MATHEMATICS OF COMPUTATION
Volume 85, Number 299, May 2016, Pages 1225–1261
http://dx.doi.org/10.1090/mcom/3022
Article electronically published on September 2, 2015

OPTIMAL ERROR ESTIMATES FOR DISCONTINUOUS GALERKIN METHODS BASED ON UPWIND-BIASED FLUXES FOR LINEAR HYPERBOLIC EQUATIONS

XIONG MENG, CHI-WANG SHU, AND BOYING WU

ABSTRACT. We analyze discontinuous Galerkin methods using upwind-biased numerical fluxes for time-dependent linear conservation laws. In one dimension, optimal a priori error estimates of order $k+1$ are obtained for the semidiscrete scheme when piecewise polynomials of degree at most k ($k \geq 0$) are used. Our analysis is valid for arbitrary nonuniform regular meshes and for both periodic boundary conditions and for initial-boundary value problems. We extend the analysis to the multidimensional case on Cartesian meshes when piecewise tensor product polynomials are used, and to the fully discrete scheme with explicit Runge–Kutta time discretization. Numerical experiments are shown to demonstrate the theoretical results.

1. Introduction

In this paper, we study the semidiscrete and fully discrete discontinuous Galerkin (DG) method using upwind-biased numerical fluxes for solving linear hyperbolic conservation laws

$$u_t + \sum_{i=1}^{d} c_i u_{x_i} = 0, \qquad (\boldsymbol{x}, t) \in \Omega \times (0, T], \tag{1.1a}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.1b}$$

where Ω is a bounded rectangular domain of $\mathbb{R}^d$, and $\boldsymbol{x} = (x_1, \ldots, x_d)$. Here, c_i are constants and $u_0(\boldsymbol{x})$ is a given smooth function. Both the periodic boundary condition and the inflow boundary condition are discussed.

Traditionally, purely upwind numerical fluxes are used for DG methods applied to linear hyperbolic equations. However, purely upwind fluxes may be difficult to construct for complex systems, as they would require exact eigenstructures of the flux Jacobian. It would be interesting to study the property of DG schemes when the more general upwind-biased fluxes are used. For such a semidiscrete DG method applied to linear conservation laws, we prove the L^2-stability and optimal convergence results in one dimension and in multidimensions for Cartesian meshes. These results are extended to the fully discrete scheme coupled with explicit third order

Received by the editor August 22, 2013 and, in revised form, October 30, 2014 and November 23, 2014.

2010 *Mathematics Subject Classification.* Primary 65M60, 65M12, 65M15.

Key words and phrases. Discontinuous Galerkin method, upwind-biased flux, conservation laws, error estimates.

The research of the second author was supported by NSF grants DMS-1112700 and DMS-1418750, and by DOE grant DE-FG02-08ER25863.

total variation diminishing Runge–Kutta (TVDRK3) time discretization, under the standard temporal-spatial CFL condition.

The DG method discussed in this paper is a class of finite element methods usually using discontinuous piecewise polynomials as the solution and the test functions. It was first introduced by Reed and Hill [19] for solving a steady-state linear hyperbolic equation in the framework of neutron transport. Later, it was developed by Cockburn et al. [8, 10–12] for solving time-dependent nonlinear conservation laws, termed Runge–Kutta discontinuous Galerkin (RKDG) method, by combining DG spatial discretization and explicit high order nonlinearly stable Runge–Kutta time discretizations [21]. The most important ingredient in designing the DG scheme is the choice of the so-called numerical fluxes, which should guarantee the stability of the scheme. Typically, the numerical fluxes of DG methods for conservation laws and wave equations are chosen as upwind or general monotone fluxes in the framework of finite volume methodology. In the present paper, we consider numerical fluxes in a more general setting, namely, the upwind-biased fluxes, which may not always be monotone, but could be easier to construct, produce less numerical dissipation, and thus give us a good approximation to smooth solutions.

