On improving transient performance in global control of multiple integrators system by bounded feedback

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The global stabilization problem for multiple integrators systems by bounded control is considered. Two classes of nonlinear feedback laws are proposed. The first one consisting of nested saturation functions is a modification and generalization of that in [A.R. Teel, Global stabilization and restricted tracking for multiple integrators with bounded controls, Systems & Control Letters 18 (3) (1992) 165–171] and the other one consists of parallel connections of saturation functions. Both these two types of nonlinear feedback laws need only $\tilde{n}(\tilde{n} = \frac{n}{2}$ if $n$ is even and $\tilde{n} = \frac{n+1}{2}$ if $n$ is odd, where $n$ is the length of the multiple integrators) saturation elements. Furthermore, the poles of the closed-loop system can be placed on any location of the left real axis when none of the saturation elements in the control laws is saturated. Both of them exhibit simpler structure, can significantly improve the transient performance of the closed-loop system, and are very superior to the other existing methods. Simulation of a fourth-order system is used to illustrate the effectiveness of the proposed methods.

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1. Introduction

Practical control systems are subject to input saturation. For the special case of linear systems subject to input saturation, several important control problems have been solved. Among these problems are global stabilizability [1,2], semi-global stabilizability [3], local stabilizability [4,5], input–output stabilization [6] and robust stabilizability [7]. For more related problems on this topic, see [8–13] and the recent monographs [14–16]. It is well-known that, for the global stabilization problem, a linear system subject to input saturation is globally stabilizable if and only if the system, in the absence of the input saturation, is asymptotically null controllable with bounded controls (ANCBC), equivalently, it is stabilizable in the ordinary sense and has all its poles located in the closed left half plane. It has been shown in [17] that a simple linear system of a chain of integrators of length $n > 2$ which is ANCBC, can not be globally stabilized by saturated linear feedback. Thus, for general linear systems, nonlinear feedback is required. Still for the multiple integrators system, Teel proposed in [18] a nonlinear state feedback law consisting of nested saturation functions which not only solves the global stabilization problem but also can be used to achieve trajectory tracking for a class of bounded trajectories. This technique of using nested saturation functions was latterly successfully applied to achieve global stabilization of general ANCBC linear systems in [1].

It has been shown in [19] that Teel’s feedback law consisting of $n$ (the order of the system) nested saturation functions exhibits good robustness and excellent disturbance rejection, and is noticeably superior to some other existing feedback laws. But on the other hand, as mentioned in [20], for larger systems and bigger initial conditions, the transient performance of the closed-loop system is degraded. Thus, due to the simplicity and the superiority of Teel’s nonlinear feedback laws for multiple integrators system, a lot of explorations and modifications have been made by some authors (see, e.g., [20–26]). Among these modified methods, the most remarkable one comes from [20] where the author has introduced a type of so-called state-dependent saturation functions to replace the standard saturation functions appearing in Teel's nonlinear feedback laws. This idea is then followed by [21] where another type of nonlinear feedback law consisting of parallel connections of saturation functions is considered. It is found that these two types of saturated nonlinear feedback laws can indeed significantly improve the transient performance of the closed-loop system. Another remarkable modification is given in [23] where the author has found that Teel’s nested nonlinear feedback law results in all the poles of the closed-loop system residing at $-1$ when none of the saturation elements in the control laws is saturated and proposed a modified control law that allows the eigenvalues to
be in any place on the left real axis. It is also found that such modifications can greatly improve the transient performance of the closed-loop system. Very recently, these two modifications have been combined to improve the transient performance of the discrete-time multiple integrators system [22].

In Teel’s original paper [18], actually two types of feedback laws consisting of nested saturation functions are proposed. Suppose \( n \) is the order of the system, the first one consists of \( n \) saturation functions and the other needs only \( \tilde{n} (\tilde{n} = \frac{n}{2} \) if \( n \) is even and \( \tilde{n} = \frac{n + 1}{2} \) if \( n \) is odd) saturation functions with the number significantly decreased. To the best of our knowledge, all of the existing discussions and modifications are based on the first type of nested nonlinear feedback law while the other one is rarely mentioned. However, the second type of nonlinear feedback law is inherently superior to the first as it requires fewer saturation elements which can increase the control energy, and thus can improve the transient performance of the closed-loop system. On the other hand, we notice that the second type of nested nonlinear feedback law also results in all the poles of the closed-loop system residing at \(-1\) when none of the saturation elements in the control laws is saturated. So one aim of the present paper is to apply the idea in [23] to the second type of nonlinear feedback law given in [18]. We succeed in doing this while the techniques used in this paper are quite different from those in [23]. The second contribution of this paper is that we have also proposed a new nonlinear feedback law consisting of the parallel connections of \( \tilde{n} \) saturation functions. This new nonlinear feedback law can also result in all the poles of the closed-loop system residing at any locations of the negative real axis when none of the saturation elements in the control laws is saturated. Simulation of a fourth order system indicates that the transient performances of the closed-loop system can be significantly improved when compared with some other existing results. Roughly speaking, they are superior to any other existing nonlinear feedback laws discussed above if the parameters in the control laws are properly chosen.

**Notations:** Throughout this paper, the state vectors are denoted by bold symbols. We use \( y_i \) to denote the \( i \)-th row of the state vector \( y \). The function sign is defined as \( \text{sign}(y) = 1 \) if \( y \geq 0 \) and \( \text{sign}(y) = -1 \) if \( y < 0 \). The saturation function can then be defined as \( \text{sat}_\alpha(y) = \text{sign}(y) \min(\alpha, |y|) \) where \( \alpha \) is the saturation level. Moreover, \( \text{sat}(y) \) is denoted by \( \text{sat}(y) \) for short. Finally, we use \( \mathbb{Z}[p, q] \) with \( p \) and \( q \) two integers satisfying \( p \leq q \) to denote the set \( \{p, p+1, \ldots, q\} \).

