Brief paper

On robustness of predictor feedback control of linear systems with input delays

Zhao-Yan Li, Bin Zhou, Zongli Lin

Abstract

This paper is concerned with the robustness of the predictor feedback control of linear systems with input delays. By applying certain equivalent transformations on the characteristic equation associated with the closed-loop system, we first transform the robustness problem of a predictor feedback control system into the stability problem of a neutral time-delay system containing an integral operator in the derivative. The range of the allowable input delay for this neutral time-delay system can be computed by exploring its delay dependent stability conditions. In particular, delay dependent stability conditions for the neutral time-delay system are established by partitioning the delay into segments. The conservatism of this method can be reduced when the number of segments in the partition is increased. Numerical examples are worked out to illustrate the effectiveness of the proposed method.

1. Introduction

In the past several decades, time-delay systems have attracted much attention because of their many applications. Many control problems for time-delay systems, especially, the stability and stabilization problems, have been extensively studied for years (see, for example, Cong & Yin, 2012, Gu & Niculescu, 2003, Gu, Kharitonov, & Chen, 2003, Gu, Zhang, & Xu, 2011, He, Wang, Lin, & Wu, 2007, Lam, Gao, & Wang, 2007, Li & De Souza, 1997, Xu, Lam, & Yang, 2002 and the references therein). Roughly speaking, studies of time-delay systems can be classified into two categories: those that deal with systems with state delays (Fridman, 2001; Li & De Souza, 1997; Xu et al., 2002; Zhou, Li, Zheng, & Duan, 2012) and those that deal with systems with input/output delays (Yue & Lam, 2004; Zhou, Li, & Lin, 2012; Zhou, Lin, & Duan, 2010, 2012).

Many memoryless controllers have been designed in the literature to deal with the input delayed systems (see, for instance, Kim, Jeung, & Park, 1996, Kolmanovskii & Myshkis, 1992, Lin & Fang, 2007, Yoon & Lin, 2013 and Zhou et al., 2012). Memoryless controllers are easy to implement. However, they may fail to control the systems that are open-loop unstable and/or when the delays are large. On the other hand, controllers with memory in general lead to better performances of the closed-loop systems than memoryless controllers. One of the most efficient approach for designing memory controllers is the so-called model reduction method, which is also known as predictor feedback and finite spectrum assignment (see Chen & Zheng, 2006, Cheres, Palmor, & Gutman, 1990, Krstic, 2010 and the references cited there). This class of controllers, however, may suffer some implementation problems. More information on the implementation and applications of this class of controllers can be found in Lozano, Castillo, and Dzul (2004), Mirkin and Raskin (2003), Zhou, Li, Lin (2012), Zhou et al. (2012) and the references therein.

Despite of the voluminous literature on stability analysis and stabilization, few results are available on the robustness of the predictor feedback control of time-delay systems. Krstic (2008) is probably the first paper that studies the robustness of the predictor feedback control systems with input delay with respect to small (constant) perturbations in the input delay. However, the allowable upper and lower perturbation bounds are only proven to exist while their computation method is quite conservative. Very recently, Karafyllis and Krstic (2013) presented a robustness analysis for the predictor feedback control of input delayed linear time-invariant systems with possibly time-varying perturbations.
in the input delay. The computation of the allowable upper and lower perturbation bounds remains conservative as norms of matrices are involved.

In this paper, we will study the robustness of the predictor feedback control of a linear system with input delay and constant perturbations in the delay. We will transform the problem into one of stability analysis for a neutral differential equation whose right-hand side is delay-free and left-hand side is recognized as an integral operator. A linear-matrix-inequalities (LMI)-based delay-dependent sufficient condition guaranteeing the stability of this neutral time-delay system is then established by combining the Lyapunov–Krasovskii functional approach and the delay partition technique. The allowable upper and lower bounds of the perturbation on the delay are obtained by testing the LMIs-based conditions with a linear search technique. Three numerical examples show that the obtained bounds are significantly less conservative than those obtained by the approaches reported in the literature and are very close to the exact bounds.

The remainder of this paper is organized as follows. The problem formulation and some preliminary results are presented in Section 2. Our main results are then given in Section 3 and a couple of numerical examples are worked out in Section 4. Section 5 concludes the paper. Finally, the proofs for several technical results and some technical lemmas are collected in the Appendix.

Notation: For a matrix A, we use $A^T$ and $\text{He}(A)$ to denote, respectively, its transpose and the symmetric matrix $A + A^T$. For a matrix $P \succeq 0$, the symbols $\lambda_{\text{min}}(P)$ and $\lambda_{\text{max}}(P)$ represent, respectively, its minimal and maximal eigenvalues. We use $0_{n \times m}$ to denote a zero matrix with dimensions $n \times m$. The symbol $\| \cdot \|$ refers to the Euclidean norm and the spectrum norm. The symbol $\otimes$ denotes the Kronecker product of two matrices. Let $B \in \mathbb{R}^{n \times m}$ with rank $(B) = p < m$. Then the matrix $B^\perp \in \mathbb{R}^{m \times (m-p)}$ with rank $m-p$ determines the right orthogonal complement of $B$. Finally, suppose $r > 0$ is a given real number; we let $\mathcal{C}_{r,n} = C([-r, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Define the norm of an element $\phi$ in $\mathcal{C}_{r,n}$ by $\| \phi \| = \sup_{-r \leq \theta \leq 0} | \phi (\theta) |$. Moreover, we denote $x_t (\theta) = x(t + \theta) \in \mathcal{C}_{r,n}$, $-r \leq \theta \leq 0$.

2. Problem formulation and preliminaries

We consider the following linear system with input delay

$$\dot{x}(t) = Ax(t) + Bu(t - r), \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant matrices and $r \geq 0$ is a constant scalar that may be unknown. Let $(A, B)$ be stabilizable. Assume that the nominal value of $r$ is $r^*$ which is exactly known. By using the nominal value $r^*$, the predictor feedback for the time-delay system (1) can be designed as (see, for example, Krstic, 2008)

$$u(t) = K \left( e^{r^*t} x(t) + \int_{t-r^*}^{0} e^{r^*(t-s)} Bu(t + s) \, ds \right), \quad (2)$$

where $K$ is chosen such that $A + BK$ is Hurwitz.