For smooth solutions of linear conservation laws, optimal a priori error estimates of order $k+1$ for one-dimensional and some multidimensional cases [7, 17, 20] can be obtained for steady-state solution or for the space-time DG discretization when upwind fluxes are used. Here and in what follows, k is the polynomial degree of the finite element space. For smooth solutions of time-dependent nonlinear conservation laws, optimal a priori error estimates of order $k+1$ are obtained for the RKDG schemes using upwind numerical fluxes [25, 26]. We would like to remark that, as far as we know, all previous optimal L^2 a priori error estimates of order $k+1$ for DG methods for first order hyperbolic equations are obtained for purely upwind fluxes, including space-time, semidiscrete, or fully discrete DG methods. When general, not purely upwind numerical fluxes are used, quasi-optimal a priori error estimates of order $k+\frac{1}{2}$ are obtained for fully discrete RKDG schemes in [3, 25, 26]. The paper [3] also contains quasi-optimal a priori error estimates for general stabilized finite element methods besides DG. We emphasize that the techniques used in [3] or in [25, 26] cannot yield optimal error estimates of order $k+1$ for DG methods with numerical fluxes which are not purely upwind. For second order wave equations, an energy conserving interior penalty DG method was proposed in [15], and optimal error estimates were derived for a fully discrete scheme coupled with the second order leap-frog time discretization [16]. In [23], Xing et al. developed an energy conserving local DG (LDG) method for solving the second order linear wave equation and showed an optimal error estimate. For high order linear wave equations, Xu and Shu [24] proposed a general approach for proving optimal error estimates by utilizing the LDG scheme and its time derivatives with different test functions and fully making use of the so-called Gauss–Radau projections.

Let us now mention in particular a work that is relevant to the current paper. In [1], following the earlier work in [5], a class of conservative DG schemes without introducing auxiliary variables that approximate the different order derivatives of the solution for the generalized KdV equation were constructed and analyzed. A key ingredient in this more recent scheme is the construction and analysis of a globally defined projection ω that is consistent with the dispersion term and possesses an important conservative property enjoyed by the fully continuous problem. In [1], some restrictions on the degree of the polynomials k and the mesh $\mathcal{I}_h$ are required to ensure the existence and optimal approximation properties of the projection ω.

The main idea in the derivation of optimal convergence results in the present paper is to obtain the energy stability of our scheme and to construct and analyze some suitable projections. In one dimension, motivated by the successful analysis of the projection ω in [1], we construct a proper global projection $P_h^\star$ according to the very definition of the upwind-biased flux, when the periodic boundary condition is considered. Unlike the projection ω, the existence and optimal approximation properties of $P_h^\star$ do not demand any special requirements on the degree of the polynomials k or the mesh $\mathcal{I}_h$, as long as the mesh is regular. Moreover, the projection $P_h^\star$ can eliminate the inter-element boundary terms involving $u - P_h^\star u$, and thus the optimal convergence rate is derived, which also works for the fully discretized scheme. For the initial-boundary value problems, we modify $P_h^\star$ to construct another projection $\tilde{P}_h$ which is no longer globally coupled, and therefore the optimal order of accuracy can be easily obtained. The proof of optimal convergence results is valid for arbitrary nonuniform regular meshes and for polynomials of degree $k \geq 0$, no matter whether the periodic boundary condition or the inflow boundary condition is concerned.

For multidimensional Cartesian meshes, we follow the same arguments in the one-dimensional case to construct a suitable projection $\Pi_h^\star$ and analyze its approximation properties, when the periodic boundary problems are considered. Again, the existence and optimal approximation properties of the projection $\Pi_h^\star$ do not require any special restrictions on the degree of the polynomials k or the mesh Ω_h, as long as the mesh is shape-regular. Although this new projection cannot eliminate the boundary terms in our error equation involving $u - \Pi_h^\star u$ that affect the convergence rate, the superconvergence result of $\Pi_h^\star$ on Cartesian meshes helps to obtain the optimal convergence results. We would like to remark that the superconvergence proof of $\Pi_h^\star$ requires some additional regularity assumption on the exact solution.

The objective of this paper is to set a solid theoretical foundation on the fact that upwind-biased numerical fluxes are comparable with the purely upwind fluxes in designing DG methods for solving linear problems in terms of stability and optimal error estimates. Indeed, we have proved L^2-stability and optimal convergence results for linear conservation laws in one dimension and in multidimensions for Cartesian meshes. To our best knowledge, this is the first optimal convergence proof for DG methods applied to conservation laws when upwind-biased, but not purely upwind, numerical fluxes are considered.