### 2. Main results

In this paper, we consider the following \( n \)-th order multiple integrators system

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
& \vdots \\
\dot{x}_n &= u,
\end{align*}
\tag{1}
\]

with bounded input \(|u| \leq u_{\text{max}}\) where \( u_{\text{max}} \) is some known parameter representing the amplitude limitation of the control. This kind of system has been well studied in the literature (see, e.g., [19,21,22] and the references therein). The problem considered in this paper is stated as follows.

**Problem 1.** Find a function \( u = u(x) \) satisfying the amplitude constraint \(|u| \leq u_{\text{max}}\) such that the system (1) is globally asymptotically stable.

To describe our main results, we need the following new state space representation of the system (1) whose proof is simple and thus omitted.

**Lemma 2.** Let \( \lambda_i, i \in \mathbb{Z}[2, n] \) be a series of given positive numbers. Then the system (1) is algebraic equivalent to

\[
\dot{y} = A_n y + b_n u, 
\tag{2}
\]

where \( A_n \) and \( b_n \) are respectively given by

\[
A_n = \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}, 
\quad b_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix},
\tag{3}
\]

via a linear change of coordinates \( y = T x \) with \( T \) given by

\[
T = \begin{bmatrix} A_n^{-1} b_n & A_n^{-2} b_n & \cdots & b_n \end{bmatrix}.
\tag{4}
\]

We give some further notations associated with the system (2). Let \( \tilde{n} = \frac{n + 1}{2} \) if \( n \) is odd and \( \tilde{n} = \frac{n}{2} \) if \( n \) is even. Then we can denote

\[
\begin{align*}
\dot{x}_i &= \begin{bmatrix} \lambda_{n+1+2(i-\tilde{n})} \\ \lambda_{n+2(i-\tilde{n})} \end{bmatrix}, 
\quad \ddot{y}_i = \begin{bmatrix} y_{n+1+2(i-\tilde{n})} \\ y_{n+2(i-\tilde{n})} \end{bmatrix}, 
\quad i \in \mathbb{Z}[2, \tilde{n}].
\end{align*}
\]

When \( i = 1 \), we define \( \tilde{x}_1 = \lambda_1, \tilde{y}_1 = y_1 \) if \( n \) is odd and we define

\[
\begin{align*}
\tilde{x}_1 &= \frac{\lambda_1}{\lambda_2}, 
\quad \tilde{y}_1 = y_1, 
\end{align*}
\]

if \( n \) is even. For two arbitrary positive scalars \( \alpha_1 \) and \( \alpha_2 \) and a vector \( x \in \mathbb{R}^2 \), we define \( P = \text{diag}(\alpha_1, \alpha_2) \) and

\[
\begin{align*}
V(x) &= \frac{1}{2} \alpha_1 x_1^2 + \frac{1}{2} \alpha_2 x_2^2, 
\quad \Omega(P, c) = \left\{ x \mid V(x) \leq \frac{1}{2} c^2 \right\}.
\end{align*}
\tag{5}
\]

#### 2.1. Nonlinear feedback by using nested saturation functions

**Lemma 3.** Consider the following second-order nonlinear system

\[
\begin{align*}
\dot{x}_1 &= 0, \\
\dot{x}_2 &= \alpha_2 \dot{x}_1 + \alpha_1 x_1 + u, \\
u &= -\varepsilon_2 \text{sat} \left( \frac{\alpha_1 \dot{x}_2 + \alpha_1 x_1 + \varepsilon_1 \text{sat} (z)}{\varepsilon_2} \right),
\end{align*}
\tag{6}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are some positive scalars and \( z = z(t) \) is an external input. If

\[
\varepsilon_2 \geq k \left( \frac{\alpha_1}{\alpha_2} \right) \varepsilon_1,
\tag{7}
\]

where

\[
k(s) = 1 + \frac{2(1 + s)(1 + s + \sqrt{s^2 - s + 1})}{3s},
\tag{8}
\]

then there exists a finite time \( T \) such that for \( \forall t \geq T \) the states \( x_1 \) and \( x_2 \) will enter and remain in the ellipsoid \( \Omega(P, c) \) with

\[
c = \sqrt{\frac{\varepsilon_2 - \varepsilon_1}{\alpha_1 + \alpha_2}}.
\tag{9}
\]

Moreover, the nonlinear system (6) can be simplified as

\[
\begin{align*}
\dot{x}_1 &= -\alpha_1 x_1 - \varepsilon_1 \text{sat} (z), \\
\dot{x}_2 &= -\alpha_1 x_1 - \alpha_2 x_2 - \varepsilon_1 \text{sat} (z).
\end{align*}
\tag{10}
\]

**Remark 4.** Note that the nonlinear system (6) can be regarded as a double integrators system controlled by a bounded input \( u \) with \( |u| \leq \varepsilon_2 \). Therefore, it is not difficult to understand that for a given initial condition, the larger the \( \varepsilon_2 \), the faster the states \( x_1 \) and \( x_2 \) will converge.
Lemma 3 can be viewed as a generalization of the result given in [18] where the parameters $\alpha_1$ and $\alpha_2$ are set to 1. The proof of Lemma 3 is given in the Subsection for clarity.

