0,\)
\[
\text{tr}(BB^T) = n\gamma, P(\gamma) BB^T P(\gamma) \leq n\gamma P(\gamma), e^{1/2} P(\gamma) e^{1/2} \leq e^{(1-\lambda)/2} P(\gamma).\]
In particular, if all the eigenvalues of \(A\) are zero, then \(A^TPA \leq 3(\gamma n)^2 P.\)
References