In this paper, we are interested in the stability of the closed-loop system consisting of (1) and (2). It is well known that if $r = r^*$, then the closed-loop system consisting of (1) and (2) is delay-free and asymptotically stable. However, if $r \neq r^*$, the closed-loop system is not delay-free and may be unstable if $|r - r^*|$ is sufficiently large. To investigate the stability of the closed-loop system (1) and (2), we first present the following lemma, which is equivalent to Corollary 2.3 in Karafyllis and Krstic (2013). For completeness, an alternative simple proof in the frequency domain is provided in Appendix A.1.

**Lemma 1.** The closed-loop system consisting of (1) and (2) is asymptotically stable if and only if

$$\dot{y}(t) = (A + BK) y(t) + BK e^{r^*t} \left( y(t - r) - y(t - r^*) \right). \quad (3)$$

is asymptotically stable.

Motivated by the above lemma, to study the stability of the closed-loop system consisting of (1) and (2), we study in this paper the stability of the following linear time-delay system

$$\dot{x}(t) = Fx(t) + G \left( x(t - r) - x(t - r^*) \right), \quad (4)$$

where $F, G \in \mathbb{R}^{n \times n}$ are constant matrices and $(r, r^*)$ is a pair of non-negative known scalars. Particularly, the problem to be studied is stated as follows.

**Problem 1.** For a fixed positive scalar $r^*$ and two given matrices $F$ and $G$ with $F$ being Hurwitz, find two non-negative scalars $r^* > r^*$ and $r^* < r^*$ such that the time-delay system (4) is asymptotically stable for all $r \in [r^*, r^*]$.

**Remark 1.** Even when $r = r^*$, the stability of the closed-loop system consisting of (1) and (2) when the controller (2) is implemented by numerical integration is related with the stability of a class of integral delay systems determined by (2) (see, for example, Engelborghs, Dambrine, & Roose, 2001 and Mondie, Dambrine, & Santos, 2002). In this paper, we do not address this problem but assume that the controller in (2) is implemented accurately.

Notice that the linear time-delay system (4) can be equivalently written as the following neutral time-delay system

$$\dot{\psi}(t) = f(t, x_t), \quad (5)$$

in which $f(t, x_t) = Fx(t)$ and

$$\psi(t) = x(t) + G \int_{t-r^*}^{t-r} x(s) \, ds. \quad (6)$$

Thus, the stability of system (4) is equivalent to the stability of system (5). Regarding the stability of (5), we have the following result cited from Hale (1977).

**Lemma 2.** Suppose that the integral delay system

$$\psi(t) = 0 \text{ is stable}, \quad f : \mathbb{R} \times [\max(r^*)] \rightarrow \mathbb{R}^n, \text{ and } \gamma(s), \beta(s) \text{ and } \delta(s) \text{ are continuous, nonnegative, and nondecreasing with } \gamma(s), \beta(s) > 0 \text{ for } s \neq 0, \text{ and } \gamma(0) = \beta(0) = 0. \text{ If there is a continuous function } V : \mathbb{R} \times [\max(r^*)] \rightarrow \mathbb{R} \text{ such that}

$$
\gamma(\| \psi(t) \|) \leq V(t, x_t) \leq \beta(\| x_t \|), \quad (7)
$$

then the solution $x = 0$ of the neutral time-delay system (5) is uniformly stable. If $\delta(s) > 0$ for $s > 0$, then the solution $x = 0$ of the neutral time-delay system (5) is uniformly asymptotically stable. The same conclusion holds if the upper bound on $V(t, x_t)$ is replaced by $-\delta(\| x(t) \|)$.

In view of Lemma 2, to obtain conditions that guarantee the stability of the neutral time-delay system (5), we will analyze the stability of the integral delay system $\psi(t) = 0$, which is a special case of the following integral delay system (Melchor-Aguilar, Kharitonov, & Lozano, 2010)

$$x(t) = \int_{t-h}^{t} L(\theta) x(t + \theta) \, d\theta, \quad (9)$$

where $h > 0$ is a constant and the matrix function $L(\theta)$ has piecewise continuous bounded elements defined in $[-h, 0]$. We have the following lemma provided in Melchor-Aguilar et al. (2010) regarding the stability of (9).
Lemma 3. The integral delay system (9) is exponentially stable if there exists a function $V : \mathbb{C}_{x,n} \to \mathbb{R}$ such that $V(x(t))$ is differentiable on $\mathbb{R}^n$ and the following conditions hold:

1. $\alpha_1 \int_{-\theta}^{0} |x(t+\theta)|^2 \, d\theta \leq V(x(t)) \leq \alpha_2 \int_{-\theta}^{0} |x(t+\theta)|^2 \, d\theta$, for some $0 < \alpha_1 \leq \alpha_2$.
2. $V(x(t)) \leq -\alpha_3 \int_{-\theta}^{0} |x(t+\theta)|^2 \, d\theta$, for some $\alpha_3 > 0$.

3. Main results

Throughout this section, we denote $F = A + BK$ and $G = BK e^{\beta\theta}$. For the neutral time-delay system (5), we first discuss, without loss of generality (see Remark 2 given later), the case that $r^* \geq r$. Denote

$$
\pi(s) = \begin{bmatrix}
x(s)
\left(s - \frac{1}{N} (r^* - r) \right)
\vdots
\left(s - \frac{N-1}{N} (r^* - r) \right)
\end{bmatrix},
$$

where $N \geq 1$ is a given integer. In addition, we define

$$
\Gamma_1 = \begin{bmatrix}
0_{n \times n} & I_n
\end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix}
0_{n \times 2n} & I_{2n}
\end{bmatrix},
$$

$$
\Gamma_3 = \begin{bmatrix}
I_n & 0_{n \times (N+1)n}
\end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix}
0_{n \times n} & I_n & 0_{n \times Nn}
\end{bmatrix}.
$$

Then we can present the following lemma which gives conditions guaranteeing the stability of the integral delay system $\phi(t) = 0$ defined in (6).

