1. Introduction

Time delay arises frequently in many engineering systems such as nuclear reactors, long transmission lines in pneumatic systems, rolling mills, sampled-data control, manufacturing processes and networked control systems (Fridman, 2010; Gu, Kharitonov, & Chen, 2003; Hale, 1977). Because of their infinite dimensional nature, control problems, especially, the problems of asymptotic stability analysis and stabilization, for time-delay systems have been recognized to be very difficult. As a result, during the past several decades, control of time-delay systems has received considerable attentions from the researchers and a large number of results have been reported in the literature (see Du, Lam, and Shu (2011), He, Wang, Lin, and Wu (2007), Lam, Gao, and Wang (2007), Xie, Fridman, and Shaked (2001), Xu, Lam, and Yang (2001), Yakoubi and Chitour (2007), Zhang, Zhang, and Xie (2004) and Zhou, Lin, and Duan (2012) and the references therein).

Among the existing methods for carrying out asymptotic stability analysis and stabilization of time-delay systems, the Lyapunov–Krasovskii (LK) functional based methods are probably the most efficient ones. In these methods, one generally needs to start with certain system transformations that bring into the original delay system additional dynamics that can be described by the so-called integral delay system (Gu & Niculescu, 2001; Kharitonov & Melchor-Aguilar, 2003). The stability of such an integral delay system is thus important in the stability analysis of the original delay system. Another efficient approach to dealing with the stabilization of time-delay systems is the predictor feedback, which is especially effective for input delay systems (Arstein, 1982; Krstic, 2010; Mirkin, 2004). However, it is now clear that the resulting infinite dimensional controllers can be safely implemented if and only if certain integral delay systems are asymptotically stable (Mondie, Dambrine, & Santos, 2001; Mondie & Melchor-Aguilar, 2012; Van Assche, Dambrine, Lafay, & Richard, 1999).

Besides the two examples mentioned above, integral delay systems appear also in some other problems associated with delay systems, for example, delay approximations of the partial differential equations for describing the propagation phenomena in excitable media (Niculescu, 2001), and stability analysis of some difference operators in neutral type functional differential equations (Hale, 1977). For more introduction on integral delay systems, the reader may refer to Melchor-Aguilar (2010), Melchor-Aguilar, Kharitonov, and Lozano (2010) and Mondie and Melchor-Aguilar (2012) and the references cited there. Because of their
many applications, integral delay systems have received much attention in recent years. LK theorems for integral delay systems have recently been introduced in Melchor-Aguilar (2010) and Melchor-Aguilar et al. (2010). With the help of this LK theorem, some sufficient conditions in terms of linear matrix inequalities (LMIs) were recently established in Mondie and Melchor-Aguilar (2012) for the exponential stability of some classes of integral delay systems with analytic kernels, which include the integral delay systems encountered in the model transformation and predictor feedback for time-delay systems as special cases.

In this paper, with the aid of the delay decomposition technique (Gouaisbaut & Peaucelle, 2006), we find that the LK functional proposed in Mondie and Melchor-Aguilar (2012) can be written as an integral quadratic form of some generalized state vector obtained by the fractionizing of the delay intervals of the state. This motivates us to propose a more general LK functional in terms of this generalized state vector. Delay dependent LMI conditions guaranteeing the exponential stability of an integral delay system are then obtained by using this new LK functional. It is proven that these conditions are less conservative than those proposed in Mondie and Melchor-Aguilar (2012). Moreover, by analyzing the characteristic equation of the considered integral delay system, necessary and sufficient conditions for stability are obtained in terms of the right-most (unstable) zeros of a certain auxiliary point-delay linear system. Consequently, for the special case that a parameter matrix in the kernel of the integral delay system is anti-stable, the stability of the integral delay system is equivalent to the stability of the auxiliary point-delay system, for which stability criteria that are easy to test are also presented. Numerical examples are worked out to illustrate the effectiveness of the obtained results.

The remainder of this paper is organized as follows. The problem formulation and some preliminary results are presented in Section 2. The LMI-based sufficient conditions are given in Section 3 while the characteristic equation based results are given in Section 4. Numerical examples are presented in Section 5 to show the effectiveness of the proposed approach and Section 6 concludes the paper.

**Notation.** The notation used in this paper is fairly standard. For a matrix \( A \in \mathbb{R}^{m \times n} \), we use \( A^T \), rank \( A \), \( \lambda (A) \), det \( (A) \) and \( \text{He}(A) \) to denote its transpose, rank, eigenvalue set, determinant, and the symmetric matrix \( A + A^T \). We use \( \text{diag}(A_1, A_2, \ldots, A_p) \) to denote a diagonal matrix whose diagonal elements are \( A_i \), \( i = 1, 2, \ldots, p \) and \( A \otimes B \) to denote the Kronecker product of matrices \( A \) and \( B \). For a semi-positive definite matrix \( P \), we use \( \lambda_{\text{min}}(P) \) and \( \lambda_{\text{max}}(P) \) to denote, respectively, its minimal and maximal eigenvalues. For a positive scalar \( h \), and an integer \( m \), let \( \mathbb{V}_{m,h} = \mathbb{V}([-h,0], \mathbb{R}^n) \) denote the Banach space of continuous vector functions mapping the interval \([-h,0]\) into \( \mathbb{R}^n \) with the topology of uniform convergence, and let \( x_t \in \mathbb{V}_{m,h} \) denote the restriction of \( x(t) \) to the interval \([t-h,t] \) translated to \([-h,0]\), that is, \( x_t(\theta) = x(t + \theta), \theta \in [-h,0] \). For any \( \psi \in \mathbb{V}_{m,h} \), the norm of \( \psi \) is defined as \( ||\psi||_h = \sup_{\theta \in [-h,0]} ||\psi(\theta)||_2 \).

**2. Problem formulation and preliminaries**

In this paper we are interested in the following integral delay system:

\[
x(t) = \int_{-h}^{0} GB(\theta) x(t + \theta) d\theta, \quad t \geq 0,
\]

where \( h > 0 \) is a constant, \( G \in \mathbb{R}^{m \times n} \), and \( B(\theta) : [-h,0] \rightarrow \mathbb{R}^{n \times m} \). Furthermore, we assume that \( B(\theta) \) is continuously differentiable for all \( \theta \in [-h,0] \) and satisfies (Mondie & Melchor-Aguilar, 2012)

\[
\dot{B}(\theta) = MB(\theta), \quad \forall \theta \in [-h,0],
\]

with \( M \in \mathbb{R}^{m \times m \times n} \) being a constant matrix. For notational simplicity, we denote \( B(0) = B \). Moreover, according to Mondie and Melchor-Aguilar (2012), we should assume that there exists a \( \gamma > 0 \) such that

\[
\min_{\theta \in [-h,0]} \{ \lambda_{\text{min}} \left( B^T(\theta) B(\theta) \right) \} > \gamma.
\]

**Remark 1.** Under the assumption in (2), the condition in (3) is equivalent to

\[
\text{rank}(B) = m.
\]

In fact, (3) is satisfied if and only if \( \text{rank}(B(\theta)) = m, \forall \theta \in [-h,0] \), which, in view of the fact that (2) is equivalent to

\[
B(\theta) = e^{MB} B(0) = e^{MB} B, \quad \forall \theta \in [-h,0],
\]

is equivalent to (4) since \( e^{MB} \) is nonsingular for all \( \theta \). Hence condition (4) instead of (3) is assumed to be satisfied hereafter.