**Theorem 5.** Let $\lambda_i, i \in \{1, n\}$ be a series of given positive numbers and $\epsilon_i, i \in \{1, \bar{n}\}$ be some positive numbers satisfying
\[
\epsilon_i \geq k \left(\frac{\lambda_{n-1+2(i-1)}}{\lambda_{n+2(i-1)}}\right) \epsilon_{i-1}, \quad i \in \{2, \bar{n}\}, \quad \epsilon_n \leq u_{\text{max}}, \quad (11)
\]
with $k(s)$ given by (8). Then the control law $u(\mathbf{x}) = \epsilon_i u_i$ with
\[
u_i = -\text{sat} \left(\frac{\lambda_i y_0}{\epsilon_i} - \frac{\epsilon_{i-1} u_{i-1}}{\epsilon_i}\right), \quad i \in \{2, \bar{n}\},
\]
\[
u_1 = -\text{sat} \left(\frac{\lambda_1 y_0}{\epsilon_1}\right),
\]
where $y$ is defined in Lemma 2 solves Problem 1. Furthermore, under such a control law, the closed-loop system will operate in a linear region at finite time with eigenvalues $-\lambda_i, i \in \{1, n\}$.

**Proof.** We prove this theorem by step by step.

**Step 1.** Consider the last two states of the system (2), i.e.,
\[
\begin{align*}
\dot{y}_n &= \lambda_n y_n - \epsilon_n \text{sat} \left(\frac{\lambda_n y_0}{\epsilon_n} - \frac{\epsilon_{n-1} u_{n-1}}{\epsilon_n}\right) \\
\dot{y}_n &= -\epsilon_n \text{sat} \left(\frac{\lambda_n y_0}{\epsilon_n} - \frac{\epsilon_{n-1} u_{n-1}}{\epsilon_n}\right) \\
\end{align*}
\]
Note that system (12) is in the form of (6). Using Lemma 3, if (11) is satisfied with $i = \bar{n}$, there exists a finite time $T$ such that for all $t \geq T$, the states $y_{n-1}$ and $y_n$ are linear in the control and the system (12) can be simplified as
\[
\begin{align*}
\dot{y}_n &= -\lambda_n y_n + \epsilon_n u_{n-1} \\
\dot{y}_n &= -\lambda_n y_n + \epsilon_n u_{n-1} \\
\end{align*}
\]
With the above simplification, the closed-loop system (2) with $u = \epsilon_i u_i$ results in
\[
\dot{y} = A_n y + b \epsilon_i u_{n-1} u_{n-1},
\]
where
\[
A_n = \begin{bmatrix}
0 & \lambda_2 & \cdots & \lambda_{n-2} & 0 & 0 \\
0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & \vdots & \ddots & \lambda_{n-2} & 0 & 0 \\
0 & \vdots & \ddots & \vdots & \lambda_{n-2} & 0 \\
0 & \vdots & \ddots & \lambda_{n-2} & \lambda_{n-1} & 0 \\
0 & \vdots & \ddots & \lambda_{n-2} & \lambda_{n-1} & -\lambda_n
\end{bmatrix}
\]

**Step 2.** We can then consider the following two states $y_{n-3}$ and $y_{n-4}$ of system (13), which are also in a differential equation in the form of (6). With the same manipulations in Step 1, the closed-loop system can be simplified. Following this procedure the closed-loop system finally becomes
\[
\dot{y} = A_1 y + b \epsilon_i u_i,
\]
where, for $n$ being even or odd, $A_1$ takes respectively the following forms
\[
A_1 = \begin{bmatrix}
0 & \lambda_2 & \cdots & 0 \\
0 & 0 & \cdots & \cdots \\
0 & 0 & -\lambda_3 & 0 & 0 \\
0 & \cdots & \cdots & \cdots \\
0 & 0 & -\lambda_3 & \cdots & -\lambda_n
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & -\lambda_2 & \cdots & 0 \\
0 & -\lambda_2 & -\lambda_3 & 0 \\
0 & -\lambda_2 & -\lambda_3 & \cdots \\
0 & -\lambda_2 & -\lambda_3 & \cdots & -\lambda_n
\end{bmatrix}
\]

**Step 3.** If $n$ is even, we consider the first two states, namely, $y_1$ and $y_2$, of system (14). These two states are also in a differential equation in the form of (6) with $z = 0$. It follows from Lemma 3 and (14) that the closed-loop system becomes
\[
\dot{y}_1 = \epsilon_1 u_1 = -\epsilon_1 \text{sat} \left(\frac{\lambda_1 y_1}{\epsilon_1}\right),
\]
at finite time for arbitrary $\epsilon_1 > 0$. If $n$ is odd, we consider the first state $y_1$ which satisfies the following differential equation
\[
\dot{y}_1 = \epsilon_1 u_1 = -\epsilon_1 \text{sat} \left(\frac{\lambda_1 y_1}{\epsilon_1}\right),
\]
which is obviously globally stable. Therefore, there exists a finite time $T_1 > 0$ such that $\max_{t \geq T_1} |\dot{y}_1| \leq 1$ for $t \geq T_1$. Hence the closed-loop system (14) also reduces to (15) for $t \geq T_1$.

Finally, we notice that (15) is a stable linear system and clearly has eigenvalues $-\lambda_i, i \in \{1, n\}$. This completes the proof. ■

**Remark 6.** It can be seen from the proof of Theorem 5 that in Step 1, $i \in \{1, \bar{n}\}$, a second-order nonlinear subsystem in the form of (6) is encountered. Therefore, by means of Remark 4, the parameters $\epsilon_i, i \in \{1, \bar{n}\}$ are expected to be as large as possible so that the states $y_i, i \in \{2, \bar{n}\}$ decrease more quickly. Thus, in view of the inequality (11), the parameters $\epsilon_i, i \in \{1, \bar{n}\}$ should be chosen as
\[
\epsilon_i = k \left(\frac{\lambda_{n-1+2(i-1)}}{\lambda_{n+2(i-1)}}\right) \epsilon_{i-1}, \quad i \in \{2, \bar{n}\}, \quad \epsilon_n = u_{\text{max}}.
\]

