A new solution to the generalized Sylvester matrix equation

\[ AV - EVF = BW \]

Bin Zhou\(^1\), Guang-Ren Duan\(^*\)

Center for Control Systems and Guidance Technology, Harbin Institute of Technology, P.O. Box 416, Harbin 150001, China

Received 22 January 2005; received in revised form 9 June 2005; accepted 20 July 2005

Available online 12 September 2005

Abstract

This note deals with the problem of solving the generalized Sylvester matrix equation \( AV - EVF = BW \), with \( F \) being an arbitrary matrix, and provides complete general parametric expressions for the matrices \( V \) and \( W \) satisfying this equation. The primary feature of this solution is that the matrix \( F \) does not need to be in any canonical form, and may be even unknown a priori. The results provide great convenience to the computation and analysis of the solutions to this class of equations, and can perform important functions in many analysis and design problems in control systems theory.

© 2005 Elsevier B.V. All rights reserved.

Keywords: Generalized Sylvester matrix equations; General solutions; Parametric solutions; Right factorization; Linear systems

1. Introduction

This note considers the generalized Sylvester matrix equation

\[ AV - EVF = BW, \] (1)

where \( A, E \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}, F \in \mathbb{R}^{p \times p} \) are given matrices, while \( V \in \mathbb{R}^{n \times p}, W \in \mathbb{R}^{r \times p} \) are matrices to be determined. In the special case of \( E = I \), the equation reduces to

\[ AV - VF = BW. \] (2)

Eq. (2) is closely related with many problems in conventional linear control systems theory, such as pole/eigenstructure assignment design [4,5,9], Luenberger-type observer design [10,11,14], robust fault detection [3,7,12], and so on, and has been investigated by several researchers [4–6,13,16]. When dealing with eigenstructure assignment, observer design and model reference control for descriptor linear systems, the more generalized Sylvester matrix equation (1), with \( E \) being usually singular, is encountered.

In solving the generalized Sylvester matrix equation (1), finding the complete parametric solutions, that is, parametric solutions consisting of the maximum number of free parameters, is of extreme importance, since many problems, such as robustness in control system design, require full use of the design freedom. For (1) with \( F \) being in Jordan form, complete parametric solutions have been proposed in [4,6]. Under the \( R \)-controllability of the matrix triple \( (E, A, B) \), [6] has given a complete and explicit solution which uses the right coprime factorization of the input-state transfer function \((sE - A)^{-1}B\), while [4] proposed a complete parametric solution which is not in a direct, explicit form but in a recursive form.

These existing solutions are directly applicable in problems like eigenstructure assignment since in such problems the matrix \( F \) is originally required to be in Jordan form. However, in some other problems the matrix \( F \) is an arbitrary square matrix. Although we can apply some similarity transformation to transform \( F \) into a Jordan form, such a process is not desirable because it not only gives additional
computational load but also may result in numerically unreliable solutions [15].

In this note, solution to the generalized Sylvester matrix equation (1), with \( F \) being an arbitrary square matrix, is considered. Like [4,6], right factorization of the input-state transfer function \((sE - A)^{-1}B\) is also adopted, and a very neat general complete parametric solution in direct, explicit form to the Eq. (1) is proposed.

2. Preliminaries

In this note, we use \( \otimes \) to denote the Kronecker product. Also, for an \( m \times n \) matrix \( R = [r_{ij}] \), the so-called stretching function \( \text{vec}(R) \) is defined as
\[
\text{vec}(R) = [r_{11}, r_{12}, \cdots, r_{m1}, r_{m2}, \cdots, r_{mn}]^T.
\]

For matrices \( M, X \) and \( N \) with appropriate dimensions, we have the following well-known result related with the stretching operation:
\[
\text{vec}(MXN) = (N^T \otimes M) \text{vec}(X). \tag{3}
\]

**Definition 1.** Let \( F \in \mathbb{R}^{p \times p} \) be an arbitrary matrix. Then

1. a pair of polynomial matrices \( N(s) \in \mathbb{R}^{n \times r}[s] \) and \( D(s) \in \mathbb{R}^{m \times r}[s] \) are said to be \( F \)-right coprime if
\[
\text{rank} \left[ \begin{bmatrix} N(\lambda) \\ D(\lambda) \end{bmatrix} \right] = r \quad \text{for any} \ \lambda \in \sigma(F), \tag{4}
\]
2. a pair of polynomial matrices \( H(s) \in \mathbb{R}^{n \times n}[s] \) and \( L(s) \in \mathbb{R}^{m \times m}[s] \) are said to be \( F \)-left coprime if
\[
\text{rank} \left[ \begin{bmatrix} H(\lambda) \\ L(\lambda) \end{bmatrix} \right] = m \quad \text{for any} \ \lambda \in \sigma(F). \tag{5}
\]

The generalized Sylvester matrix equation (1) is homogeneous, so its solution is not unique. Using relation (3) the following result about the freedom of solution \((V, W)\) can be obtained.