**Remark 2.** In Mondie and Melchor-Aguilar (2012), the matrix \( M \) is of dimension \((N + 1)\) for some integer \( N \geq 0 \). Here in (1) the dimension of \( M \) is \( n \), which is not necessarily a multiple of \( m \). Of course, to ensure that (4) is satisfied, we must have \( n \geq m \).

In this paper we are interested in the stability analysis of the integral delay system (1). To this end, we first introduce the stability principle for this class of systems. Let \( \psi(\theta) \in \mathbb{V}_{m,h}, \theta \in [-h,0] \), be an initial condition for system (1) and \( x(t) = x(t, \psi), t \geq 0 \), be a solution of (1) satisfying \( x(t) = \psi(t), t \in [-h,0] \). Then we can give the definition of exponential stability with a guaranteed decay rate for the integral delay system (1).

**Definition 1.** The integral delay system (1) is said to be exponentially stable with a guaranteed decay rate \( \beta \geq 0 \) if there exists a \( \mu > 0 \) and an \( \alpha > 0 \) such that every solution of (1) satisfies the inequality

\[
\|e^{\beta t} x(t)\|_e \leq \mu \|\psi\|_h e^{-\alpha t}, \quad t \geq 0.
\]

When \( \beta = 0 \), the above definition reduces to the definition of exponential stability as given in Mondie and Melchor-Aguilar (2012).

**Lemma 1 (Melchor-Aguilar et al. (2010) and Mondie and Melchor-Aguilar (2012).** The integral delay system (1) is exponentially stable with a guaranteed decay rate \( \beta \geq 0 \) if there exists a function \( V : \mathbb{V}_{m,h} \rightarrow \mathbb{R} \) such that \( t \mapsto V(x_t) \) is differentiable on \( \mathbb{R}_+ \) and the following conditions hold:

\[
(1) \alpha_1 \int_{-h}^{0} |x(t + \theta)|^2 d\theta \leq V(x_t) \leq \alpha_2 \int_{-h}^{0} |x(t + \theta)|^2 d\theta,
\]

for some \( 0 < \alpha_1 \leq \alpha_2 \);

\[
(2) V(x_t) + 2\beta V(x_t) \leq -\alpha_3 \int_{-h}^{0} |x(t + \theta)|^2 d\theta,
\]

for some \( \alpha_3 > 0 \).

In order to study the exponential stability of (1), in Mondie and Melchor-Aguilar (2012), the following LK functional

\[
V_h(x_t) = \int_{-h}^{0} \chi^T(t + \theta) B^T(\theta) e^{2\beta \theta} R(\theta) B(\theta) x(t + \theta) d\theta,
\]

was constructed, where \( R(\theta) = P(\theta + h) Q \) and \( P, Q \in \mathbb{R}^{m \times m} \) are two positive definite matrices. With this LK functional (7), the following LMI based conditions for testing the exponential stability of (1) were obtained in Mondie and Melchor-Aguilar (2012).

**Lemma 2.** The integral delay system in (1) which satisfies (2) and (3) is exponentially stable with a guaranteed decay rate \( \beta \geq 0 \) if there exist two positive definite matrices \( P, Q \in \mathbb{R}^{m \times m} \) such that the following two LMIs, where \( R = P + hQ \), are satisfied:

\[
Q + M^T R + RM - hG^T B^T R B G > 0,
\]

\[
Q + M^T P + MP - h e^{2\beta h} G^T B^T R B G > 0.
\]
Finally, for easy reference, we recall from Melchor-Aguilar (2010) and Mondie and Melchor-Aguilar (2012) the following result regarding algebraic conditions for testing the exponential stability of (1).

Lemma 3. The integral delay system in (1) is exponentially stable if

\[ h < h_{\text{max}} = \left( \max_{s \in [0, b]} \| GB(s) \| \right)^{-1}. \]

3. LK functional based sufficient conditions

3.1. Construction of the LK Functional

In order to reduce the conservatism of the LK functional approach, motivated by the work of Gouaisbaut and Peaucelle (2006), we consider the fractions \( \frac{1}{n} \), \( i = 1, 2, \ldots, N \), of the delay \( h \), where \( N \geq 1 \) is a given integer. For an \( s \in \mathbb{R} \), we define

\[
\pi(s) = \begin{bmatrix}
B_1(s-t)x_1(s) \\
B_2(s-t)x_2(s) \\
\vdots \\
B_N(s-t)x_N(s)
\end{bmatrix},
\]

in which, for \( i = 1, 2, \ldots, N \),

\[
x_i(s) = x\left( s - \frac{h}{N}(i-1) \right),
\]

\[
B_i(s) = B\left( s - \frac{h}{N}(i-1) \right).
\]

We notice that \( \pi(s) \) is also dependent on \( t \). Let \( \theta + \frac{h}{N}(i-1) = s, i = 1, 2, \ldots, N \). Then it follows from (7) that

\[
V_h(x_t) = \sum_{i=1}^{N} \int_{-h/(i-1)}^{0} x^T(t + \theta) B^T(\theta) x(t + \theta) d\theta \times e^{2\beta_0 R t} \Phi(\theta) B(\theta) x(t + \theta) d\theta = \sum_{i=1}^{N} \int_{-h/(i-1)}^{0} x_i^T(t + s) B_i^T(s) e^{2\beta_0 L_i(s)} B_i(s) x_i(t + s) ds = \int_{-h/1}^{0} \pi^T(t + s) e^{2\beta_0} (\Phi + \left( s + \frac{h}{N} \right) \mathcal{D}) \pi(t + s) ds,
\]

where

\[
L_i(s) = e^{-2\beta_0 s(i-1)} \left( P + \left( h - \frac{h}{N} \right) Q \right) + e^{2\beta_0 s(i-1)} \left( s + \frac{h}{N} \right) Q
\]

and we have denoted

\[
\Phi = \text{diag} \{ P_1, P_2, \ldots, P_N \}, \quad \mathcal{D} = \text{diag} \{ Q_1, Q_2, \ldots, Q_N \},
\]

in which

\[
P_i = e^{-2\beta_0 s(i-1)} \left( P + \left( h - \frac{h}{N} \right) Q \right),
\]

\[
Q_i = e^{2\beta_0 s(i-1)} Q.
\]

Notice that both \( \Phi \) and \( \mathcal{D} \) are block diagonal matrices. Hence the conservatism of Lemma 2 can be further reduced by increasing the degrees of freedom of \( \Phi \) and \( \mathcal{D} \). More specifically, we can respectively use two symmetric matrices \( \Phi \) and \( \mathcal{D} \) instead of the block diagonal matrices \( \Phi \) and \( \mathcal{D} \) defined in (12)–(13). Namely, we consider the following LK functional:

\[
V(x_t) = \int_{-h/1}^{0} \pi^T(t + s) e^{2\beta_0 s} \times \left( \Phi + \left( \frac{h}{N} + s \right) \mathcal{D} \right) \pi(t + s) ds,
\]

where \( \Phi, \mathcal{D} \in \mathbb{R}^{Nn \times Nn} \) are two given symmetric matrices.