The paper is organized as follows. In section 2, we present the semidiscrete DG method using the upwind-biased numerical flux for the one-dimensional linear conservation laws. In this simple setting, the main ideas of how to perform stability and convergence analysis are clearly displayed. Thus, the L^2-stability and optimal convergence results are obtained. Extensions of the analysis to the multidimensional case are carried out in section 3, in which optimal error estimates are proved, essentially following the same idea of the one-dimensional case. In section 4, by taking one-dimensional linear conservation laws for example, we provide stability analysis and optimal error estimates for the fully discrete RKDG scheme coupled with TVDRK3 time discretization. Numerical experiments confirming the optimality of our theoretical results are given in section 5. We end in section 6 with some concluding remarks and perspectives for future work.

2. The DG method for the one-dimensional case

In this section, to display the main idea of our optimal error estimates, we present and analyze the semidiscrete DG method using upwind-biased fluxes for the following one-dimensional linear conservation law

$$u_t + u_x = 0, \qquad (x,t) \in [0,2\pi] \times (0,T], \tag{2.1a}$$

$$u(x,0) = u_0(x), \quad x \in R, \tag{2.1b}$$

where $u_0(x)$ is a smooth function. Both the periodic boundary condition $u(0,t) = u(2\pi,t)$ and the inflow boundary condition $u(0,t) = g(t)$ are discussed.

2.1. Notation and definitions in the one-dimensional case.

In this subsection, we shall first introduce some notation and definitions in the one-dimensional case, which will be used in our analysis for one-dimensional linear conservation laws.

2.1.1. *The meshes.* Let us denote by $\mathcal{I}_h$ a tessellation of the computational interval $I = [0, 2\pi]$, consisting of cells $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ with $1 \le j \le N$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

The following standard notation of DG methods will be used. Denote $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$, $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $h = \max_j h_j$, and $\rho = \min_j h_j$. The mesh is assumed to be regular in the sense that h/ρ is always bounded during mesh refinements, namely, there exists a positive constant γ such that $\gamma h \le \rho \le h$. We denote by $p^-_{j+\frac{1}{2}}$ and $p^+_{j+\frac{1}{2}}$ the values of p at the discontinuity point $x_{j+\frac{1}{2}}$, from the left cell, I_j, and from the right cell, I_{j+1}, respectively. In what follows, we employ $[\![p]\!] = p^+ - p^-$ and $\{\!\{p\}\!\} = \frac{1}{2}(p^+ + p^-)$ to represent the jump and the mean value of p at each element boundary point. The following discontinuous piecewise polynomials space is chosen as the finite element space:

$$V_h \equiv V_h^k = \left\{ v \in L^2(I) : v|_{I_j} \in P^k(I_j),\ j = 1, \dots, N \right\},$$

where $P^k(I_j)$ denotes the set of polynomials of degree up to $k \ge 0$ defined on the cell I_j.

2.1.2. *Function spaces and norms.* For any integer $l \ge 0$, denote by $\|\cdot\|_{W^{l,2}(I_j)}$ and $\|\cdot\|_{W^{l,\infty}(I_j)}$ the standard Sobolev norms on the cell I_j. Then, the norms of the broken Sobolev spaces $W^{l,p}(\mathcal{I}_h) := \{u \in L^2(I) : u|_{I_j} \in W^{l,p}(I_j),\ \forall j = 1, \dots, N\}$ with $p = 2, \infty$ are given by

$$\|u\|_{W^{l,2}(\mathcal{I}_h)} = \|u\|_{H^l(\mathcal{I}_h)} = \left(\sum_{j=1}^N \|u\|^2_{H^l(I_j)} \right)^{\frac{1}{2}}, \quad \|u\|_{W^{l,\infty}(\mathcal{I}_h)} = \max_{1 \le j \le N} \|u\|_{W^{l,\infty}(I_j)}.$$

In the case $l = 0$, we denote $\|u\|_{L^2(I)} = \|u\|_{H^0(\mathcal{I}_h)}$.

2.2. **The DG method and stability analysis.**

2.2.1. *The DG scheme.* The approximation of the semidiscrete DG scheme to solve (2.1) is as follows. Find, for any time $t \in (0, T]$, the unique function $u_h = u_h(t) \in V_h^k$ such that

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} u_h (v_h)_x dx + \hat{u}_h v_h^- |_{j+\frac{1}{2}} - \hat{u}_h v_h^+ |_{j-\frac{1}{2}} = 0 \tag{2.2}$$

holds for all $v_h \in V_h^k$ and all $j = 1, \ldots, N$.