Hence, the numbers $k \left(\frac{\lambda_{n-1+2(i-1)}}{\lambda_{n+2(i-1)}}\right), i \in \{2, \bar{n}\}$ are expected to be as small as possible. The function $k(s)$ is illustrated in the left hand side of Fig. 1. It is easy to see and prove that the minimal value is $k_{\text{min}} = k(1) = 5$. Then the parameters $\epsilon_i, i \in \{1, \bar{n}\}$ are minimized under the equality constraints (16) if and only if
\[
\lambda_{n-1+2(i-1)} = \lambda_{n+2(i-1)} = \lambda_i, \quad i \in \{2, \bar{n}\},
\]
The above series of equations give a guide for choosing the parameters $\lambda_i, i \in \{1, n\}$. However, how to choose $\lambda_i, i \in \{2, \bar{n}\}$ is still unknown. Simulation results show that the optimal values of $\lambda_i, i \in \{2, \bar{n}\}$ in the sense that the transient performance of the closed-loop system is the best, are even dependent on the initial conditions.

**2.2 Nonlinear feedback by using parallel connections of saturation functions**

The following lemma is parallel to Lemma 3 and will be proven in Section 4.2.

**Lemma 7.** Consider the following second-order nonlinear system
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & \alpha_2 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + u, u = -\epsilon_2 \text{sat} \left(\frac{\alpha_2 x_2 + \alpha_1 x_1}{\epsilon_2}\right) + z,
\]
where $\alpha_1$ and $\alpha_2$ are some positive scalars and $z$ is an external input satisfying $|z| \leq \epsilon_1 < \epsilon_2$. If
\[
\epsilon_2 \geq p \left(\frac{\alpha_1}{\alpha_2}\right) \epsilon_1,
\]
where

\[ p(s) = \frac{2(s+1)^2 + (s+1)\sqrt{s^2 - s + 1}}{3s}, \]  

(20)

then there exists a finite time \( T \) such that for all \( t \geq T \) the states \( x_1 \) and \( x_2 \) will enter and remain in the ellipsoid \( \Omega \) \((p, c)\) with

\[ c = \frac{\lambda_2}{\sqrt{\lambda_1 + \lambda_2}}. \]  

(21)

Moreover, the nonlinear system (18) can be simplified as

\[
\begin{align*}
\dot{x}_1 &= -\alpha_1 x_1 + z \\
\dot{x}_2 &= -\alpha_2 x_2 + \alpha_3 y + z.
\end{align*}
\]

(22)

By using the above lemma recursively, we can prove the following result regarding solution of Problem 1. The details of the proof are quite similar to that of Theorem 5 and thus are omitted for simplicity.

**Theorem 8.** Let \( \lambda_i, i \in \{1, n\} \) be a series of given positive numbers and \( \varepsilon_i, i \in \{1, n\} \) be some positive numbers satisfying

\[ \varepsilon_i \geq \frac{\lambda_i}{\lambda_{i+1} + \lambda_{i+2}} \sum_{k=2}^{i} \varepsilon_{k-1}, \quad i \in \{2, n\}, \sum_{i=1}^{n} \varepsilon_i \leq u_{\max}, \]

with \( p(s) \) given by (20). Then the control law \( u(x) = u \) where

\[ u = -\sum_{i=1}^{n} \varepsilon_i \text{sat}(\frac{\lambda_i y}{\varepsilon_i}), \]

with \( y \) defined in Lemma 2 solves Problem 1. Furthermore, under such a control law, the closed-loop system will operate in a linear region at finite time with eigenvalues \(-\lambda_i, i \in \{1, n\} \).

**Remark 9.** Similar to Remark 6, the parameters \( \varepsilon_i, i \in \{1, n\} \) should be chosen as

\[ \varepsilon_i = \frac{\lambda_i}{\lambda_{i+1} + \lambda_{i+2}} \sum_{k=2}^{i} \varepsilon_{k-1}, \quad i \in \{2, n\}, \sum_{i=1}^{n} \varepsilon_i = u_{\max}, \]

to improve the transient performances of the closed-loop system. Again, the numbers \( p\left(\frac{\lambda_i}{\lambda_{i+1} + \lambda_{i+2}}\right), i \in \{2, n\} \) are expected to be minimized. It is also easy to show that \( p(s) \) is minimized when \( s = 1 \), i.e. \( p_{\min} = p(1) = 4 \). Therefore, we can also let Eq. (17) be satisfied.

### 3. Example

Consider a fourth-order integrator system in the form of (1) with \( |u| \leq u_{\max} = 2 \). Let \( y = T x \) where \( T \) is defined in (4) with \( \lambda_i, i \in \{1, 4\} \) some given positive scalars. We consider the following 8 types of control laws.

**Control Law 1:** This control law is proposed in Teel's original paper [18] and has the form

\[ u_1 = -\text{sat}_{\alpha_1}(y_3 + \text{sat}_{\alpha_2}(y_4 + \text{sat}_{\alpha_3}(y_1, y_2, y_3))), \]

where \( \lambda_i = 1, i \in \{1, 4\} \). Taking \( \alpha_i = \frac{1}{\sqrt{\lambda_i}} \) for \( i \in \{1, 4\} \) and \( \alpha_4 = u_{\max} = 2 \) to satisfy the stability condition and bounded condition.

**Control Law 2:** This control law is given in [20] and is a generalization of \( u_1 \) having the form

\[ u_2 = -\text{sat}_{\alpha_2}(y_4 + \text{sat}_{\alpha_1}(y_3 + \text{sat}_{\alpha_3}(y_1, y_2, y_3))), \]

with \( \gamma_i = 2 \) and

\[ y = \begin{cases} 
1, & \text{if } |y_{i+1}| > |y_{i+1}|, \quad i \in \{1, 3\} \\
\frac{1}{2.001} \left[ \alpha_{i+1} + |y_{i+1}| - y_{i+1} \right], & \text{otherwise,} 
\end{cases} \]

where \( \alpha_i, i \in \{1, 4\} \) and \( y \) are the same as in \( u_1 \) and \( L_i = \frac{1.001}{2.001} \alpha_i, i \in \{1, 2, 4\} \).