**Lemma 4.** Assume that $r^* \geq r$. Then the integral delay system $\phi(t) = 0$ defined in (6) is asymptotically stable if there exist a positive integer $N$, and matrices $T_1, Q_1 \in \mathbb{R}^{n \times n}, T_2, Q_2 \in \mathbb{R}^{n \times n}, S \in \mathbb{R}^{n \times n}$ such that $S > 0$, $T_1 > 0$, $T_2 > 0$, $Q_2 > 0$, and the following two LMIs

$$
\mathcal{Z}_{\epsilon_1} + \mathcal{Z}_{\epsilon_2} + \mathcal{Z}_{Q_1} + \mathcal{Z}_{Q_2}
$$

are satisfied, in which $\epsilon_i$ denotes a column vector whose $i$th column is 1 and the other elements are zero and

$$
\mathcal{Z}_{T_1} = \Gamma_1^T T_1 \Gamma_1 - \Gamma_1^T T_2 \Gamma_2, \quad \mathcal{Z}_{T_2} = \Gamma_2^T T_2 \Gamma_2 - \Gamma_2^T T_4 \Gamma_4,
$$

$$
\mathcal{Z}_{Q_1} = \left( r^* - r \right)^2 \Gamma_1^T Q_1 \Gamma_1, \quad \mathcal{Z}_{Q_2} = r \Gamma_2^T Q_2 \Gamma_2.
$$

**Proof.** From (14) we know that $Q_1 \geq 0$. Hence the following Lyapunov–Krasovskii functional

$$
V(x(t)) = \int_{t-r}^{t-r} \pi^T(s) \Phi(t,s) \pi(s) \, ds + \int_{t-r}^{t} x^T(s) (T_2 + (r + s - t) Q_2) x(s) \, ds,
$$

where

$$
\Phi(t,s) = T_1 + \left( \frac{1}{N} (r^* - r) + s - (t - r) \right) (r^* - r) Q_1.
$$

satisfies Condition 1 of Lemma 3. Then direct computation gives

$$
\dot{V}(x(t)) = \pi^T(t-r) \left( T_1 + \frac{1}{N} (r^* - r) Q_1 \right) \pi(t-r) - \pi^T(t-r - \frac{1}{N} (r^* - r)) T_1 \pi \left( t-r - \frac{1}{N} (r^* - r) \right)
$$

$$
+ x^T(t) (T_2 + Q_2) x(t) - x^T(t-r) T_2 x(t-r)
$$

$$
+ q_2(t) - (r^* - r) \int_{t-r}^{t} \pi^T(s) Q_1 \pi(s) \, ds,
$$

where $q_2(t) = -\int_{t-r}^{t} x^T(\theta) Q_2 x(\theta) \, d\theta$. If we define a generalized state vector as

$$
\xi(t) = \begin{bmatrix}
x^T(t) 
\pi^T(t-r) 
x^T(t-r^*)
\end{bmatrix}^T,
$$

then $\dot{V}(x(t))$ can be written as

$$
\dot{V}(x(t)) = \xi^T(t) \left( \mathcal{Z}_{\epsilon_1} + \mathcal{Z}_{\epsilon_2} + \mathcal{Z}_{Q_1} + \mathcal{Z}_{Q_2} \right) \xi(t) + q_2(t)
$$

$$
- (r^* - r) \int_{t-r}^{t} \pi^T(s) Q_1 \pi(s) \, ds,
$$

where $\mathcal{Z}_{\epsilon_1}, \mathcal{Z}_{\epsilon_2}, \mathcal{Z}_{Q_1}$, and $\mathcal{Z}_{Q_2}$ are defined in (15) and (16). By using the Jensen inequality (Lemma 7 in Appendix A.3), it follows from (13) and (14) that

$$
\dot{V}(x(t)) \leq \xi^T(t) S \xi(t) + q_2(t)
$$

$$
- (r^* - r) \int_{t-r}^{t} \pi^T(s) Q_1 \pi(s) \, ds
$$

$$
= \left( \int_{t-r}^{t} \xi^T(\theta) S \xi(\theta) \, d\theta \right) + q_2(t)
$$

$$
- (r^* - r) \int_{t-r}^{t} \pi^T(s) Q_1 \pi(s) \, ds
$$

$$\leq \xi^T(t) S \xi(t) + q_2(t)
$$

$$
- (r^* - r) \int_{t-r}^{t} \pi^T(s) Q_1 \pi(s) \, ds
$$

$$= \left( \int_{t-r}^{t} \xi^T(\theta) S \xi(\theta) \, d\theta \right) + q_2(t)
$$

$$\leq -\epsilon_1 (r^* - r) \int_{t-r}^{t} \xi^T(\theta) S \xi(\theta) \, d\theta + q_2(t),
$$

where $\epsilon_1$ is some constant and we have used the inequality (14). Then, for both $r^* = r$ and $r^* > r$, the second condition of Lemma 3 is satisfied and hence the integral delay system $\phi(t) = 0$ defined in (6) is asymptotically stable. The proof is completed.

We are now ready to study the stability of the neutral time-delay system (5). Assume again, without loss of generality, that $r^* \geq r$ in system (5). For a given integer $N \geq 1$, we denote

$$
g = \begin{bmatrix}
G & G & \cdots & G
\end{bmatrix} \in \mathbb{R}^{n \times n},
$$

$$
K_N = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0
0 & 1 & -1 & \cdots & 0
\vdots & \vdots & \vdots & \ddots & \vdots
0 & 0 & 0 & \cdots & 1
\end{bmatrix} \in \mathbb{R}^{n \times (N+1)},
$$

and a series of matrices $A_i, i = 1, 2, \ldots, N$, as follows

$$
A_1 = \begin{bmatrix}
0_{n \times n} & I_n & 0_{n \times 4n}
\end{bmatrix},
$$

$$
A_2 = \begin{bmatrix}
I_n & 0_{n \times 4n+1}
\end{bmatrix}.
$$
$A_1 = \begin{bmatrix} 0_{n \times n} & 0_{n \times 2n} & l_{1n} \\ 0_{2n \times (n+1)n} & l_{2n} & 0_{2n \times 2n} \end{bmatrix}$, \quad (27)

$A_2 = \begin{bmatrix} 0_{n \times n} & 0_{n \times 2n} & l_{1n} \\ 0_{2n \times (n+1)n} & l_{2n} & 0_{2n \times 2n} \end{bmatrix}$, \quad (28)

$A_3 = \begin{bmatrix} 0_{2n \times (n+1)n} & l_{1n} \\ 0_{n \times 2n} & 0_{n \times 2n} \end{bmatrix}$, \quad (29)

$A_4 = \begin{bmatrix} 0_{2n \times (n+1)n} & l_{1n} \\ 0_{n \times 2n} & 0_{n \times 2n} \end{bmatrix}$, \quad (30)

$A_5 = \begin{bmatrix} 0_{2n \times (n+1)n} & l_{1n} \\ 0_{n \times 2n} & 0_{n \times 2n} \end{bmatrix}$, \quad (31)

$A_6 = \begin{bmatrix} 0_{n \times 4n} & l_{1n} \\ 0_{n \times 2n} & -l_{1n} \end{bmatrix}$, \quad (32)

$A_7 = \begin{bmatrix} 0_{n \times 4n} & l_{1n} \\ 0_{n \times 2n} & -l_{1n} \end{bmatrix}$, \quad (33)

$A_{10} = \begin{bmatrix} 0_{n \times (N+1)n} & l_{1n} \\ 0_{n \times (N+1)n} & 0_{n \times (4N-1)n} \end{bmatrix}$, \quad (34)

We have the following result regarding the stability of the neutral

time-delay system (5).