In this paper, instead of using the purely upwind flux, we adopt the so-called upwind-biased flux. To be more specific, we choose

$$\hat{u}_h = \theta u_h^- + (1-\theta) u_h^+ \quad \text{at } x_{j+\frac{1}{2}}, \quad j = 0, \ldots, N \tag{2.3}$$

for the periodic boundary condition, and

$$(\hat{u}_h)_{j+\frac{1}{2}} = \begin{cases} \theta u_h^- + (1-\theta) u_h^+, & j = 1, \ldots, N-1, \\ (u_h)_{\frac{1}{2}}^- = g(t), & j = 0, \\ (u_h)_{N+\frac{1}{2}}^-, & j = N \end{cases} \tag{2.4}$$

for the inflow boundary condition. Here and in what follows, $\theta > \frac{1}{2}$. Note that fluxes (2.3) and (2.4) are not even monotone when $1 > \theta > \frac{1}{2}$. For the initial condition, we take $u_h(0) = \mathbb{P}_h u_0$, and it holds that

$$\|u_0 - \mathbb{P}_h u_0\|_{L^2(I)} \leq C h^{k+1} \|u_0\|_{H^{k+1}(\mathcal{I}_h)}, \tag{2.5}$$

where $\mathbb{P}_h$ is the L^2 projection into V_h^k.

2.2.2. *Stability analysis.* The DG scheme using the upwind-biased numerical flux for the one-dimensional linear conservation laws satisfies the following L^2-stability.

Proposition 2.1 (stability). *The solution of the semidiscrete DG method defined by* (2.2) *with the numerical flux* (2.3) *or* (2.4) *satisfies*

$$\|u_h(T)\|_{L^2(I)} \leq \|u_h(0)\|_{L^2(I)} + \|g\|_{L^2([0,T])},$$

where $\|g\|_{L^2([0,T])} = \left(\int_0^T g^2(t) dt \right)^{\frac{1}{2}}$.

Proof. Taking $v_h = u_h$ in the DG scheme (2.2) and summing over all j, we obtain, for the periodic boundary condition,

$$\int_I (u_h)_t u_h dx + \sum_{j=1}^{N} ((\{\!\{u_h\}\!\} - \hat{u}_h)[\![u_h]\!])_{j+\frac{1}{2}} = 0.$$

Using the upwind-biased numerical flux (2.3), we get

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_{L^2(I)}^2 + (\theta - \frac{1}{2}) \sum_{j=1}^{N} [\![u_h]\!]_{j+\frac{1}{2}}^2 = 0.$$

For the initial-boundary value problem, we follow the same line and take into account the upwind-biased numerical flux (2.4) to obtain, after some simple algebraic calculations:

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_{L^2(I)}^2 + (\theta - \frac{1}{2}) \sum_{j=1}^{N-1} [\![u_h]\!]_{j+\frac{1}{2}}^2 + \frac{1}{2}((u_h)_{N+\frac{1}{2}}^-)^2 + \frac{1}{2}[\![u_h]\!]_{\frac{1}{2}}^2 = \frac{1}{2} g^2(t).$$

In both cases, since $\theta > \frac{1}{2}$, we get

$$\frac{d}{dt}\|u_h\|^2_{L^2(I)} \leq g^2(t).$$

The L^2-stability result follows immediately by integrating the above inequality with respect to time between 0 and T. This completes the proof. □

2.3. **A priori error estimates.** In this subsection, we state the a priori error estimate of the DG method using upwind-biased fluxes for solving one-dimensional conservation laws and then briefly discuss our main results. The detailed proofs will be provided in the next subsection, in which problems with the periodic boundary condition and the inflow boundary condition are discussed separately.

Theorem 2.2 (error estimate)**.** *Assume that the exact solution u of* (2.1) *is sufficiently smooth with bounded derivatives, i.e., $\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}$ and $\|u_t\|_{W^{k+1,\infty}(\mathcal{I}_h)}$ are bounded uniformly for any time $t \in [0,\ T]$. Let u_h be the numerical solution of the semidiscrete DG scheme* (2.2) *when the upwind-biased numerical flux* (2.3) *or* (2.4) *with $\theta > \frac{1}{2}$ is used, which corresponds, respectively, to the periodic boundary problem or the initial-boundary value problem. For regular triangulations of $I = [0, 2\pi]$, if the finite element space V_h^k of piecewise polynomials with arbitrary degree $k \geq 0$ is used, then for $T > 0$ there holds the error estimate*

$$\|u(T) - u_h(T)\|_{L^2(I)} \leq C(1+T)h^{k+1}, \tag{2.6}$$

where C depends on θ, $\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}$ and $\|u_t\|_{W^{k+1,\infty}(\mathcal{I}_h)}$, but is independent of h.