**Theorem 1.** Let \( A, E \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{p \times p}, B \in \mathbb{R}^{n \times r} \), \( \text{rank}B = r \). Then the degree of freedom in the solution \((V, W)\) to the generalized Sylvester matrix equation (1) is \( rp \) if and only if \((sE - A) \) and \( B \) are \( F \)-left coprime, that is,
\[
\text{rank} \left[ \lambda E - A \quad B \right] = n \quad \text{for any} \ \lambda \in \sigma(F). \tag{6}
\]

**Proof.** Putting \( \text{vec}(\cdot) \) on both sides of Eq. (1) and using the Eq. (3), yields
\[
\begin{bmatrix} I_n \otimes A - F^T \otimes E & -I_p \otimes B \end{bmatrix} \begin{bmatrix} \text{vec}(V) \\ \text{vec}(W) \end{bmatrix} = 0, \tag{7}
\]
which is equivalent to Eq. (1). Let \( P \) and \( J \) be the eigenvector matrix and the Jordan form of the matrix \( F^T \), respectively, then we have
\[
F^T = PJ P^{-1}. \tag{8}
\]

By substituting (8) into (7), we obtain
\[
\begin{bmatrix} I_n \otimes A - F^T \otimes E & -I_p \otimes B \end{bmatrix} = P \otimes I_n[ -I_n \otimes A + J \otimes E \quad I_p \otimes B ] \otimes \text{diag}( -P^{-1} \otimes I_n, -P^{-1} \otimes I_r ).
\]

Noting that \( P \otimes I_n, P^{-1} \otimes I_n \) and \( P^{-1} \otimes I_r \) are all nonsingular, we have
\[
\text{rank} \left[ I_n \otimes A - F^T \otimes E \quad -I_p \otimes B \right] = \text{rank} \left[ J \otimes E - I_n \otimes A \quad I_p \otimes B \right] = \text{rank} \left[ \begin{bmatrix} s_1 E - A & * & \cdots & 0 & B \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_p E - A & * & \cdots & 0 & B \end{bmatrix} \\ s_1 E - A & B & * & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_p E - A & B \\ \end{bmatrix} \right] = \text{rank} \left[ \begin{bmatrix} s_1 E - A & B & * & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_p E - A & B \end{bmatrix} \right], \tag{9}
\]
where the terms denoted by \( * \) may be zero and \( s_i, i = 1, \ldots, p \), are the eigenvalues of the matrix \( F \), which are not necessarily distinct. It clearly follows from (9) that
\[
\text{rank} \left[ I_n \otimes A - F^T \otimes E \quad -I_p \otimes B \right] = np
\]
if and only if (6) holds. According to linear equation theory, the number of free parameters of the solution \((V, W)\) is then given by
\[
\pi = np + rp - \text{rank} \left[ I_n \otimes A - F^T \otimes E \quad -I_p \otimes B \right] = rp.
\]
Finally, recalling that \( F^T \) and \( F \) have the same Jordan form, we complete the proof. □

The above theorem clearly has the following corollary.

**Corollary 1.** Let \( A, E \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{p \times p}, B \in \mathbb{R}^{n \times r} \), \( \text{rank}(B) = r \). Then the maximum number of free parameters of the solution \((V, W)\) is \( rp \) if one of the following conditions holds:

1. the matrix triple \((E, A, B)\) is \( R \)-controllable,
2. the eigenvalues of \( F \) are different from the finite poles of the system, i.e., \( \Omega \cap \sigma(F) = \emptyset \), where
\[
\Omega = \{ s | \det(sE - A) = 0 \}.
\]

To verify the \( F \)-coprimeness of two polynomial matrices according to Definition 1, we need to get the eigenvalues of the matrix \( F \). This may involve bad conditioning. Before ending this section, we now introduce a criterion which needs only the coefficient matrices \( N_i, D_i, i = 0, 1, \ldots, \omega \) and \( F \).
Lemma 1. Let $F \in \mathbb{R}^{p \times p}$ be an arbitrary matrix, and

$$N(s) = \sum_{i=0}^{\omega} N_i s^i, \quad N_i \in \mathbb{R}^{n \times r},$$

$$D(s) = \sum_{i=0}^{\omega} D_i s^i, \quad D_i \in \mathbb{R}^{r \times r}. \quad (10)$$

Then $N(s)$ and $D(s)$ are $F$-right coprime, if and only if

$$\text{rank}\left[\sum_{i=0}^{\omega} \left[F^i \otimes \left[N_i D_i\right]\right]\right] = rp. \quad (11)$$

The proof of this lemma is given in the appendix.