**Control Law 3:** This control law is also proposed in Teel’s paper [18] and has the form

\[ u_3 = -\alpha_2 \text{sat}\left(\frac{y_4 + y_3}{\alpha_2} + \frac{y_1}{\alpha_1}\right), \]

where \( y \) is also the same as in \( u_1 \). According to Theorem 5, we take \( u_{\max} = \alpha_2 = k(1) \alpha_1 = 5 \alpha_1 \) to satisfy the stability condition and bounded condition.

**Control Law 4:** This control law is given by Theorem 5 and has the form

\[ u_4 = -\alpha_2 \text{sat}\left(\frac{\lambda_4 y_4 + \lambda_3 y_3}{\alpha_2} + \frac{\alpha_1}{\alpha_2} \text{sat}\left(\frac{\lambda_2 y_2 + \lambda_1 y_1}{\alpha_1}\right)\right), \]

where \( \lambda_i = 0.15, i \in \{1, 4\} \). We take \( \epsilon_2 = 2 \) and \( \epsilon_3 = k(\frac{\alpha_1}{\alpha_2}) \) to satisfy the stability condition and bounded condition.

**Control Law 5:** This control law is given by Theorem 8 and has the form

\[ u_5 = -\alpha_2 \text{sat}\left(\frac{\lambda_4 y_4 + \lambda_3 y_3}{\alpha_2} - \epsilon_1 \text{sat}\left(\frac{\lambda_2 y_2 + \lambda_1 y_1}{\epsilon_1}\right)\right). \]

We take \( \epsilon_3 = p(\frac{\alpha_1}{\alpha_2}) \) and \( \epsilon_2 + \epsilon_1 = 2 \) to satisfy the stability condition and bounded condition. Again, \( \lambda_i = 0.15, i \in \{1, 4\} \).
Control Law 6: This control law is proposed in [21] where the authors have extended the results given in [1]. This control law is in the form of

$$u_b = -\eta_{\max} \sum_{i=1}^4 \epsilon^{4i+1+1} \text{sat}_{\mu_i}(y_i),$$

where \( y = T \sigma \) by setting \( \lambda_i = \lambda^{4i+1+1}, i \in [1, 4] \) and \( \beta_i = 1, i \in [1, 4] \). As analyzed in [21], the parameter \( \epsilon \) should be chosen as \( \epsilon \leq 0.5437 \) to ensure stability.

Control Law 7: This control law \( u_7 \) is also proposed in [21] by using the state-dependent saturation functions and is also in the form of (23) with \( \beta_i = 1 \) and

$$\beta_i = 1 + \frac{1}{\epsilon}(\beta_i - |\text{sat}_{\mu_i}(y_i)|), \quad k \in [1, 3].$$

Control Law 8: The original version of this class of control laws is proposed in [22] recently to globally stabilize the discrete-time multiple integrators. The corresponding continuous-time version of this class of nonlinear control laws can be given as follows

$$u_b = -\sigma \sum_{i=1}^4 \theta_i \text{sat}_{\mu_i}(y_i),$$

where \( \sigma = \eta_{\max} \sum_{i=1}^4 \beta_i = 1 \) and \( \theta_i, i \in [1, 3] \) are given by

$$\theta_i = 1 + \frac{\theta_i + 1}{\theta_i} \left[ \theta_i - |\text{sat}_{\mu_i}(y_i)| \right], \quad \chi_i \in [0, 1].$$

The parameters \( \theta_i, i \in [1, 4] \) should be chosen as \( 0 < \sum_{i=1}^4 \theta_i < \theta_i \) to ensure stability. It is shown in [22] that the transient performances can be improved if \( \chi_i = 1, i \in [1, 3] \) which is assumed to be true here. Take \( \theta_i = 1.01 \sum_{i=1}^4 \theta_i, i \in [2, 4] \) and \( \theta_1 = 0.15 \) to ensure stability and better transient performance which is validated by simulations.

The initial condition is chosen as \( x_0 = \{5 \quad 8 \quad -5 \quad -8\}^T \). The simulation results are shown in the right hand side of Fig. 1 where the y-axis is \( \|x(t)\| \). It can be seen that the transient performances of the closed-loop system under the control laws \( u_4 \) and \( u_5 \) are significantly improved with respect to the other feedback laws. Since the control laws \( u_4 \) and \( u_5 \) have the same structure but different parameters \( \lambda_i, i \in [1, 4] \), the simulation results show that small parameters \( \lambda_i, i \in [1, 4] \) will lead to better transient performances. However, this is not always the case. We will do some further comparisons to illustrate this point.