Remark 2.3. The proof of Theorem 2.2 is valid for the DG scheme (2.2) using the numerical flux (2.3) or (2.4) with any $\theta > \frac{1}{2}$ and for any $k \geq 0$. In particular, we conclude that the optimal convergence results hold true not only for the DG method with general monotone fluxes and the upwind flux corresponding to $\theta \geq 1$, but for the DG method with numerical fluxes that are not even monotone corresponding to $1 > \theta > \frac{1}{2}$.

Remark 2.4. On the extreme case $\theta = \frac{1}{2}$, the numerical flux (2.3) reduces to the central flux $\{\!\{u_h\}\!\}$, and thus the DG method (2.2) becomes an energy conservative scheme, when the periodic boundary condition is concerned. In this case, existence and the optimal approximation property of the global projection $P_h^\star$ can also be proved under the same assumptions on the polynomial degree k and the mesh $\mathcal{I}_h$ as those in [1]. Moreover, the inverse of the coefficient matrix is an $N \times N$ circulant matrix with $[1, -1, 1, -1, \cdots, 1]$ as its first row, and thus the optimal convergence results can be obtained under the aforementioned conditions. In the numerical experiments, we can see that the convergence rate is $k+1$ for even values of k, but only k for odd values of k.

Remark 2.5. Even though the proof in this section is presented only for simple scalar linear conservation laws (2.1), the same optimal convergence results can be easily obtained for one-dimensional linear systems along the same lines. This is due to the fact that the one-dimensional linear hyperbolic system can be decoupled to several scalar equations.

2.4. **Proof of the error estimates.** This section is devoted to the proof of Theorem 2.2 stated in the previous section. In subsection 2.4.1, we recall some *local* projections and define a *global* projection we are going to use in our analysis. The optimal convergence result for the periodic boundary condition is thus proved in subsection 2.4.1. For the initial-boundary value problem, the analysis is much easier since the corresponding projection is no longer coupled, and the optimal convergence result is derived in subsection 2.4.2.

2.4.1. *The periodic boundary condition case.* It is well known that the locally defined Gauss–Radau projections $P_h^{\pm}$ play an important role in deriving optimal error estimates of DG methods using upwind fluxes applied to hyperbolic problems and LDG methods using alternating fluxes applied to elliptic as well as parabolic problems. Taking the definition of P_h^- in, for example, [4], for any given function $u \in H^1(I_j)$ and an arbitrary cell $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, the restriction of $P_h^- u$ to I_j is the unique polynomial in V_h^k satisfying

$$\int_{I_j} P_h^- u(x) v_h dx = \int_{I_j} u(x) v_h dx \quad \forall v_h \in P^{k-1}(I_j), \tag{2.7a}$$

$$(P_h^- u)^- = u^- \quad \text{at } x_{j+\frac{1}{2}}. \tag{2.7b}$$

In the case $k = 0$, we set $P^{-1}(I_j) = \{0\}$. From the above definition, we clearly see that the projection P_h^- can be solved explicitly on each element.

The projection defined above has the following approximation error estimates. Let $u \in W^{k+1,p}(I_j)$ $(p = 2, \infty)$; then by the standard approximation theory [2,6]

$$\|u - P_h^- u\|_{L^p(I_j)} \le C h_j^{k+1} \|u\|_{W^{k+1,p}(I_j)}, \quad p = 2, \infty, \tag{2.8}$$

where C is independent of h_j.

For the DG scheme (2.2) using the upwind-biased flux (2.3) solving conservation laws with periodic boundary conditions, we need to construct a globally defined projection $P_h^\star$, since we have made use of the information of the numerical solution at cell interfaces from both the left and the right. For $u \in H^1(\mathcal{I}_h)$, the projection $P_h^\star u$ is defined as the element of V_h^k that satisfies

$$\int_{I_j} P_h^\star u(x) v_h dx = \int_{I_j} u(x) v_h dx \qquad \forall v_h \in P^{k-1}(I_j), \tag{2.9a}$$

$$\widehat{P_h^\star u} = \widehat{u} \qquad \text{at } x_{j+\frac{1}{2}},\ j = 1, \dots, N \tag{2.9b}$$

with $\theta > \frac{1}{2}$. Here and below, parallel to the definition of the upwind-biased numerical flux $\hat{u}_h$, we denote $\widehat{w} := \theta w^- + (1-\theta) w^+$ for any $w \in H^1(\mathcal{I}_h)$. In particular, when $\theta = 1$, $\widehat{w} = w^-$ and the projection $P_h^\star$ reduces to the standard Gauss–Radau projection P_h^-.