**Theorem 1.** Assume that $r^* \geq r$. Then the neutral time-delay

system (5) is asymptotically stable if there exist an integer $N \geq 1$, 

positive definite matrices $S, P, U, T_2, Q_2, Y \in \mathbb{R}^{n \times n}$, $T, Z, Q \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{2n \times 2n}$ and matrices $Q_1 \in \mathbb{R}^{Nn \times Nn}$ such that the

LMIs (13), (14) and the following LMI

$$
(L^*)^T (Q_D + \Omega + Q_D + \Omega_G + Q_G + \Omega_G + \Omega_Y) L^* < 0,
$$

are satisfied, where

$$
Q_D = A_1^T (P^T F + F^T P) A_1 - 2 \Re \{A_1^T F P \beta \},
$$

$$
Q_T = A_1^T A_1 - A_1^T T A_4,
$$

$$
Q_G = \frac{(r^* - r)^2}{N} A_1^T Q A_1 - N A_1^T Q A_2,
$$

$$
Q_W = \frac{(r^* - r)^2}{N} A_1^T W A_5 - N A_1^T W A_6,
$$

$$
Q = A_1^T Z A_1 - A_1^T Z A_2,
$$

$$
Q_9 = \frac{1}{N} A_1^T U A_6 + r A_1^T A_2,
$$

$$
Q_Y = A_1^T Y A_1 - A_1^T Y A_1,
$$

in which $A_1$ are given by (25)-(34) and $L^* \in \mathbb{R}^{2n \times 2n}$ takes the

following form

$$
L^* = \begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix} = \begin{bmatrix} I_{2n \times (n+1)n} & 0_{2n \times 2n} \\ 0_{2n \times (n+1)n} & I_{2n \times 2n} \end{bmatrix},
$$

with

$$
L_{11} = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} \end{bmatrix}, \quad (44)
$$

$$
L_{12} = \begin{bmatrix} I_{2n \times 2n} & 0_{2n \times 2n} \end{bmatrix}, \quad (45)
$$

**Proof.** Since the set of LMIs stated in this theorem contains those in

Lemma 4, it follows from Lemma 4 that the integral delay system $s \varphi (t) = 0$ defined in (6) is asymptotically stable. So in the following we will apply Lemma 3 to test the stability of the neutral time-delay system (5). Choose the following Lyapunov–Krasovskii functional

$$
V (x_i) = \sum_{i=1}^{6} V_i (x_i),
$$

in which $V_i (x_i), i = 1, 2, \ldots, 6, $ are defined as

$$
V_1 (x_i) = \psi (t) P \psi (t),
$$

$$
V_2 (x_i) = \int_{t-r^*}^{t} \pi (\theta) \Phi (t, s) \pi (\theta) d\theta,
$$

$$
V_3 (x_i) = \int_{t-r^*}^{t} \pi (\theta) Z \pi (\theta) d\theta,
$$

$$
V_4 (x_i) = (r^* - r) \int_{t-r^*}^{t} \pi (\theta) W \pi (\theta) d\theta ds,
$$

$$
V_5 (x_i) = \int_{t-r^*}^{t} x^T (\theta) Y x (\theta) d\theta,
$$

$$
V_6 (x_i) = \int_{t-r^*}^{t} x^T (\theta) U \dot{x} (\theta) d\theta ds,
$$

with $\Phi (t, s) = T + \frac{1}{2} (r^* - r) + \frac{1}{2} (s - (t - r)) (r^* - r)Q$ and $\pi (\theta) = [\pi (\theta - r), \pi (\theta - r^*)]^T$. It is obvious that $V_i (x_i) \geq \lambda_{\text{max}} (P) \psi (t)^2$. On the other hand, it is not hard to show that there exists a $\beta > 0$ such that $V_i (x_i) \leq \beta \|x_i\|^2, i = 1, 2, 3, 4, 5, 6$. We show that this inequality also holds for $i = 4$. Notice that

$$
V_4 (x_i) = \delta \beta \int_{t-r^*}^{t} \pi (\theta) \pi (\theta - r) d\theta,
$$

where $\beta = \frac{1}{2} (r^* - r).$ By some computation we can get

$$
\int_{t-r^*}^{t} \pi (\theta) \pi (\theta - r) d\theta = \sum_{i=1}^{N} \int_{t-r^*}^{t} \pi (\theta - r) \pi (\theta - r) d\theta
$$

$$
= \sum_{i=1}^{N} \int_{t-r^*}^{t} \frac{1}{i} \pi (\theta - r) d\theta
$$

$$
= \sum_{i=1}^{N} \int_{t-r^*}^{t} \frac{1}{i} \pi (\theta - r) d\theta
$$

$$
\leq (N + 1) \int_{t-r^*}^{t} \pi (\theta) d\theta
$$

Similarly, we have

$$
\int_{t-r^*}^{t} \pi (\theta - r^*) \pi (\theta - r^*) d\theta
$$

$$
\leq (N + 1) \int_{t-r^*}^{t} \pi (\theta) d\theta
$$

Thus, it follows from (53)-(56) that

$$
V_4 (x_i) \leq \frac{2r^*}{N} (r^* - r)^2 \lambda_{\text{max}} (W) \|x_i\|^2.
$$

Notice that the delay interval has been expanded as $ [-2r^*, 0]$. Next, we need to prove

$$
\dot{V} (x_i) \leq -\delta \|x (t)\|^2,
$$
for some $\delta > 0$. Direct computation gives

$$
\dot{V}_1 (x_t) = 2 p^T (t) P \dot{w} (t) = x^T (t) (PF + F^T P) x (t) - 2x^T (t) F \int_{t-r-\frac{1}{2}}^{t-r} \pi (\theta) \, d\theta ,
$$