3. Main results

Let $N(s) \in \mathbb{R}^{n \times r}[s]$ and $D(s) \in \mathbb{R}^{r \times r}[s]$ be some polynomial matrices satisfying

$$(sE - A)N(s) + BD(s) = 0. \quad (12)$$

When $(E, A)$ is regular, i.e., $(sE - A)$ is invertible, the above equation can be rewritten in the form of the following so-called right factorization

$$(A - sE)^{-1}B = N(s)D^{-1}(s). \quad (13)$$

If we denote

$$D(s) = [d_{ij}(s)]_{r \times r}, \quad N(s) = [n_{ij}(s)]_{n \times r} \quad \text{and} \quad \omega_1 = \max (\text{deg} (d_{ij}(s)), i, j = 1, 2, \cdots, r),$$

$$\omega_2 = \max (\text{deg} (n_{ij}(s)), i = 1, 2, \cdots, n, j = 1, 2, \cdots, r),$$

$$\omega = \max (\omega_1, \omega_2),$$

then $N(s)$ and $D(s)$ can be rewritten in the form of (10).

Regarding the general solution to the generalized Sylvester matrix equation (1), we have the following result.

Theorem 2. Let $E$, $A \in \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{n \times r}$, rank $B = r$, and $(sE - A)$ and $B$ be $F$-left coprime. Further, let $N(s)$ and $D(s)$ be in the form of (10) and satisfy (12). Then

1. the matrices $V \in \mathbb{R}^{n \times p}$, $W \in \mathbb{R}^{r \times p}$ given by

$$V = N_0 Z + N_1 ZF + \cdots + N_{\omega} ZF^{\omega},$$
$$W = D_0 Z + D_1 ZF + \cdots + D_{\omega} ZF^{\omega} \quad (14)$$

satisfy the generalized Sylvester matrix equation (1) for arbitrary matrix $Z \in \mathbb{R}^{r \times p}$, and

2. all the matrices $V \in \mathbb{R}^{n \times p}$, $W \in \mathbb{R}^{r \times p}$ satisfying the matrix equation (1) can be parameterized as (14) if and only if $N(s)$ and $D(s)$ are $F$-right coprime.

Proof. Substituting (10) into (12), and comparing the coefficients of $s^i$, gives the following group of relations:

$$-AN_0 + BD_0 = 0, \quad EN_1 - AN_{i+1} + BD_{i+1} = 0,$$
$$i = 0, 1, \ldots, \omega - 1, \quad EN_{\omega} = 0. \quad (15)$$

Now let us show that the matrices $V$ and $W$ given by (14) satisfy the generalized Sylvester matrix equation (1).

Using the first expression in (14), we have

$$AV - EVF = A \left(\sum_{i=0}^{\omega} N_i ZF^i\right) - E \left(\sum_{i=0}^{\omega} N_i ZF^i\right) F$$
$$= (AN_0)Z + \sum_{i=1}^{\omega} (AN_i - EN_{i-1})ZF^i - EN_{\omega}ZF^{\omega+1}. \quad (16)$$

Substituting (15) into (16), and using (14) again, produces

$$AV - EVF = BD_0 Z + \sum_{i=1}^{\omega} BD_i ZF^i$$
$$= B \left(\sum_{i=0}^{\omega} D_i ZF^i\right)$$
$$= BW.$$ 

This states that the matrices $V$ and $W$ given by (14) satisfy the matrix equation (1).

Now let us prove the second conclusion. Putting vec$(\cdot)$ on both sides of (14) and using Eq. (3), we obtain

$$\text{vec}(V) = \sum_{i=0}^{\omega} ((F^T)^i \otimes N_i) \text{vec}(Z),$$

$$\text{vec}(W) = \sum_{i=0}^{\omega} ((F^T)^i \otimes D_i) \text{vec}(Z),$$

or equivalently,

$$\begin{bmatrix} \text{vec}(V) \\ \text{vec}(W) \end{bmatrix} = \sum_{i=0}^{\omega} \left(\sum_{i=0}^{\omega} ((F^T)^i \otimes N_i) \right) \text{vec}(Z). \quad (17)$$