We next show how the parameters \( \lambda_i, i \in [1, 4] \) in control laws \( u_4 \) and \( u_5 \) influence the transient performances of the corresponding closed-loop systems. For simplicity, we define

$$\tilde{\lambda} = [\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4].$$

Simulation results imply that \( \lambda_1 = \lambda_3 = \lambda_4 = 3 \) will lead to better transient performances. So in the following such a relation is assumed to be satisfied. The initial condition is chosen as before. For different parameter vectors \( \lambda_i, i \in [1, 5] \), the simulation results are shown in Fig. 2 where the legend “Parameters \( i, i = 1, 2, 3, 5 \)” means that the corresponding parameters \( \lambda_i, i \in [1, 5] \) are chosen as

$$[\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5] = [0.5 t \quad 0.2 t \quad 0.15 t \quad 0.1 t \quad 0.08 t],$$

where \( t \) is a \( 1 \times 5 \) vector whose elements are \( 1 \). It follows from Fig. 2 that in contrast to the control laws 1-3 and 6-8, these two control laws may lead to decreasing oscillations (this phenomenon can be explained in the proofs of Lemmas 3 and 7. See Remark 10 in Section 4). Note that when \( \lambda \) reduces from \( 0.5 \) to \( 0.2 \), the peak value and the regulating time of the trajectory of the closed-loop system are also reduced. But, reduce \( \lambda \) further and the peak value and regulating time of the trajectory are increasing! This observation indicates that there exists an optimal \( \lambda \) such that the peak value and regulating time are minimized. However, it is very difficult to determine this critical value in theory.

4. Proof of the main technical lemmas

4.1. Proof of Lemma 3

Consider the following state space partition

$$[I : \alpha_2 x_2 + \alpha_1 x_1 \geq \varepsilon_1 + \varepsilon_2 \quad \|x\| = 0 \quad \|x\| = 0 \quad \|x\| = 0 \quad \|x\| = 0]$$

which means that the \( x_1 - x_2 \) plane is divided into three regions by two lines \( \alpha_2 x_2 + \alpha_1 x_1 = \pm (\varepsilon_1 + \varepsilon_2) \). See Fig. 3. We will prove the following two statements as done in [18].

(1) Any bounded initial condition in Region I or III yields a trajectory that enters the boundary of Region II at finite time.

(2) Any state in the boundary of Region I or III that will enter Region I will return to the boundary of Region II at finite time and has a lower energy level with respect to the energy function \( V(x) \).

4.1.1. Proof of statement 1

Note that the system (6) in Region I reduces to

$$\dot{x}_1(t) = \begin{bmatrix} 0 & \alpha_2 & 0 & 0 \end{bmatrix} x_1(t) - \varepsilon_2.$$

(25)
It is easy to obtain the closed form solution of the linear system (25) as

\[
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} = \begin{bmatrix}
x_1(t_0) \\
x_2(t_0)
\end{bmatrix} + \left( t - t_0 \right) \begin{bmatrix}
(\alpha_2 - \alpha_2 x_2(t_0)) + (\alpha_1 + \alpha_2) \varepsilon_2 \\
(\varepsilon_2 + \varepsilon_1) - (\alpha_1 x_1(t_0) + \alpha_2 x_2(t_0))
\end{bmatrix}.
\]

(26)

If we set \(\alpha x_1(t) + \alpha x_2(t) - (\varepsilon_2 + \varepsilon_1) = 0\), we can get the equation

\[
(\alpha x_1(t_0) + \alpha x_2(t_0)) = 0.
\]

Now we consider the difference of the Lyapunov function between the two states \(x(t_0)\) and \(x(t)\). Evaluating the closed form solution (26) at \(t_0\) and simplifying, gives

\[
x_1^2(t_0) - x_1^2(t_0) = \frac{4}{\alpha_1^2} (\alpha_1 \varepsilon_1 - \alpha_2 \varepsilon_2) \left( \alpha_1 \varepsilon_1 - \alpha_2 x_2(t_0) - \alpha_2 x_2(t_0) \right).
\]

(29)

\[
x_2^2(t_0) - x_2^2(t_0) = -\frac{4}{\alpha_2^2} \left( \varepsilon_2 (\alpha_1 + \alpha_2) + (\alpha_1 x_2(t_0)) \right).
\]

(30)

Obviously, if \(\alpha_1 \varepsilon_1 - \alpha_2 \varepsilon_2 < 0\), i.e.,

\[
\varepsilon_2 > \frac{\alpha_1}{\alpha_2},
\]

then we have

\[
x_1^2(t_0) - x_1^2(t_0) < 0.
\]

(31)

Since \(x_2(t_0)\) satisfies (27), it follows from (29) that

\[
x_2^2(t_0) - x_2^2(t_0) < 0.
\]

(32)

Inequalities (31) and (32) clearly result in

\[
V(x(t_0)) - V(x(t_0)) < 0,
\]

provided (30) is satisfied. The same argument holds for Region III by symmetry.

**Remark 10.** It follows from Statement 2 that this type of nonlinear control law may lead to decreasing oscillations. The curve in Fig. 2 clearly verifies this statement. Such a phenomenon cannot be observed in the other nonlinear feedback laws studied in [20–26]. Furthermore, (28) implies that the “oscillation period” is linear in the initial condition \(x_2(t_0)\) that is on the boundary of Region III.

4.1.3. The negativity of \(\dot{V}(x)\) in Region III \(\setminus \Omega(P, c)\)

We first give the following lemma whose proof is omitted due to space limitation.

**Lemma 11.** Let \(|t| < \varepsilon_2 + \varepsilon_1\). Then the following inequality holds

\[
|f(t)| = \left| t - \varepsilon_2 \text{sat}_\phi \left( \frac{t + \varepsilon_1 \text{sat}(z)}{\varepsilon_2} \right) \right| \leq \varepsilon_1, \quad \phi \geq 1.
\]

(33)

We now consider the trajectories of the nonlinear system (6) in Region III. Denote

\[
h(x, z) = \text{sat} \left( \frac{\alpha x_2 + \alpha x_1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \text{sat}(z) \right).
\]

(34)
Using Lemma 11, the time derivative of $V(x)$ along the trajectories of (6) is given by

$$
V'(x) = \alpha x_1 x_1 + \alpha x_2 x_2 \\
= (\alpha x_1 + \alpha x_2) (\alpha x_1 + \alpha x_2 - \varepsilon_2 h(x, z)) \\
- \left(\alpha^2 x_1^2 + \alpha^2 x_2^2 + \alpha x_1 x_2 \right) \\
\leq |\alpha x_1 + \alpha x_2| \varepsilon_1 - \left(\alpha^2 x_1^2 + \alpha^2 x_2^2 + \alpha x_1 x_2 \right).
$$