(58)

$$
\dot{V}_2 (x_t) = \pi^T (t-r) \left( T + \frac{1}{N} (r^* - r)^2 Q \right) \pi (t-r) - (r^* - r) \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) Q \pi (\theta) \, d\theta - \pi^T \left( t - r - \frac{1}{N} (r^* - r) \right) T \pi \times \left( t - r - \frac{1}{N} (r^* - r) \right) \leq \pi^T (t-r) \left( T + \frac{1}{N} (r^* - r)^2 Q \right) \pi (t-r) - N \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) \, d\theta \pi (t-r) \pi^T (\theta) \, d\theta - N \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) \, d\theta \pi (t-r) \pi^T (\theta) \, d\theta \leq \pi^T (t-r) \left( T + \frac{1}{N} (r^* - r)^2 Q \right) \pi (t-r)
$$

(59)

$$
\dot{V}_3 (x_t) = \pi^T (t-r) Z \pi (t-r) - \pi^T (t-r^*) Z \pi (t-r^*) ,
$$

(60)

$$
\dot{V}_4 (x_t) = \frac{1}{N} (r^* - r)^2 \pi^T (t) \pi (t) + (r^* - r) \int_{t-r}^{t} \pi^T (\theta) \, d\theta - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta = \frac{1}{N} (r^* - r)^2 \pi^T (t) \pi (t) - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta \leq \frac{1}{N} (r^* - r)^2 \pi^T (t) \pi (t) - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta \leq \frac{1}{N} (r^* - r)^2 \pi^T (t) \pi (t) - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta - \int_{t-r-\frac{1}{2}}^{t-r} \pi^T (\theta) (r^* - r) \pi (t-r) \, d\theta
$$

(61)

$$
\dot{V}_5 = x^T (t) \dot{y} (t) - x^T (t-r) \dot{y} (t-r) ,
$$

(62)

and

$$
\dot{V}_6 (x_t) = r^* \ddot{x}^T (t) U \ddot{x} (t) - \int_{t-r}^{t} \ddot{x}^T (s) U \ddot{x} (s) \, ds \leq r^* \ddot{x}^T (t) U \ddot{x} (t) - \frac{1}{r^*} (x(t) - x(t-r^*))^T U (x(t) - x(t-r^*)) ,
$$

(63)

where we have used the Jensen inequality in Lemma 7 in Appendix A.3. Define a vector $\eta (t)$ as follows

$$
\eta (t) = \begin{bmatrix}
\int_{t-r-\frac{1}{2}}^{t-r} \pi (\theta) \, d\theta \\
\int_{t-r-\frac{1}{2}}^{t-r} \pi (\theta) \, d\theta \\
\end{bmatrix}
$$

(64)

Then we can write

$$
\dot{V} (x_t) \leq \eta^T (t) \Omega \eta (t) ,
$$

(65)

in which $\Omega = \Omega_2 + \Omega_2 + \Omega_w + \Omega_u + \Omega_y$. It is obviously from the Newton–Leibniz formula that, for $i = 1, 2, \ldots, N$,

$$
\psi \left( t - r - \frac{i-1}{N} (r^* - r) \right) - \psi \left( t - r - \frac{i}{N} (r^* - r) \right)
$$

(66)

which means that

$$
x \left( t - r - \frac{i-1}{N} (r^* - r) \right) + G \int_{t-r-\frac{1}{2}}^{t-r} \pi (r^* - r) \, x (s) \, ds - x \left( t - r - \frac{i}{N} (r^* - r) \right) - G \int_{t-r-\frac{1}{2}}^{t-r} \pi (r^* - r) \, x (s) \, ds
$$

(67)

which is further equivalent to

$$
x \left( t - r - \frac{i-1}{N} (r^* - r) \right) - x \left( t - r - \frac{i}{N} (r^* - r) \right) + G \int_{t-r-\frac{1}{2}}^{t-r} \pi (r^* - r) \, x (s) \, ds - x \left( t - r - \frac{i}{N} (r^* - r) \right) - G \int_{t-r-\frac{1}{2}}^{t-r} \pi (r^* - r) \, x (s) \, ds
$$

(68)

By applying some variable substitutions, for any $i = 1, 2, \ldots, N$, we can get

$$
G \int_{t-r-\frac{1}{2}}^{t-r} \left( \theta - r^* - \frac{i-1}{N} (r^* - r) \right) \theta \, d\theta
$$

(69)

which can be rewritten in the following compact form

$$
(l_N \otimes G) \int_{t-r-\frac{1}{2}}^{t-r} \left( \theta - r^* - \frac{i-1}{N} (r^* - r) \right) \theta \, d\theta + \pi (t-r) - \pi \left( t - r - \frac{1}{N} (r^* - r) \right)
$$

(70)

It follows from (70) that

$$
L \eta (t) = 0 ,
$$

(71)

where $L$ is defined as

$$
L = [ -I_N \otimes F, -I_N \otimes G, I_N \otimes G ] ,
$$

(72)

$$
\Xi = [ 0_{n \times n}, K_N \otimes I_n, 0_{n \times (N-1)n} ] .
$$

(73)
It is easy to verify that
\[
\begin{align*}
[-I_N \otimes \mathbf{F}] \mathbf{Z}^+ &= \left[ I_{(2N+1)} \otimes \mathbf{F} \right], \\
[-I_N \otimes \mathbf{F}] \left[ I_{(2N+1)} \otimes \mathbf{F} \right]^{-1} - I_N \otimes \mathbf{G} &= 0, \\
[-I_N \otimes \mathbf{F}] \left[ I_{(2N+1)} \otimes \mathbf{F} \right]^{-1} \mathbf{I}_N \otimes \mathbf{G} &= 0.
\end{align*}
\]

Therefore, by using Lemma 6 in Appendix A.3, we can get \( L^1 \) as expressed in (43). Moreover, from inequality (35), (71) and Lemma 5 in Appendix A.3, we can get
\[
\dot{V}(x) \leq \eta^2(t) \Omega \eta(t) < 0,
\]
which implies (57). The asymptotic stability of the neutral time-delay system (5) by swapping \( r^* \) and \( r \) in Theorem 1.

Remark 2. In the above we have assumed that \( r \leq r^* \). When \( r > r^* \), we can get the same LMI conditions to guarantee the stability of the neutral time-delay system (5) by swapping \( r^* \) and \( r \) in Theorem 1.

Notice that if \( r = r^* \), the time-delay system (5) becomes \( \dot{x}(t) = Fx(t) \) and is obviously asymptotically stable since \( F \) is Hurwitz. This fact suggests that the set of LMIs in Theorem 1 may be feasible for any \( r^* \geq 0 \) when \( r = r^* \). This is indeed the case as stated in the following proposition whose proof is provided in Appendix A.2 for clarity.