Before giving our main results, we present the following technical lemma whose proof is provided in Appendix for the sake of clarity.

Lemma 4. Let \( P \in \mathbb{R}^{n \times n} \) and \( Q \in \mathbb{R}^{n \times n} \) be two symmetric matrices and \( R \in \mathbb{R}^{n \times n} \) be a semi-positive definite matrix. Let

\[
\Phi(s) = P + sQ + f(s) R, \quad \forall s \in [s_1, s_2],
\]

where \( s_1 \) and \( s_2 \) are two constants with \( s_2 > s_1 \) and \( f(s) \) is a concave function. Then \( \Phi(s) > 0, \forall s \in [s_1, s_2] \) if and only if \( \Phi(s_1) > 0 \) and \( \Phi(s_2) > 0 \).

Remark 3. Let \( \Upsilon(s) = \mathcal{P} + \left( \frac{h}{N} + s \right) \mathcal{D} \). Then the LK functional \( V(x_t) \) in (14) is positive definite if and only if \( \Upsilon(s) \) is positive definite for all \( s \in \left[ -\frac{h}{N}, 0 \right] \), which, according to Lemma 4, is the case if and only if \( \Upsilon\left( -\frac{h}{N} \right) = \mathcal{P} > 0 \) and

\[
\Upsilon(0) = \mathcal{P} + \frac{h}{N} \mathcal{D} > 0.
\]

For the same reason given in the above remark, the condition that \( Q > 0 \) in Lemma 2 can be weakened as

\[
P + hQ > 0.
\]

For easy reference, we state the following corollary.

Corollary 1. The integral delay system in (1) is exponentially stable with a guaranteed decay rate \( \beta \geq 0 \) if there exist a positive definite matrix \( P \in \mathbb{R}^{n \times n} \) and a symmetric matrix \( Q \in \mathbb{R}^{n \times n} \) such that (8)–(9) and (17) are satisfied.

For future use, for any integer \( N \geq 1 \), we rewrite the integral delay system (1) as

\[
x(t) = \sum_{i=1}^{N} \int_{-h/(i-1)}^{0} GB(\theta) x(t + \theta) d\theta = \sum_{i=1}^{N} \int_{-h/(i-1)}^{0} GB\left( s - \frac{h}{N}(i-1) \right) x\left( t + s - \frac{h}{N}(i-1) \right) ds = G I_{Nn} \int_{-h/1}^{0} \pi(t + s) ds,
\]

in which \( I_{Nn} = [ I_n \ I_n \ \cdots \ I_n ] \in \mathbb{R}^{n \times Nn} \).

3.2. Some sufficient conditions

In this subsection, we provide some sufficient conditions for testing the exponential stability of the integral delay system (1).

Theorem 1. The integral delay system in (1) is exponentially stable with a guaranteed decay rate \( \beta \geq 0 \) if there exist an integer \( N \geq 1 \), two positive definite matrices \( \Phi \in \mathbb{R}^{Nn \times Nn} \) and \( S \in \mathbb{R}^{n \times n} \), and a symmetric matrix \( \mathcal{D} \in \mathbb{R}^{Nn \times Nn} \) such that (16) and

\[
\Theta = \begin{bmatrix}
I_n \\
0_{nN \times nN}
\end{bmatrix} S \begin{bmatrix}
I_n \\
0_{nN \times nN}
\end{bmatrix}^T < 0,
\]

\[
\mathcal{M}^T \left( \Phi + \frac{h}{N} \mathcal{D} \right) + \left( \Phi + \frac{h}{N} \mathcal{D} \right) \mathcal{M} + \mathcal{D} - \frac{h}{N} I_{Nn} C^T B^T S B G I_{Nn} > 0,
\]

\[
\mathcal{M}^T \Phi + \mathcal{P} \mathcal{M} + \mathcal{D} - \frac{h}{N} I_{Nn} C^T B^T S B G I_{Nn} > 0,
\]
are satisfied, where $\mathcal{M} = I_N \otimes M$ and $\Theta$ is defined as

$$
\Theta = \begin{bmatrix} I_{Nn} & \rho + \frac{h}{N} \rho \end{bmatrix} \begin{bmatrix} I_{Nn} & 0 \end{bmatrix}^T - \begin{bmatrix} 0 & I_{Nn} \end{bmatrix} \begin{bmatrix} e^{-2\beta \frac{h}{N} \rho} & \rho \end{bmatrix} \begin{bmatrix} 0 & I_{Nn} \end{bmatrix}^T.
$$

(22)

**Proof.** Consider the LK functional in (14). Similarly to the derivation of (18), we can derive

$$
\int_{-\frac{h}{\pi}}^{0} \pi^T (t + s) \pi (t + s) ds
= \int_{0}^{\infty} x^T (t + \theta) B^T (\theta) B (\theta) x (t + \theta) d\theta.
$$

(23)

With this identity and (3), it is easy to verify that $V (x_t)$ satisfies Item 1 of Lemma 1.