Now consider the level set $\Omega(P, c)$ defined in (5) where $c$ will be determined later. It is easy to verify that

$$
|\alpha x_1 + \alpha x_2| \leq c\sqrt{\alpha_1 + \alpha_2}, \quad \forall x \in \Omega(P, c). \tag{35}
$$

Assume that $\Omega(P, c) \subset \text{Region } II$. \tag{36}

Then the region $\Omega(P, c)$ can be divided into the following two parts (see Fig. 3)

$$
\mathcal{A} = \left\{ x \in \mathbb{R}^n \mid |\alpha x_1 + \alpha x_2| > c\sqrt{\alpha_1 + \alpha_2} \right\},
$$

and

$$
\mathcal{B} = \left\{ x \in \mathbb{R}^n \mid |\alpha x_1 + \alpha x_2| \leq c\sqrt{\alpha_1 + \alpha_2} , \right\}
$$

To ensure that the sign of $V(x)$ in $\mathcal{A}$ is negative, we should ensure the signs of $V(x)$ are negative in both $\mathcal{A}$ and $\mathcal{B}$. We first consider region $\mathcal{A}$. Note that the region $\mathcal{A}$ can be equivalently rewritten as

$$
\mathcal{A} = \bigcup_{\beta \in \mathbb{R}} \left\{ x \mid |\alpha x_1 + \alpha x_2| = \beta c\sqrt{\alpha_1 + \alpha_2} \right\},
$$

where $\omega = \left(1, \frac{\alpha x_1 + \alpha x_2}{c\sqrt{\alpha_1 + \alpha_2}}\right)$. Then it follows that

$$
\min_{\alpha x \in \mathcal{A}} \left( \frac{1}{2} (|\alpha x_1 + \alpha x_2|)^2 + \frac{1}{2} (\alpha x_2)^2 \right) = \min_{\beta \in \mathbb{R}} \left( \frac{\beta^2}{4} (\alpha_1 + \alpha_2) \right).
$$

Note that for an arbitrary $\beta$, we have

$$
V(x) \leq |\alpha x_1 + \alpha x_2| \varepsilon_1 - \left(\alpha^2 x_1^2 + \alpha^2 x_2^2 + \alpha x_1 x_2 \right)
= \beta c\sqrt{\alpha_1 + \alpha_2} - \frac{3}{4} \beta^2 c^2 (\alpha_1 + \alpha_2).
$$

To guarantee $V(x) < 0$ for all $\beta \in \omega$, we must have

$$
c > \frac{4}{3}\beta \sqrt{\alpha_1 + \alpha_2} \varepsilon_1, \quad 1 < \beta < \frac{\varepsilon_2 + \varepsilon_1}{c\sqrt{\alpha_1 + \alpha_2}}
$$

which is equivalent to

$$
\varepsilon_2 > \frac{1}{3} \varepsilon_1. \tag{37}
$$

We next consider the region $\mathcal{B}$. To this end, we firstly consider the minimal value of the function $f(x) = \alpha x_1^2 + \alpha x_2^2 + \alpha x_1 x_2$ in region $\mathcal{B}$. It is easy to see that $\mathcal{B} \subset \overline{\Omega}(P, c)$ where

$$
\overline{\Omega}(P, c) = \left\{ x \in \mathbb{R}^n \mid \frac{m}{\sqrt{\alpha_1}} \sin(\theta), x_2 = \frac{m}{\sqrt{\alpha_2}} \cos(\theta) \right\}.
$$

with $m \geq c$ and $\theta \in [0, 2\pi]$. Thus we have $\min_{x \in \mathcal{B}} f(x) \geq \min_{x \in \overline{\Omega}(P, c)} f(x)$. It is easy to obtain

$$
\min_{x \in \overline{\Omega}(P, c)} f(x) = \frac{c^2}{2} \left( \alpha_1 + \alpha_2 - \sqrt{\alpha_1^2 + \alpha_2^2 - \alpha_1 \alpha_2} \right).
$$

Using (35) and the above formulation produces

$$
V(x) < c\sqrt{\alpha_1 + \alpha_2} \varepsilon_1 - \frac{c^2}{2} \left( \alpha_1 + \alpha_2 - \sqrt{\alpha_1^2 + \alpha_2^2 - \alpha_1 \alpha_2} \right).
$$

So the inequality $V(x) < 0$ holds in $\mathcal{B}$ provided the following inequality is guaranteed

$$
c \geq \frac{2\sqrt{\alpha_1 + \alpha_2} \varepsilon_1}{\alpha_1 + \alpha_2 - \sqrt{\alpha_1^2 + \alpha_2^2 - \alpha_1 \alpha_2}}. \tag{39}
$$

4.1.4. Positive invariant set and the corresponding conditions

Now note that $V(x) < 0$ in the region $\overline{\Omega}(P, c)$ is positive invariant, i.e., there exists a finite time $T$ such that for $t \geq T$ the state $x$ will enter the level set $\Omega(P, c)$ and remain there forever. In view of inequality (35), if $c\sqrt{\alpha_1 + \alpha_2} < \varepsilon_1$, \tag{40}

it is satisfied, then the nonlinear function $h(x, z)$ defined in (34) is linear in $x$. Consequently, the system (6) reduces to (10). Finally, inequality (40) clearly implies $c\sqrt{\alpha_1 + \alpha_2} < \varepsilon_1 \varepsilon_2$ which indicates that the assumption (36) is true.