Proposition 1. If \( r = r^* \), then the LMIs in Theorem 1 are feasible for any \( r^* > 0 \).

We also give the following proposition regarding the continuity of the LMIs in Theorem 1 with respect to \( r \).

Proposition 2. If the LMIs in Theorem 1 are feasible for some \( r = r^* > 0 \), then they are also feasible for any \( r \in [r^*, r^*] \).

Proof. Let \( (S^1, P^1, U^1, T^1, Q^1, Y^1, T^1, Z^1, Q^1, W^1, Q^1) \) be a feasible solution to the LMIs in Theorem 1 with \( r = r^* \). Notice that for any \( r \in [r^*, r^*] \), we have
\[
\frac{(r^* - r)^2}{N} \leq \frac{(r^* - r)^2}{N}.
\]

It then follows from \( Q_1 \geq 0, Q > 0, W > 0 \) that \( (S^1, P^1, U^1, T^1, Q^1, Y^1, T^1, Z^1, Q^1, W^1, Q^1) \) is a feasible solution to the LMIs in Theorem 1 with any \( r \in [r^*, r^*] \), where \( Q_2 = r_1^2 \cdot Q_1 > 0 \) is such that \( rQ_2 = r^*Q_2 \). The proof is completed.

For a given \( r^* \), by the continuity property stated in Proposition 2, a pair of \( (r^-, r^+) \) that solves Problem 1 can be computed by using a bisection approach based on solving the set of LMIs in Theorem 1.

4. Numerical examples

In this section, we present three examples to demonstrate the effectiveness of the proposed approaches.

Example 1. We consider a scalar time-delay system in the form of (1) with \( A = 1, B = 1, r^* = 1 \) and \( K = -p \), where \( p > 1 \). This example is taken from Karayiannis and Krstic (2013).

Example 2. We consider a planar time-delay system in the form of (1) with
\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K^T = \begin{bmatrix} -1 \\ -3 \end{bmatrix},
\]
and \( r^* = 1 \). It follows that \( \lambda(A) = \{ \frac{1}{2}(1 \pm \sqrt{3}) \} \) and \( \lambda(A + BK) = \{ -1 \pm j \} \).

Example 3. We consider a third-order time-delay system in the form of (1) with \( r^* = 1 \) and
\[
A = \begin{bmatrix} -2 & -1 & 5 \\ 0 & -1 & 0 \\ 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix},
\]
\[
F = \begin{bmatrix} 0.3572 & -0.4853 & 1.1281 \\ 0.3925 & -0.566 & 0.4235 \end{bmatrix},
\]
in which \( \lambda(A) = \{-1, -1, 1\} \) and \( F \) is obtained such that \( \lambda(A + BF) = \{-1 \pm j, -2\} \) by the Matlab function place.

For Example 1, with \( p \in (1, 10) \), the obtained upper bound \( r^+ \) and the lower bound \( r^- \) for \( r \) by using, respectively, the method in Karayiannis and Krstic (2013) and the method in this paper with \( N = 1 \) and \( N = 5 \) are recorded in Fig. 1. Also shown in this figure are the exact upper and lower bounds for \( r \) computed by using the software package DDE-BIFTOOL (Engelborghs, Luzyanina, & Samaey, 2001). The upper and lower bounds \( r^+ \) and \( r^- \) in Example 1 with \( p = 2 \), Examples 2 and 3 by using the approach in the present paper with \( N = 1, 3, 5 \), the approach in Karayiannis and Krstic (2013), the discretized Lyapunov functional approach in Li and Gu (2010) and the software package DDE-BIFTOOL (Engelborghs, Luzyanina et al., 2001) are also listed in Table 1. For the approach in Li and Gu (2010), the numbers of gridding segments in \([-r, 0] \) and \([r^*, 0] \) are 3. For comparison purpose, the computational time \( \tau_p \), which denotes the time used for finding a feasible solution with \( r = r^* \) (Intel i5-3320M CPU @2.6 GHz), by using different approaches is also recorded in Table 1.

In Fig. 1 and Table we can make the following observations:

1. The approach established in the present paper leads to less conservative results than the approaches proposed in Karayiannis and Krstic (2013) and Li and Gu (2010) even when \( N = 1 \). Moreover, compared with the discretized Lyapunov functional approach in Li and Gu (2010), our approach requires less computational time.

2. When \( N = 5 \), the bounds obtained by the proposed approach are closer to the exact bounds than that with \( N = 1 \), that
is, increasing $N$ can reduce the conservatism of the computed bounds.

(3) The gaps between the obtained bounds by using the proposed approach and the exact bounds become smaller and smaller as $N$ increases.

5. Conclusions

In this paper, the robustness problem of the predictor feedback control of linear systems with input delays was studied. The basic idea is to transform the robustness problem of the predictor feedback into the stability problem of a kind of neutral time-delay systems by using the equivalent transformation on the characteristic equation of the time-delay systems. Then a delay dependent stability condition for the stability of the neutral time-delay systems was proposed by using a delay partition method, which can gradually reduce the conservatism of the obtained results when the numbers of partition increase. Numerical examples showed the effectiveness of the proposed method.

Acknowledgments

The authors would like to thank Professor Keqin Gu for providing the Matlab function for Li and Gu (2010). This work was supported in part by the National Natural Science Foundation of China under Grant numbers 61104124, 61273028 and 61322305, by the Fundamental Research Funds for the Central Universities under Grant HIT.NSRIF.2011007, by Program for Innovation Research of Science in Harbin Institute of Technology (PIRS of HIT A201407), and by the National Science Foundation of the United States under Grant number CMSI-1129752. Zhao-Yan Li and Bin Zhou would like to thank the support of the University of Virginia when they were on leave with Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, USA, from July 2012 to August 2013.