According to Theorem 1, the maximum number of free parameters of the solution $(V, W)$ is $rp$. Recalling the fact that $Z \in \mathbb{R}^{r \times p}$ is an arbitrary parameter matrix, we need only to validate that each element in $Z$ contributes to $(V, W)$ independently if and only if condition (4) is met. It follows from (17) that each element in $Z$ contributes to $(V, W)$ independently if and only if

$$\text{rank} \left[\sum_{i=0}^{\omega} \left(\sum_{i=0}^{\omega} ((F^T)^i \otimes N_i) \right) \right] = rp. \quad (18)$$

Therefore, in the following we suffice only to show that (18) holds if and only if $N(s)$ and $D(s)$ are $F$-right coprime.
By the definition of the Kronecker product, we have
\[
\text{rank} \left[ \sum_{i=0}^{a_0} ((F^T)^j \otimes N_i) \right] = \text{rank} \left[ \sum_{i=0}^{a_0} ((F^T)^j \otimes D_i) \right] = \text{rank} \left[ \sum_{i=0}^{a_0} \left( (F^T)_{ij}N_i \cdots (F^T)_{ip}N_i \right) \right].
\]
By exchanging certain rows of the matrix in the right-hand side of the above equation, we further get
\[
\text{rank} \left[ \sum_{i=0}^{a_0} ((F^T)^j \otimes N_i) \right] = \text{rank} \left[ \sum_{i=0}^{a_0} ((F^T)^j \otimes D_i) \right] = \text{rank} \left[ \sum_{i=0}^{a_0} \left( (F^T)^j \otimes \left[ \begin{array}{c} N_i \\ D_i \end{array} \right] \right) \right],
\]
where \((F^T)^j\) denotes the element in the \(i\)th row and \(j\)th column of the matrix \(F^T\). By Lemma 1, (18) holds if and only if
\[
\text{rank} \left[ \begin{array}{c} N(\lambda) \\ D(\lambda) \end{array} \right] = r \quad \text{for any } \lambda \in \sigma(F^T).
\]
Since \(F\) and \(F^T\) have the same eigenvalues, the above condition is equivalent to the \(F\)-right coprimeness of \(N(s)\) and \(D(s)\). With this we complete the proof. \(\square\)

**Remark 1.** When the right coprime factorization (13) is established, the solution (14) can be written out immediately. Furthermore, this solution allows the matrix \(F\) to be set undetermined. Such a property may give great convenience and advantages to some analysis and design problems in control systems theory. In some practical applications, the matrix \(F\), together with the parameter matrix \(Z\), can be utilized as the design degrees of freedom, and optimized to achieve some additional performance.

**Remark 2.** In order to derive the complete parametric solutions to the generalized Sylvester matrix equation (1), we need the coefficient matrices \(N_i, D_i, i = 1, \ldots, a_0\), of the right factorization \(N(s)\) and \(D(s)\) satisfying (4) and (13). For general numerical algorithms solving such right factorization, one can refer to [1,2], and also [6].

### 4. Example

Consider a generalized Sylvester matrix equation in the form of (1) with the parameters \([6]\)
\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

It is easy to verify that \((E, A, B)\) is \(R\)-controllable, and a pair of right coprime factorization (13) can be obtained as follows:
\[
D(s) = \begin{bmatrix} s^2 & -1 \\ 0 & 1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} -1 & 0 \\ -s & 0 \\ 0 & -1 \end{bmatrix}.
\]

Thus
\[
D_0 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad D_1 = 0, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{(19)}
\]

and
\[
N_0 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \text{(20)}
\]

According to Theorem 2, for an arbitrary matrix \(F \in \mathbb{R}^{p \times p}\), a complete analytical and explicit solution to the generalized Sylvester matrix equation (1) can be parameterized as
\[
V = N_0Z + N_1ZF, \quad W = D_0Z + D_1ZF + D_2ZF^2,
\]
where \(Z \in \mathbb{R}^{2 \times p}\) is an arbitrary parameter matrix. Particularly, let
\[
F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},
\]
and denote
\[
Z = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix},
\]
we get the general solution for this equation as
\[
V = \begin{bmatrix} -\gamma_{11} & -\gamma_{12} \\ -\gamma_{21} & -\gamma_{22} \end{bmatrix}, \quad W = \begin{bmatrix} -\gamma_{21} + \gamma_{11} + \gamma_{12} & -\gamma_{22} + 2\gamma_{12} + \gamma_{11} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}.
\]
Specially choosing
\[
Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]
gives the special solution
\[
V = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.
\]
5. Conclusion

An explicit parametric general solution to the matrix equation \( AV = EVF = BW \), with \( F \) being an arbitrary square matrix, is established, which possesses the following features:

1. It is in a very neat form and can be immediately obtained as soon as a pair of so-called \( F \)-right coprime polynomial matrices satisfying Eq. (12) is derived;
2. It gives all the degrees of freedom to the equation, which are represented by the parameter matrix \( Z \);
3. It does not require the matrix \( F \) to be prescribed, and thus allows the matrix \( F \) to be set undetermined and used as a parameter matrix.