Inequalities (39) and (40) are equivalent to (7) with $s = \frac{m}{\sqrt{\alpha_1}}$. Also, inequality (37) implies (31). and, finally, we note that when (7) is satisfied, the inequality (30) holds automatically. The proof is completed.

4.2. Proof of Lemma 7

Consider the following state space partition

$$
\Pi \cup \Omega(P, c) \cup \Omega(P, c).
$$

Similarly to the proof of Lemma 3, we will prove the statements given below Eq. (24).

4.2.1. Proof of statement 1

Note that the system (18) in Region $\Pi$ becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & x_1(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \frac{u(t)}{u(t)} + u(t),
$$

where $u(t) = \varepsilon_2 - z(t) > \varepsilon_2 - \varepsilon_1 > 0$. The closed form solution of (42) is

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t_0) + \alpha_2 (t - t_0) x_2(t_0) \\ x_2(t_0) \end{bmatrix}
$$

$$
- \int_{t_0}^t \left(1 + \alpha_2 (t - \tau) \right) u(\tau) \, d\tau,
$$

$$
- \int_{t_0}^t u(\tau) \, d\tau.
$$

If we set $\alpha x_1(t) + \alpha x_2(t) - \varepsilon_2 = 0$, then we can get the equation $G(t) = F(t, u)$, \tag{44}

where

$$
\begin{cases}
F(t, u) = (\alpha_1 + \alpha_2) \int_{t_0}^t u(\tau) \, d\tau + \alpha_1 \int_{t_0}^t (\alpha_2 - u(\tau)) \, d\tau \\
G(t) = (\alpha x_1(t_0) + \alpha x_2(t_0) - \varepsilon_2) + \alpha x_2(t_0) \rangle \langle x_2(t_0) \rangle.
\end{cases}
$$

Since $\varepsilon_2 - \varepsilon_1 \leq |u| \leq \varepsilon_2 + \varepsilon_1$, straightforward manipulation shows that

$$
F(t, \varepsilon_2 - \varepsilon_1) \leq F(t, u) \leq F(t, \varepsilon_2 + \varepsilon_1), \tag{45}
$$

where $F(t, \varepsilon) = \varepsilon (t - t_0) (\alpha_1 + \alpha_2 - \alpha x_1(t_0) + \alpha x_2(t_0)) < 0$. Since $\varepsilon_2 - (\alpha x_1(t_0) + \alpha x_2(t_0)) < 0$, both the equations $G(t) = F(t, \varepsilon_2 - \varepsilon_1)$ and $G(t) = F(t, \varepsilon_2 + \varepsilon_1)$ have solutions $t_1, t_2 \geq 0$. By continuity of the functions $G(t)$, $F(t, u)$, $F(t, \varepsilon_2 - \varepsilon_1)$, and $F(t, \varepsilon_2 + \varepsilon_1)$, and the relation (45), there exists a time $t_0 > t_0$ such that Eq. (44) holds. In other words, the trajectory of the system (18) will join the boundary of Region $\Pi$ at finite time. The same argument holds for Region $\Pi$ by symmetry.
2.4.2. Proof of statement 2
To prove Statement 2, we need the following lemma whose proof is omitted.

Lemma 12. Let $\alpha_1, \alpha_2 > 0$, $0 < \gamma \leq u(t) \leq \mu$, $\forall t \geq t_0$ and $\beta$ be a constant number satisfying $\beta > \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} u(t_0)$. For a fixed function $u(t)$, let $\tau^*_{n}$ be the minimal positive real solution of the following equation

$$\alpha_1 \int_{t_0}^{\tau^*_{n}} (1 + \alpha_2 (\tau^*_{n} - \tau)) u(\tau) d\tau + \alpha_2 \int_{t_0}^{\tau^*_{n}} u(\tau) d\tau = \beta \alpha_1 \alpha_2 (\tau^*_{n} - t_0).$$

Then there holds

$$\int_{t_0}^{\tau^*_{n}} u(\tau) d\tau \leq \int_{t_0}^{\tau^*_{n}} \gamma d\tau = 2\beta - \frac{2(\alpha_1 + \alpha_2)}{\alpha_1 \alpha_2} \gamma.$$

We now assume an initial state $\pi(t_0)$ on the boundary of Region 1, i.e.,

$$\pi(t_0) = \pi_{x_1}(t_0) + \pi_{x_2}(t_0) = \epsilon_2. \tag{46}$$

Note that for this initial condition, the nonlinear system (18) can be also rewritten as (42). To enter Region II, we must have $\pi_{x_2}(t_0) + \pi_{x_1}(t_0) > 0$, i.e., $\pi_{x_1}(t_0) > 0$ which gives $\pi_{x_2}(t_0) > 0$. Assume that the state $\pi$ has been in Region I. We have shown in Statement 1 that the state $\pi$ will return to the boundary of Region I at finite time $t_0 > t_0$, namely, $\pi_{x_2}(t_0) + \pi_{x_1}(t_0) = \pi_{x_2}(t_0) + \pi_{x_1}(t_0) = \epsilon_2$. Using this relation and setting $t = t_0$ in (43), we get

$$\pi(t_0) = \frac{1}{2} \alpha_1 \left( x_1^2(t_0) - x_1^2(t_0) \right) + \frac{1}{2} \alpha_2 \left( x_2^2(t_0) - x_2^2(t_0) \right) = \left( \frac{1}{2} \alpha_2 \int_{t_0}^{t_0} u(\tau) d\tau \right) h(t_0),$$

where $h(t_0)$ is given by

$$h(t_0) = 2 \pi_{x_1}(t_0) - \pi_{x_2}(t_0) + \left( \frac{\alpha_2}{\alpha_1} + 1 \right) \int_{t_0}^{t_0} u(\tau) d\tau.$$
References