We next prove that $V (x_t)$ also satisfies Item 2 of Lemma 1. To do so, let $t + s = \theta$. Then

$$
V (x_t) = \int_{-\frac{h}{\pi}}^{t} \pi^T (\theta) e^{2\beta (\theta - t)} \times \left( \rho + \left( \frac{h}{N} + \theta - t \right) \rho \right) \pi (\theta) d\theta,
$$

(24)

from which it follows that

$$
\dot{V} (x_t) = \pi^T (t) \left( \rho + \frac{h}{N} \rho \right) \pi (t) - \pi^T (t - \frac{h}{N}) e^{-2\beta t} \times \rho \pi (t - \frac{h}{N}) + 2 \int_{-\frac{h}{\pi}}^{t} \pi^T (\theta) e^{2\beta (\theta - t)} \left( \rho + \left( \frac{h}{N} \right) \right) \pi (\theta) d\theta - \int_{-\frac{h}{\pi}}^{t} \pi^T (\theta) e^{2\beta (\theta - t)} \times \rho \pi (\theta) d\theta - 2\beta \int_{-\frac{h}{\pi}}^{t} \pi^T (\theta) e^{2\beta (\theta - t)} \times \left( \rho + \left( \frac{h}{N} + \theta - t \right) \rho \right) \pi (\theta) d\theta.
$$

(25)

In view of (2), we can compute

$$
\dot{\pi} (\theta) = -\mathcal{M} \pi (\theta).
$$

(26)

Define a generalized state vector

$$
\eta (t) = \begin{bmatrix} \pi (t) \\ B (-h) x (t - h) \end{bmatrix} = \begin{bmatrix} B (0) x (t) \\ \pi (t - h) \end{bmatrix}.
$$

(27)

Then, the equation in (25) can be rewritten as

$$
\dot{\eta} (t) = \eta^T (t) \Theta \eta (t)
- \int_{-\frac{h}{\pi}}^{t} \pi^T (t + s) e^{2\beta t} \Gamma (s) \pi (t + s) ds,
$$

(28)

where $\Theta$ is defined in (22) and

$$
\Gamma (s) = \mathcal{M}^T \left( \rho + \left( \frac{h}{N} + s \right) \rho \right) + \left( \rho + \left( \frac{h}{N} + s \right) \rho \right) \mathcal{M} \rho.
$$

(29)

By applying the inequality in (19), the Jensen inequality (Gu et al., 2003) and the system model (18), we can get

$$
\dot{V} (x_t) + 2\beta V (x_t) \leq x^T (t) B^T SBG x (t)
- \int_{-\frac{h}{\pi}}^{t} \pi^T (t + s) e^{2\beta t} \Gamma (s) \pi (t + s) ds
= \left( \int_{-\frac{h}{\pi}}^{t} \pi^T (t + s) ds \right)^T I_{Nn} \mathcal{M}^T B^T SBG I_{Nn}
\times \left( \int_{-\frac{h}{\pi}}^{t} \pi (t + s) ds \right)
- \int_{-\frac{h}{\pi}}^{t} \pi^T (t + s) e^{2\beta t} \Gamma (s) \pi (t + s) ds
\leq - \int_{-\frac{h}{\pi}}^{t} e^{2\beta t} \pi^T (t + s) \Phi (s) \pi (t + s) ds,
$$

(30)

where $\Phi (s), s \in \left[ -\frac{h}{\pi} , 0 \right]$, is defined as

$$
\Phi (s) = \Gamma (s) - e^{-2\beta s} \frac{h}{N} I_{Nn} \mathcal{M}^T B^T SBG I_{Nn}.
$$

(31)

Since $-e^{-2\beta s}$ is a concave function and $\frac{h}{\pi} I_{Nn} \mathcal{M}^T B^T SBG I_{Nn} \geq 0$, it follows from Lemma 4 that $\Phi (s) > 0, \forall s \in \left[ -\frac{h}{\pi} , 0 \right]$ if and only if $\Phi (0) > 0$ and $\Phi ( - \frac{h}{\pi} ) > 0$, which are respectively equivalent to (20) and (21). Hence, there exists a sufficiently small number $\alpha_3 > 0$ such that

$$
\dot{V} (x_t) + 2\beta V (x_t) \leq -\alpha_3 \int_{-\frac{h}{\pi}}^{t} \pi^T (t + s) \pi (t + s) ds
\leq -\alpha_3 \gamma \int_{-\frac{h}{\pi}}^{t} \| x (t + s) \|^2 ds,
$$

(32)

which completes the proof.

The following proposition shows that Theorem 1 is less conservative than Lemma 2. The proof of this proposition is given in Appendix for the clarity of the presentation.

**Proposition 1.** For any given $\beta \geq 0$, if there exist two positive definite matrices $P, Q \in \mathbb{R}^{n \times n}$ such that (8) and (9) are satisfied, then, for any given integer $N \geq 1$, there exist two positive definite matrices $\rho \in \mathbb{R}^{n \times n}$ and $S \in \mathbb{R}^{n \times n}$, and a symmetric matrix $\rho \in \mathbb{R}^{n \times n}$ such that (16) and (19)–(21) are satisfied.

In the particular case that $M = 0$, namely $B (\theta) = \bar{B}$ is a constant matrix for all $\theta \in \left[ -h, 0 \right]$, we can rewrite the integral delay system (1) as

$$
x (t) = \int_{-h}^{t} GB x (s) ds,
$$

(33)

where $G \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant matrices. In this case, by noting that inequality (21) implies (20), we obtain the following corollary of Theorem 1.

**Corollary 2.** The integral delay system (33) is exponentially stable with a guaranteed decay rate $\beta \geq 0$ if there exist an integer $N \geq 1$ and matrices $0 < \rho \in \mathbb{R}^{n \times n}$, $\bar{B} \in \mathbb{R}^{n \times n}$, and $0 \leq S \in \mathbb{R}^{n \times n}$ such that (16), (19) and the following inequality are satisfied:

$$
\rho + \frac{h}{N} \rho \in \mathbb{R}^{n \times n} \mathcal{M}^T \mathcal{M} + \mathcal{L} \mathcal{L}.
$$

(34)
In this particular case, we can add the following functional:
\[
W (x_t) = \int_0^1 \int_{t+v}^t e^{2b(t-\tau)} \pi^T (\theta) \pi (\theta) \, d\theta \, dv, \tag{35}
\]
where \(0 \leq \pi \in \mathbb{R}^{n \times n}\) is a semi-positive definite matrix, to the LK function (14) to get new sufficient conditions guaranteeing stability. In fact, by letting
\[
F (v) = \int_0^1 e^{2b(t-\tau)} \pi^T (\theta) \pi (\theta) \, d\theta \geq 0, \tag{36}
\]
we can compute
\[
W (x_t) \leq \int_0^1 \max_{\pi \leq 0} \{ F (v) \} \, dv = \frac{h}{N} \int_0^1 \pi^T (t + s) \pi (t + s) \, ds
\leq \frac{h}{N} \lambda_{\max} (\pi) \int_0^1 \pi (t + s) \pi (t + s) \, ds
= \frac{h}{N} \lambda_{\max} (\pi) \lambda_{\max} (B^T B) \int_0^1 \| x (t + \theta) \|^2 \, d\theta. \tag{37}
\]
Hence \(W (x_t)\) satisfies Item 1 of Lemma 1 and \(V (x_t) = W (x_t)\), is indeed an LK functional candidate for the integral delay system (33). For this LK functional, by applying a technique quite similar to the one used in the proof of Theorem 1, we can obtain the following result.