Due to the above advantages, the provided solution may play important roles in linear systems theory.

Acknowledgements

This work has been partially supported by the Chinese Outstanding Youth Foundation under Grant no. 69504002.

Appendix

Let the Jordan form of matrix \( F \) be as follows:

\[
J = \text{Blockdiag} \left( J_1, \ldots, J_w \right),
\]

\[
J_i = \begin{bmatrix}
   s_i & 1 & & \\
   & s_i & \ddots & \\
   & & \ddots & 1 \\
   & & & s_i
\end{bmatrix}_{p_i \times p_i},
\]

where \( s_i, i = 1, \ldots, w \), are obviously the eigenvalues of the matrix \( F \) (which are not necessarily distinct). Further, let the corresponding eigenvector matrix of \( F \) be \( P \), then we have (8). Substituting Eq. (8) into the left side of (11), yields

\[
\sum_{i=0}^{\omega} \left[ F^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} \right] = \sum_{i=0}^{\omega} \left( PJP^{-1} \right)^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} = (P \otimes I_{n+r}) \sum_{i=0}^{\omega} \left[ J_i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} \right] (P^{-1} \otimes I_r).
\]

Since \( P \otimes I_{n+r} \) and \( P^{-1} \otimes I_r \) are nonsingular, we obviously have

\[
\text{rank} \left[ \sum_{i=0}^{\omega} F^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} \right] = \sum_{j=0}^{w} \text{rank} \left[ \sum_{i=0}^{\omega} J_i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} \right].
\]

Define the nilpotent matrix

\[
E_j = \begin{bmatrix} 0 & l_{pj}^{-1} \\ 0 & 0 \end{bmatrix}_{p_j \times p_j}, \quad j = 1, \ldots, w.
\]

It is easy to verify that

\[
E_j^l = \begin{bmatrix} 0 & l_{pj}^{l-1} \\ 0 & 0 \end{bmatrix}, \quad l = 0, 1, \ldots, p_j.
\]

Noting that

\[
J_j = s_j I_{pj} + E_j, \quad j = 1, \ldots, w,
\]

and the matrices pair \( s_j I_{pj} \) and \( E_j \) are commutative, by using the binomial theorem, we further have

\[
J_j^l = s_j^l I_{pj} + s_j^{l-1} C_j E_j + \cdots + s_j^0 C_j E_j^l, \quad j = 1, \ldots, w.
\]

Substituting the above matrices into (21), gives

\[
\sum_{i=0}^{\omega} \left[ J_j^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} \right] = \sum_{i=0}^{\omega} [s_j^i I_{pj} + \cdots + s_j^0 C_j E_j^i] \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} = \sum_{i=0}^{\omega} [s_j^i I_{pj} C_j^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}]
\]

\[
\quad + \sum_{i=0}^{\omega-1} s_j^i E_j C_j^i \otimes \begin{bmatrix} N_i+1 \\ D_i+1 \end{bmatrix} + \cdots + \sum_{i=0}^{0} s_j^i E_j^{\omega-1} C_j^{\omega-1} \otimes \begin{bmatrix} N_i+\omega-1 \\ D_i+\omega-1 \end{bmatrix}.
\]

Denote

\[
\theta_j^{\omega-k} = \sum_{i=0}^{k} [s_j^i C_j^{\omega-k} \otimes \begin{bmatrix} N_i+\omega-k \\ D_i+\omega-k \end{bmatrix}], \quad k = 0, \ldots, \omega.
\]

Comparing (10) with the expression of \( \theta_j^0 \), we clearly have

\[
\theta_j^0 = \begin{bmatrix} N(s_j) \\ D(s_j) \end{bmatrix}.
\]
So (21) can be simplified as
\[
\sum_{i=0}^{\omega} \left[ J_i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} \right] = I_{p_j} \otimes \theta_j^0 + \cdots + E_j^0 \otimes \theta_j^\omega.
\]

Since the above matrix has full column rank if and only if \( \theta_j^0, j = 1, 2, \ldots, w, \) has full column rank, i.e., relation (4) holds, the conclusion of the lemma clearly holds true.

References