Appendix

A.1. Proof of Lemma 1

The closed-loop system consisting of (1) and (2) in frequency domain can be written as

$$
\begin{bmatrix}
    s_l - A & -B e^{-rs} \\
    -Ke^{e\tau} & I_m - K \int_{-r}^{0} e^{-\tau B} e^{\tau s} d\tau
\end{bmatrix}
\begin{bmatrix}
x(s) \\
u(s)
\end{bmatrix} = 0.
$$

(82)

Then the closed-loop system is asymptotically stable if and only if all the zeros of the following characteristic equation have negative real parts (Hale, 1977).

$$
\det
\begin{bmatrix}
s_l - A & -B e^{-rs} \\
-K e^{e\tau} & I_m - K \int_{-r}^{0} e^{-\tau B} e^{\tau s} d\tau
\end{bmatrix} = 0,
$$

(83)

$$
\text{det}
\begin{bmatrix}
s_l - A & -B e^{-rs} \\
-K e^{e\tau} & I_m - K \int_{-r}^{0} e^{-\tau B} e^{\tau s} d\tau
\end{bmatrix} = 0.
$$

A.2. Proof of Proposition 1

First, we need to prove that inequalities (13) and (14) are feasible for any $r^* = r > 0$. Let $T_{i,j} = t_{ij}T_0$ for some $t_{ij} > 0$, $j = 1, 2, \ldots, N$, and $T_1 = \text{diag}(T_{1,1,1}, T_{1,1,2}, \ldots, T_{1,N,N})$. Let $t_{ij}, j = 1, 2, \ldots, N$, be such that

$$
T_{i(j+1,j+1)} < T_{i,j}, \quad j = 1, 2, \ldots, N - 1,
$$

(85)

$$
T_{i(1,1)} < T_2,
$$

(86)

are satisfied. Moreover, we choose $Q_2, S \in \mathbb{R}^{n \times n}$ such that

$$
T_2 + r^* Q_2 < S,
$$

(87)
and \( Q_i = q_{in} \) where \( q > G^T S G \). Then direct manipulation shows that inequalities (13) and (14) are satisfied with these matrices chosen above.

Next, let us prove that inequality (35) can be satisfied for any \( r^* = r > 0 \). It is obvious that we only need to prove that

\[
\Omega_r + \Omega_T + \Omega_Q + \Omega_w + \Omega_z + \Omega_U + \Omega_Y < 0, 
\]

holds true for any \( r^* = r > 0 \). We choose positive definite matrices \( T \) and \( Z \) which are block diagonal matrices and the diagonal elements are, respectively, denoted by \( T_{(i,0)} \in \mathbb{R}^r \) and \( Z_{(i,0)} \in \mathbb{R}^r \), \( i = 1, 2, \ldots, N \). Then, under the assumption \( r^* = r \), \( \Omega_r + \Omega_T + \Omega_Q + \Omega_w + \Omega_z + \Omega_U + \Omega_Y \) can be rewritten as

\[
\begin{bmatrix}
-NQ & \Delta^T & 0_{N \times 2N} \\
0_{2N \times N} & 0_{2N \times (2N+1)n} & -NW
\end{bmatrix},
\]

in which

\[
\Delta = [-g^T PF \ 0_{N \times 2N}], \quad \Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix}
\]

where

\[
\Phi_1 = \begin{bmatrix} \Phi_{11} & 1/r & U + r^* F^T U G \\ \frac{1}{r^*} U + r^* G^T U F & \Phi_{12} \end{bmatrix},
\]

\[
\Phi_2 = \text{diag}(\Phi_{21}, \Phi_{22}, \Phi_{23}),
\]

\[
\Phi_3 = \begin{bmatrix} \Phi_{31} & \Phi_{32} \\ \Phi_{33} & \Phi_{34} \end{bmatrix},
\]

with

\[
\Phi_{11} = PF + F^T P + Y + \frac{1}{r} U + r^* F^T U G,
\]

\[
\Phi_{12} = -Y + r^* G^T U G + T_{(1,1)} + Z_{(1,1)},
\]

\[
\Phi_{21} = \text{diag}(T_{(2,2)} + Z_{(2,2)} - T_{(1,1)}, T_{(3,3)} + Z_{(3,3)} - T_{(2,2)}, \ldots, T_{(N,N)} + Z_{(N,N)} - T_{(N-1,N-1)}),
\]

\[
\Phi_{22} = Z_{(1,1)} - T_{(N,N)} + r^* G^T U G - \frac{1}{r^*} U,
\]

\[
\Phi_{23} = \text{diag}(Z_{(2,2)}, \ldots, Z_{(N,N)}),
\]

\[
\Phi_{31} = \Phi_{33} = 0_{2N \times (N-1)n},
\]

\[
\Phi_{32} = [-r^* G^T U U + \frac{1}{r^*} U - r^* G^T U G]^T.
\]

By using a Schur complement, we know that (89) is equivalent to the following two inequalities:

\[
\begin{bmatrix}
\Phi & 0_{(1+2N)n \times 2N} \\
0_{2N \times (1+2N)n} & -NW
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\Delta^T \\
0_{2N \times (2N+1)n}
\end{bmatrix}
\begin{bmatrix}
Q^{-1} & 0_{N \times 2N} \\
0_{N \times 2N} & 0_{N \times N}
\end{bmatrix} < 0.
\]

Inequality (101) is obviously true for any \( Q \) > 0. Furthermore, (102) can be rewritten as

\[
\begin{bmatrix}
\Phi^a & 0_{(2N+1)n \times 2N} \\
0_{2N \times (2N+1)n} & -NW
\end{bmatrix} < 0,
\]

in which \( \Phi^a \) is obtained by replacing \( \Phi_{11} \) with \( \Phi_{11}^a = \Phi_{11} + F^T P g Q^{-1} g^T P F \) in \( \Phi \). We also denote \( \Phi^a \) by replacing \( \Phi_{11} \) with

\[
\Phi_{11}^a = \Phi_{11} + F^T P g Q^{-1} g^T P F \in \Phi_1.
\]

It is clear that the inequality in (103) is equivalent to

\[
\text{diag}(\Phi_{23} - NW) < 0, \quad (104)
\]

\[
\begin{bmatrix}
\Phi_{31}^a & \Phi_{32} \\
\Phi_{31}^a & 0_{(N-1)n \times N}
\end{bmatrix} < 0. \quad (105)
\]