**Lemma 5.** The integral delay system (33) is exponentially stable with a guaranteed decay rate \(\beta \geq 0\) if there exist matrices \(0 < \pi \in \mathbb{R}^{n \times n}, 0 \leq \pi \in \mathbb{R}^{n \times n}\) and \(0 \leq S \in \mathbb{R}^{n \times n}\) such that (16), (19) and (34) are satisfied, where \(\pi\) is replaced by \(\pi + \lambda\) in (19) and (34).

However, the following proposition indicates that the introduction of \(W (x_t)\) is not helpful in the reduction of conservatism of Corollary 2.

**Proposition 2.** For any given \(\beta \geq 0\) and \(N \geq 1\), the LMIs in Corollary 2 are feasible if and only if the LMIs in Lemma 5 are feasible.

**Proof.** We need only to show that if the LMIs in Lemma 5 are feasible then the LMIs in Corollary 2 are also feasible. In fact, it is straightforward to verify that if \((\pi, S, \lambda)\) is a feasible solution to the LMIs in Lemma 5, then \((\pi, S, \lambda, S)\) is a feasible solution to the LMIs in Corollary 2.

**4. Characteristic equation based necessary and sufficient conditions**

For the integral delay system (1), we provide in this section necessary and sufficient conditions for stability by analyzing its associated characteristic equation. We rewrite (5) as
\[
B (\theta) = e^{A \theta} B (0) = e^{-A \theta} B, \quad \forall \theta \in [-h, 0], \tag{39}
\]
where \(A = -M\). The integral delay system (1) can then be rewritten as
\[
x (t) = \int_{t-h}^t e^{-A \theta} B x (t + \theta) \, d\theta, \tag{40}
\]
whose characteristic eigenvalue set can be obtained as
\[
\lambda_{ID} = \left\{ \lambda : \det \left( I_n - \int_{t-h}^t e^{-A \theta} B e^{\lambda \theta} \, d\theta \right) = 0 \right\}. \tag{41}
\]
It is known that the integral delay system (40) is asymptotically stable if and only if \(\lambda_{ID} \subseteq \mathbb{C}^-\) (Hale, 1977; Melchor-Aguilar et al., 2010). Then we have the following result.

**Proposition 3.** Let \(\lambda_D\) be defined as
\[
\lambda_D = \left\{ \lambda : \det \left( I_n - (A + B G) + e^{A \theta} B e^{-\lambda \theta} \right) = 0 \right\}, \tag{42}
\]
and \(\lambda_{ID} \subseteq \mathbb{C}^-\) defined in (41).

Consequently, if \(A\) is asymptotically stable (namely, \(M\) is anti-stable), then the integral delay system (40) is asymptotically stable if and only if the following delay system is so,
\[
\dot{y} (t) = (A + B G) y (t) - e^{A \theta} B G y (t - h). \tag{44}
\]

**Proof.** Since \(\lambda_D\) is the characteristic eigenset value for the delay system (44), we need only to show \(\lambda_{ID} \subseteq \mathbb{C}^-\) can be guaranteed decay rate \(\beta\) if and only if the LMIs in Proposition 3 are feasible. In fact, it is straightforward to verify that if \(\lambda_{ID} \subseteq \mathbb{C}^-\), then the system is asymptotically stable.

From Proposition 3 we can see that if \(\lambda_D \subseteq \mathbb{C}^-\), then the delay system (44) is asymptotically stable, then we must have \(\lambda_{ID} \subseteq \mathbb{C}^-\). Notice that \(\lambda_D \subseteq \mathbb{C}^-\) implies that matrix \(A\) is Hurwitz. Hence, in the case that \(A\) is Hurwitz, the exponential stability of the integral delay system (1) is equivalent to the exponential stability of the delay system in (44), which is a point-delay system and its exponential stability has been extensively studied in the literature (see, for example, Gouaisbaut and Peaucelle (2006), Gu et al. (2003), He et al. (2007) and Niculescu (2001)). In particular, the exact bound on \(h\) for the stability of the system can be determined by computing the right-most elements in \(\lambda_D\), which can be accomplished by the software package DDE-BIFTOOL (Engelborghs, Luyzanina, & Samaey, 2001).

However, it is still possible that \(\lambda_{ID} \subseteq \mathbb{C}^-\) (namely, the delay system (40) is asymptotically stable) if \(\lambda_D\) does not belong to \(\mathbb{C}^-\) (namely, the delay system (44) is not asymptotically stable). This is the case if \(A\) is not Hurwitz. In this case, let us define
\[
\lambda_D^+ = \left\{ \lambda : \det ( I_n - (A + B G) - e^{A \theta} B e^{-\lambda \theta} ) = 0 \right\}. \tag{46}
\]
and \(\lambda_{ID}^+ = \left\{ \lambda : \det ( I_n - (A + B G) - e^{A \theta} B e^{-\lambda \theta} ) = 0 \right\}. \tag{47}
\]
It then follows from Proposition 3 that the stability of the integral delay system (1) is equivalent to \(\lambda_D^+ \subseteq \mathbb{C}^-\), which can again be verified by computing the unstable elements in \(\lambda_D\) with the help of the software package DDE-BIFTOOL (Engelborghs et al., 2001).

**Remark 4.** The result in Proposition 3 can also be used to test the exponential stability of (40) with a guaranteed decay rate \(\beta \geq 0\). To this end, we let \(z (t) = e^{\beta t} x (t)\). Then \(x (t)\) converges with a guaranteed decay rate \(\beta\) if and only if the dynamics of \(z (t)\) is stable. Notice that
\[
z (t) = \int_{t-h}^t e^{(A + \beta I_n) \theta} B z (t + \theta) \, d\theta. \tag{48}
\]
Hence, to test the stability of the dynamics of \(z (t)\), we need only to compute the right-most zeros of the point-delay system (44) in which the matrix \(A\) is replaced with \(A + \beta I_n\).
Based on Proposition 3, the following algorithm can be stated.

**Algorithm 1.** (For three given matrices $A$, $B$ and $G$, compute the maximal delay $h_{\text{sup}}$ such that the integral delay system (1) is exponentially stable for all $h \in [0, h_{\text{sup}}]$).

1. Compute $\lambda^+(A)$ according to (46) and set $h = (\max_{s \in [-h, 0]} \frac{1}{\|GB(s)\|})^{-1}$.
2. Compute $\lambda^+_D$ according to (47) by using the software package DDE-BIFTOOL.
3. If $\lambda^+_D = \lambda^+(A)$, then set $h = h + \Delta h$ ($\Delta h$ is a small enough number representing the step length) and go to Step 2. Otherwise, $h_{\text{sup}} = h - \Delta h$.

**Remark 5.** We mention that Algorithm 1 may never stop in the case that the integral delay system (1) is exponentially stable for all $h$ (see Corollary 3). To avoid this problem, we may prescribe a large enough integer $\bar{h}$ and rewrite the third step in Algorithm 1 as follows.

4. If $\lambda^+_D = \lambda^+(A)$ and $h < \bar{h}$, then $h = h + \Delta h$ and go to Step 2. Otherwise, $h_{\text{sup}} = \bar{h} - \Delta h$.

This indicates that the integral delay system (1) is exponentially stable as least for $h \in [0, \bar{h}]$.

**Remark 6.** Although Proposition 3 and Algorithm 1 have provided a numerical method to check the necessary and sufficient conditions for the stability of the integral delay system (1). Theorem 1 possesses the advantage that it provides for the integral delay system (1) a novel LK functional that can be used for other purposes in the analysis of system (1). Particularly, it is possible to generalize the LK functional approach to the case that the delay in the integral delay system (1) is time-varying. We note that Proposition 3 and Algorithm 1 are only applicable to systems with constant delays.

In the particular case that $G = B = I_2$ (hence $m = n$), namely, for the following integral delay system:

$$x(t) = \int_{t-h}^{t} e^{-\theta h} x(t+\theta) \, d\theta, \quad (49)$$

can we obtain the following corollary of Proposition 1.

**Corollary 3.** Assume that $A$ is Hurwitz and let $\sigma_{\max} < 0$ be the maximal real part of the eigenvalues of $A$. Then the integral delay system (49) is asymptotically stable for all $h \geq h_{\max} \geq 0$, where $h_{\max}$ is such that

$$\sigma_{\max} + e^{h_{\max} \sigma_{\max}} < -1. \quad (50)$$

Moreover, the integral delay system (49) is asymptotically stable for all $h \geq 0$ if $\sigma_{\max} \leq -2$.

**Proof.** Notice that

det \{sh_n - (A + I_n) + e^{sh}e^{-\theta h}\} = \prod_{i=1}^{n} (s - \lambda_i - 1 + e^{-\theta (s - \lambda_i) h}).

Let $s = \sigma + \omega i$ and $\lambda_i = \sigma_i + \omega_i i$. Then, for any $\sigma \geq 0$, we have

\[ \begin{align*}
&\text{Re} \{s - \lambda_i - 1 + e^{-\theta (s - \lambda_i) h}\} \\
&= \sigma - \sigma_i - 1 + e^{h \sigma_i} \cos (h (\omega_i - \omega)) \\
&\geq \sigma - \sigma_i - 1 - e^{h \sigma_i} \\
&\geq -\sigma_{\max} - 1 - e^{h \sigma_{\max}}.
\end{align*} \]

Hence, if (50) is satisfied, we know that

$$s - \lambda_i - 1 + e^{-\theta (s - \lambda_i) h} = 0, \quad (52)$$

does not exist on $s : \Re(s) \geq 0$ and the integral delay system (49) is thus asymptotically stable. Finally, by noticing that (50) is guaranteed for all $h_{\max} \geq 0$ if $\sigma_{\max} \leq -2$, we complete the proof.

### 5. Numerical examples

In this section, we present some examples to validate the effectiveness of the proposed approaches.

**Example 1** (Example 1 in Mondie and Melchor-Aguilar (2012)). Consider an integral delay system in the form of (1) with $G = B = I_2$ and

$$M = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}. \quad (53)$$

Notice that in this case the matrix $-M$ is Hurwitz since $\lambda(M) = \{1, 2\}$. For different values of $\beta$, by applying different approaches, the corresponding maximal values of the delay $h$ are recorded in Table 1. From this table we can observe the following facts.

1. Corollary 1 is less conservative than Lemma 2 (Theorem 7 in Mondie and Melchor-Aguilar (2012)), which indicates that the weakened condition (17) instead of $Q > 0$ can indeed reduce the conservativeness. In fact, for $\beta = 1$ and $h = 0.64$, we obtain the following numerical values for matrices $P$ and $Q$:

\[ P = \begin{bmatrix} 27.7386 & -16.1260 \\ -16.1260 & 10.6060 \end{bmatrix}, \]

\[ Q = \begin{bmatrix} -18.9847 & 10.6050 \\ 10.6050 & -6.5418 \end{bmatrix}, \]

which means that $Q$ is negative definite.

2. Theorem 1 is less conservative than Lemma 2 (Theorem 7 in Mondie and Melchor-Aguilar (2012)) for any $\beta$ and $N$. In particular, a larger $N$ leads to a larger maximal delay value, which is reasonable since a larger $N$ means more decision variables in the conditions.

3. Since $-M$ is Hurwitz (thus $A = -M$ is Hurwitz), Proposition 3 can be used directly to obtain the exact maximal delay value by computing the right-most zero of the point-delay system (44) when $\beta = 0$. The computation result shows that there still is room for the improvement of the obtained sufficient conditions when $\beta = 0$.

**Example 2** (Example 2 in Mondie and Melchor-Aguilar (2012)), consider an integral delay system in the form of (1) with $G = B = I_2$ and

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (54)$$

Notice that in this case the matrix $M$ is marginally unstable since $\lambda(M) = \{0, 0\}$. Again, for different values of $\beta$, the corresponding maximal values of the delay $h$ by applying different approaches are recorded in Table 2. From this table we again find that Theorem 1 is less conservative than Lemma 2 (Theorem 7 in Mondie and Melchor-Aguilar (2012)) for any $\beta$ and $N$. Moreover, even though Corollary 8 in Mondie and Melchor-Aguilar (2012), where an argumentation has been utilized to increase the size of the decision variables, is better than Theorem 1 with $N = 1$, it is however more conservative than Theorem 1 with $N = 5$.

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$P_{\text{Corollary}1}$</td>
<td>$P_{\text{Corollary}1}$</td>
</tr>
<tr>
<td>$P_{\text{Lemma}2}$</td>
<td>$P_{\text{Lemma}2}$</td>
</tr>
<tr>
<td>$Q_{\text{Corollary}1}$</td>
<td>$Q_{\text{Corollary}1}$</td>
</tr>
<tr>
<td>$Q_{\text{Lemma}2}$</td>
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</tbody>
</table>
stable. Numerical examples demonstrate the effectiveness of the rightmost zeros of a certain auxiliary point-delay linear system and sufficient conditions for stability are obtained in terms of the delay intervals (namely, the number of delays). It is shown that the proposed LMI conditions are less conservative than the existing set of conditions for any number of divisions of the delay intervals (namely, the number $N$). Moreover, necessary and sufficient conditions for stability are obtained in terms of the rightmost zeros of a certain auxiliary point-delay linear system by analyzing the characteristic equation of the considered integral delay system. It is shown that the stability of the integral delay system is equivalent to the stability of the auxiliary point-delay system if a parameter matrix in the kernel of the integral delay system is anti-stable. Numerical examples demonstrate the effectiveness of the obtained results.