Inequality (104) is obviously true for any \( Z > 0 \) and \( W > 0 \). For inequality (105), we know it is equivalent to the following two inequalities

\[
\Phi_{22} < 0,
\]

\[
\begin{bmatrix}
\Phi_{31}^a & \Phi_{32} \\
\Phi_{31}^a & 0_{(N-1)n \times N}
\end{bmatrix} < 0,
\]

by applying a Schur complement. It is easy to choose \( Z_{(1,1)} > 0, T_{(N,N)} > 0, \) and \( U > 0 \) such that

\[
\Phi_{22} = -Z_{(1,1)} - T_{(N,N)} + r^* G^T U G - \frac{1}{r^*} U < 0, \quad (108)
\]

namely, inequality (106) is satisfied. The inequality in (107) is further equivalent to

\[
\begin{bmatrix}
\Phi_{31}^a & \Phi_{32} \\
\Phi_{31}^a & 0_{(N-1)n \times N}
\end{bmatrix} < 0,
\]

in which

\[
\Phi_1^b = \begin{bmatrix} \Phi_{11}^b & \Phi_{12}^b \\ \Phi_{12}^b & \Phi_{13}^b \end{bmatrix},
\]

\[
\Phi_{11}^b = \Phi_{11}^* - \left(-r^* F^T U G + \frac{1}{r^*} U \right) \Phi_{22}^{-1} \times \left(-r^* G^T U F + \frac{1}{r^*} U \right),
\]

\[
\Phi_{12}^b = \Phi_{12} - \left(r^* \right)^2 G^T U G \Phi_{22}^{-1} G^T U G,
\]

\[
\Phi_{13}^b = \frac{1}{r^*} U + r^* F^T U G + \left(-r^* \right)^2 F^T U G + U \Phi_{22}^{-1} G^T U G.
\]

As \( \Phi_{31}^a \) is a zero matrix, the inequality in (109) is equivalent to

\[
\Phi_1^b < 0, \quad (114)
\]

\[
\Phi_{21} < 0. \quad (115)
\]

Here, for any \( T_{(1,1)} > 0, \) we can choose \( T_{(i+1,i+1)} < T_{(i,0)} \) and \( Z_{(i+1,i+1)} \) sufficient small for \( i = 1, 2, \ldots, N - 1, \) such that

\[
T_{(i+1,i+1)} + Z_{(i+1,i+1)} - T_{(i,0)} < 0, \quad i = 1, 2, \ldots, N - 1, \quad (116)
\]

namely, inequality (115) is satisfied. Then, by applying a Schur complement again, inequality (114) holds true if and only if the following inequalities hold true

\[
\Phi_{12}^b < 0, \quad (117)
\]

\[
\Phi_{11}^b - \Phi_{12}^b \Phi_{12}^{-1} \Phi_{13}^b < 0. \quad (118)
\]

It is obviously that there exists \( Y \) and \( T_{(1,1)} \) such that (117) is satisfied. Because \( F \) is Hurwitz, then there exists a \( P > 0 \) such that (118) is satisfied since the term \( P F + F^T P + F^T P g Q^{-1} g^T P F \) can be made as negative as possible by choosing \( Q = q_{in} \) with \( q \) sufficiently large and the other terms in (118) do not involve \( F \). The proof is thus completed.
A.3. Some technical lemmas

We list some technical lemmas that has been used in this paper.

\textbf{Lemma 5} (Finsler's Lemma). Consider real matrices $B$ and $M$ such that $B$ has full row rank and $M = M^T$. Then $\forall x \in \mathbb{R}^n$, $x^T M x < 0$, for all $x \neq 0$, if and only if $B^T M B^T < 0$.

\textbf{Lemma 6}. Let $n_1$, $n_2$, and $n_3$ be four positive integers and the matrix $F \in \mathbb{R}^{n_3 \times (n_1 + n_2 + n_3)}$ have the following form

$$F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix},$$

where $f_1 \in \mathbb{R}^{n_1 \times n_2}$, $f_2 \in \mathbb{R}^{n_2 \times n_3}$, and $f_3 \in \mathbb{R}^{n_3 \times n_1}$, and rank ($f_1$) = rank ($f_2$) = rank ($f_3$) = $n_0$. Then the right orthogonal complement of $f$ is

$$f^\perp = \left[\begin{array}{c} f_1^\perp \\ 0_{n_2 + n_3} \\ 0_{n_2 + n_3} \end{array} \right],$$

in which $f_1^\perp \in \mathbb{R}^{n_1 \times (n_1 - n_0)}$ represents the right orthogonal complement of $f_1$, and $(X_2, X_3)$ is a pair of solutions of the linear equations $f_2 X_2 + f_3 = 0$ and $f_1 X_2 + f_3 X_3 = 0$.

The proof of the above lemma is straightforward and is omitted. We finally recall the so-called Jensen Inequality from Gu (2000).

\textbf{Lemma 7}. Given two integers $a$ and $b$, for a vector $x(\theta)$ and a positive definite matrix $Z$ of appropriate dimensions, there holds

$$\int_a^b x(\theta) d\theta \leq |a - b| \int_a^b \sqrt{Z x(\theta) Z x(\theta) d\theta}.$$

References


Zhao-Yan Li was born in Hebei Province, PR China, on August 13, 1982. She received her B.Sc. degree from the Department of Electrical Engineering at Harbin Institute of Technology. She was born in Luotian County, Huanggang, Hubei Province, PR China, on July 28, 1981. He received the Bachelor’s degree, the Master’s degree and the Ph.D. degree from the Department of Control Science and Engineering at Harbin Institute of Technology, Harbin, China, in 2004, 2006 and 2010, respectively. He was a Research Associate at the Department of Electrical and Computer Engineering, University of Virginia from July 2012 to August 2013. He is currently an associate editor on the Conference Editorial Board of the IEEE Control Systems Society and an associate editor of Journal of System Science and Mathematical Science.
Zongli Lin is a Professor of Electrical and Computer Engineering at University of Virginia. He received his B.S. degree in Mathematics and Computer Science from Xiamen University, Xiamen, China, in 1983, his Master of Engineering degree in Automatic Control from Chinese Academy of Space Technology, Beijing, China, in 1989, and his Ph.D. degree in Electrical and Computer Engineering from Washington State University, Pullman, Washington, USA, in 1994. His current research interests include nonlinear control, robust control, and control applications. He was an Associate Editor of the IEEE Transactions on Automatic Control (2001–2003), IEEE/ASME Transactions on Mechatronics (2006–2009) and IEEE Control Systems Magazine (2005–2012). He was an elected member of the Board of Governors of the IEEE Control Systems Society (2008–2010) and has served on the operating committees and program committees of several conferences. He currently chairs the IEEE Control Systems Society Technical Committee on Nonlinear Systems and Control and serves on the editorial boards of several journals and book series, including Automatica, Systems & Control Letters, Science China Information Sciences, and Springer/Birkhauser book series Control Engineering. He is a Fellow of the Institute of Electrical and Electronics Engineers (IEEE), the International Federation of Automatic Control (IFAC) and the American Association for the Advancement of Science (AAAS).