Appendix

A.1. Proof of Lemma 4

Let us write $\Phi(s_1) = P + s_1 Q + f(s_1) R$, and $\Phi(s_2) = P + s_2 Q + f(s_2) R$. Since $f(s)$ is a concave function, for any $s \in [s_1, s_2]$, we have

$$
\frac{s - s_1}{s_2 - s_1} f(s_2) + \frac{s - s_2}{s_2 - s_1} f(s_1) \\
\leq f\left(\frac{s - s_1}{s_2 - s_1} s_2 + \frac{s - s_2}{s_2 - s_1} s_1\right).
$$

As a result, we can compute

$$
\frac{s - s_1}{s_2 - s_1} \Phi(s_2) + \frac{s - s_2}{s_2 - s_1} \Phi(s_1) \\
= \left(\frac{s - s_1}{s_2 - s_1} + \frac{s - s_2}{s_2 - s_1}\right) P + \left(\frac{s - s_1}{s_2 - s_1} s_2 + \frac{s - s_2}{s_2 - s_1} s_1\right) Q \\
+ \left(\frac{s - s_1}{s_2 - s_1} f(s_2) + \frac{s - s_2}{s_2 - s_1} f(s_1)\right) R \\
\leq P + s Q + f(s) R \\
= \Phi(s), \quad \forall s \in [s_1, s_2].
$$

Hence $\Phi(s) > 0$ for all $s \in [s_1, s_2]$ if and only if $\Phi(s_1) > 0$ and $\Phi(s_2) > 0$. The proof is completed.

A.2. Proof of Proposition 1

Let $(P, Q)$ be a solution to the LMIs in (8) and (9). We choose $S = P + h Q$. (58)

Define a function $\Delta(\theta) : [-h, 0] \rightarrow \mathbb{R}^{2,2}$ as follows:

$$
\Delta(\theta) = Q + M^T \left( P + (\theta + h) Q \right) + \left( P + (\theta + h) Q \right) M \\
- h e^{-2\theta h} G^T B^T S B G.
$$

Since $\Delta(\theta)$ is in the form of (15), by using Lemma 4, inequality (8) and inequality (9), we get

$$
\Delta(\theta) > 0, \quad \forall \theta \in [-h, 0].
$$

Hence, for $\theta = -\frac{i}{N} h$, $i = 0, 1, \ldots, N$, we can obtain

$$
\Delta\left(-\frac{i}{N} h\right) = T_i - h e^{-\frac{i}{N} h} G^T B_i^T S B G > 0,
$$

in which $T_i, i = 0, 1, \ldots, N$, are defined as

$$
T_i = Q + M^T \left( P + \frac{N - i}{N} h Q \right) + \left( P + \frac{N - i}{N} h Q \right) M.
$$

With the application of the Schur complement, (61) can be equivalently transformed into

$$
\frac{1}{h} S^{-1} - e^{\frac{i}{N} h} B G T_i^{-1} G^T B_i > 0, \quad i = 0, 1, \ldots, N,
$$

from which it follows that

$$
0 < \sum_{i=0}^{N-1} \left( \frac{1}{h} S^{-1} - e^{\frac{i}{N} h} B G T_i^{-1} G^T B_i \right) \\
= \left( \frac{h}{N} S \right)^{-1} - B G T_0^{-1} T_i^{-1} G^T B_i,
$$

where

$$
\mathcal{T} = \text{diag}\left\{ T_0, e^{-2\beta h} T_1, \ldots, e^{-2\beta \frac{(N-1)h}{N}} T_{N-1} \right\}.
$$

### Table 2

Maximal delay value by using different approaches for Example 2.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Corollary 3</th>
<th>Lemma 2</th>
<th>Lemma 1</th>
<th>Theorem 1 with $N = 1$</th>
<th>Theorem 1 with $N = 5$</th>
<th>Theorem 1 with $N = 10$</th>
<th>Theorem 1 with $N = 15$</th>
<th>Proposition 3</th>
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### Table 3

Maximal delay value by using different approaches for Example 3.

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<tr>
<th>$\beta$</th>
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<th>Lemma 2</th>
<th>Corollary 1</th>
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<th>Corollary 2 with $N = 5$</th>
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<th>Corollary 2 with $N = 15$</th>
<th>Proposition 3</th>
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</table>

Example 3 (System (16) in Melchor-Aguilar (2010)). We consider an integral delay system in the form of (33) with $B = I_2$ and $G = \begin{bmatrix} -4 & 1 \\ -13 & 2 \end{bmatrix}$. (55)

This system corresponds to (1) with $M = 0$. Then Lemmas 2, 3 and Corollary 1 are applicable to this example. The results are recorded in Table 3, from which we again see that Corollary 2 can indeed lead to less conservative results than Lemmas 2, 3 and Corollary 1.

6. Conclusions

This paper proposes some delay-dependent sufficient conditions for the exponentially stability of a class of integral delay systems which are frequently encountered in studying stability problems of time-delay systems. The basic idea is to divide the delay interval into $N$ small intervals so that more information of the delayed state can be utilized to construct the LK functional, which in turns can reduce the conservatism of the resulting sufficient conditions expressed in linear matrix inequalities (LMIs). It is shown that the proposed LMIs conditions are less conservative than the existing set of conditions for any number of divisions of the delay intervals (namely, the number $N$). Moreover, necessary and sufficient conditions for stability are obtained in terms of the rightmost zeros of a certain auxiliary point-delay linear system by analyzing the characteristic equation of the considered integral delay system. It is shown that the stability of the integral delay system is equivalent to the stability of the auxiliary point-delay system if a parameter matrix in the kernel of the integral delay system is anti-stable. Numerical examples demonstrate the effectiveness of the obtained results.
Again with the application of the Schur complement, inequality (64) is equivalent to
\[
T = \frac{h}{N} I_{nN} G B^T S B G I_{nN} > 0. \tag{65}
\]
Now let \( \rho \) and \( \delta \) be defined in (12) and (13), respectively. Then, for \( i = 1, 2, \ldots, N \), we can compute
\[
\begin{align*}
\text{He} & \left( \frac{h}{N} \left( P + \frac{Q}{h} \right) M \right) + Q_i \\
& = e^{-\frac{\epsilon}{N} (i-1)P} \left( \text{He} \left( \frac{h}{N} \left( P + \frac{N - (i - 1)}{N} h Q \right) M \right) + Q \right) \\
& = e^{-\frac{\epsilon}{N} (i-1)P} T_{i-1}.
\end{align*}
\]
Hence the inequality in (65) can be exactly rewritten as (20), namely \((\rho, \delta, S)\) is a solution to the LMI (20).

Similarly to the derivation of inequality (64), we get
\[
0 < \sum_{i=1}^{N} \left( \frac{1}{h} \left( S - e^{\frac{\epsilon}{h} hP} B G I_{nN} T_i \right)^{-1} - e^{\frac{\epsilon}{h} hP} B G I_{nN} T_i \right) G B^T,
\]
where
\[
T = \text{diag} \left\{ T_1, e^{-\frac{\epsilon}{h} hP} T_2, \ldots, e^{-\frac{\epsilon}{h} hP} T_N \right\}.
\]
Via a Schur complement, (67) is equivalently to
\[
T - \frac{h}{N} e^{\frac{\epsilon}{h} hP} I_{nN} G B^T S B G I_{nN} > 0. \tag{68}
\]
Again, if we let \( \rho \) and \( \delta \) be defined in (12) and (13), respectively, then, for \( i = 1, 2, \ldots, N \), we can compute
\[
\begin{align*}
\text{He} (P M) + Q_i \\
& = e^{-\frac{\epsilon}{h} hP} \left( P + \frac{N - i}{N} h Q \right) M + e^{-\frac{\epsilon}{h} hP} hQ \\
& = e^{-\frac{\epsilon}{h} hP} T_i.
\end{align*}
\]
Hence (68) can be written as (21), namely, \((\rho, \delta, S)\) is a also the solution to the LMI (21).

Since (20) is a strict LMI with a feasible solution \((\rho, \delta, S)\), for some sufficiently small number \( \epsilon > 0 \), we have
\[
\begin{align*}
-\mathcal{M}^T \left( \rho + \frac{h}{N} \delta \right) - \left( \rho + \frac{h}{N} \delta \right) \mathcal{M} - \mathcal{M}^T \\
+ \frac{h}{N} I_{nN} G B^T S B G I_{nN} + \left( 2 \frac{h}{N} ||\mathcal{M}|| + 1 \right) \epsilon I_{nN} < 0.
\end{align*}
\]
Let \((\rho_\epsilon, \delta_\epsilon, S_\epsilon) = (\rho, \delta - \epsilon I_{nN}, S)\). Then we can obtain
\[
\begin{align*}
-\mathcal{M}^T \left( \rho_\epsilon + \frac{h}{N} \delta \right) - \left( \rho_\epsilon + \frac{h}{N} \delta \right) \mathcal{M} \\
- \delta_\epsilon + \frac{h}{N} I_{nN} G B^T S B G I_{nN} \\
- \mathcal{M}^T \left( \rho + \frac{h}{N} \delta \right) - \left( \rho + \frac{h}{N} \delta \right) \mathcal{M} - \mathcal{M}^T \\
+ \frac{h}{N} I_{nN} G B^T S B G I_{nN} \\
+ \left( \frac{h}{N} \mathcal{M}^T + \frac{h}{N} M + I_{(N+1)n} \right) \epsilon \\
\leq -\mathcal{M}^T \left( \rho + \frac{h}{N} \delta \right) - \left( \rho + \frac{h}{N} \delta \right) \mathcal{M} - \mathcal{M}^T \\
+ \frac{h}{N} I_{nN} G B^T S B G I_{nN} + \left( \frac{2}{N} ||\mathcal{M}|| + 1 \right) \epsilon I_{nN} < 0,
\end{align*}
\]
which indicates that \((\rho_\epsilon, \delta_\epsilon, S_\epsilon)\) is also a solution to the LMI (20). Similarly, we can show that \((\rho_\epsilon, \delta_\epsilon, S_\epsilon)\) is also a solution to the LMI (21) provided that \( \epsilon \) is sufficiently small.

Now, by substituting \((\rho, \delta, S)\) into the left-hand side of (19), we can compute
\[
\Theta = \begin{bmatrix} \frac{h}{n} \left\{ S_\epsilon \right\} & \frac{h}{n} \left\{ S_\epsilon \right\} \end{bmatrix} = \begin{bmatrix} \rho + \frac{h}{N} (\delta - \epsilon I_{nN}) \end{bmatrix} \begin{bmatrix} 0_{nN \times n} & 0_{nN \times n} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix},
\]
which implies that \((\rho_\epsilon, \delta_\epsilon, S_\epsilon)\) is also a solution to the LMI (19). The proof is completed.

References

Zhao-Yan Li was born in Hebei Province, PR China, on August 13, 1982. She received her B.Sc. Degree from the Department of Information Engineering at North China University of Water Conservancy and Electric Power, Zhengzhou, PR China, in 2005, and her M.Sc. and Ph.D. Degrees in Department of Mathematics, Harbin Institute of Technology, PR China, in 2007 and 2010, respectively. She is a Research Associate at the Department of Electrical and Computer Engineering, University of Virginia from July 2012 to August 2013. She is now a lecturer in the Department of Mathematics at Harbin Institute of Technology, PR China. Her research interest includes stochastic system theory and time-delay systems.

Bin Zhou was born in Luotian County, Huanggang, Hubei Province, PR China, on July 28, 1981. He received the Bachelor’s degree, the Master’s Degree and the Ph.D. degree from the Department of Control Science and Engineering at Harbin Institute of Technology, Harbin, China, in 2004, 2006 and 2010, respectively. He was a Research Associate at the Department of Mechanical Engineering, University of Hong Kong from December 2007 to March 2008, a Visiting Fellow at the School of Computing and Mathematics, University of Western Sydney from May 2009 to August 2009, and a Visiting Research Scientist at the Department of Electrical and Computer Engineering, University of Virginia from July 2012 to August 2013. In February 2009, he joined the School of Astronautics, Harbin Institute of Technology, where he has been a Professor since December 2012. He is a reviewer for American Mathematical Review and is an active reviewer for a number of journals and conferences. He was selected as the “New Century Excellent Talents in University”, the Ministry of Education of China in 2011. He received the “National Excellent Doctoral Dissertation Award” in 2012 from the Academic Degrees Committee of the State Council and the Ministry of Education of PR China. He is currently an associate editor on the Conference Editorial Board of the IEEE Control Systems Society.

Zongli Lin is a Professor of Electrical and Computer Engineering at University of Virginia. He received his B.S. degree in Mathematics and Computer Science from Xiamen University, Xiamen, China, in 1983, his Master of Engineering degree in Automatic Control from Chinese Academy of Space Technology, Beijing, China, in 1989, and his Ph.D. degree in Electrical and Computer Engineering from Washington State University, Pullman, Washington, USA, in 1994. He was an Associate Editor of the IEEE Transactions on Automatic Control (2001–2003), IEEE/ASME Transactions on Mechatronics (2006–2009), IEEE Control Systems Magazine (2005–2012). He has served on the operating committees and program committees of several conferences and was an elected member of the Board of Governors of the IEEE Control Systems Society (2008–2010). He currently chairs the IEEE Control Systems Society Technical Committee on Nonlinear Systems and Control and serves on the editorial boards of several journals and book series, including Automatica, Systems & Control Letters, Science China: Information Science, and Springer/Birkhauser book series Control Engineering. He is a Fellow of the Institute of Electrical and Electronics Engineers (IEEE), the International Federation of Automatic Control (IFAC) and the American Association for the Advancement of Science (AAAS).