Existence and the optimal approximation properties of the *global* projection $P_h^\star$ are established in the following lemma

Lemma 2.6. *Assume that u is sufficiently smooth and periodic. Then, there exists a unique $P_h^\star$ satisfying the conditions (2.9). Moreover, there holds the following approximation properties*

$$\|u - P_h^\star u\|_{L^\infty(I_j)} \le C h^{k+1} \|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}, \tag{2.10a}$$

$$\|u - P_h^\star u\|_{L^2(I_j)} \le C h^{k+\frac{3}{2}} \|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}, \tag{2.10b}$$

where $C = C(\theta)$ is independent of the cell I_j and the mesh size h.

Proof. Let us first prove existence and uniqueness of $P_h^\star u$.

For $u \in H^1(\mathcal{I}_h)$, let $P_h^- u$ and $P_h^\star u$ be defined by (2.7) and (2.9), respectively. Denote $P_h^\star u - u = P_h^\star u - P_h^- u + P_h^- u - u := E + P_h^- u - u$. If we can prove the existence and uniqueness of E, then $P_h^\star u = E + P_h^- u$ will exist and is unique. To do that, we start by combining (2.7) and (2.9) to obtain

$$\int_{I_j} E v_h dx = 0 \qquad \forall v_h \in P^{k-1}(I_j), \tag{2.11a}$$

$$\widehat{E} = (1-\theta)(u - P_h^- u)^+ \qquad \text{at } x_{j+\frac{1}{2}}, \tag{2.11b}$$

which holds for $j = 1, \dots, N$. Let $P_l(\xi)$ be the lth-order Legendre polynomials that are orthogonal on $[-1,\ 1]$ with $\xi = \frac{2(x-x_j)}{h_j}$. Next, on each element, we denote $P_l(\xi) = P_l(\frac{2(x-x_j)}{h_j}) := P_{j,l}(x)$ for $x \in I_j$. Thus, the restriction of E to each I_j, denoted by E_j, can be expressed in the form

$$E_j(x) = \sum_{l=0}^{k} \alpha_{j,l} P_{j,l}(x) = \sum_{l=0}^{k} \alpha_{j,l} P_l(\xi).$$

The equality (2.11a) and the orthogonality of the Legendre polynomials yield that

$$\alpha_{j,l} = 0, \quad l = 0, \dots, k-1,\ j = 1, \dots, N.$$

Hence, $E_j(x) = \alpha_{j,k} P_k(\xi)$. It follows from the equality (2.11b) that, for $j = 1, \dots, N$,

$$\theta \alpha_{j,k} + (1-\theta)(-1)^k \alpha_{j+1,k} = (1-\theta)(u - P_h^- u)^+_{j+\frac{1}{2}} \tag{2.12}$$

since $P_k(1) = 1$ and $P_k(-1) = (-1)^k$. For any given u, $P_h^- u$ is uniquely defined, and thus (2.12) is an $N \times N$ linear system for $\alpha_{j,k}$ $(j = 1, \dots, N)$ with a known right-hand side. For periodic boundary conditions discussed in this subsection, if we denote

$$\eta_{j+1} = (u - P_h^- u)^+_{j+\frac{1}{2}} \tag{2.13}$$

for $j = 0, \dots, N-1$ with $\eta_{N+1} = \eta_1$, then the linear system (2.12) can be written in the matrix form

$$A \alpha_k = (1-\theta)\eta, \tag{2.14}$$

where $A = \mathrm{circ}(\theta, (1-\theta)(-1)^k, 0, \cdots, 0)$ is an $N \times N$ circulant matrix, and

$$\alpha_k = \begin{pmatrix} \alpha_{1,k} \\ \alpha_{2,k} \\ \vdots \\ \alpha_{N,k} \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_2 \\ \eta_3 \\ \vdots \\ \eta_{N+1} \end{pmatrix}.$$

Here and from now on, $\mathrm{circ}(a_1, a_2, \cdots, a_N)$ denotes an $N \times N$ circulant matrix with the first row $(a_1, a_2, \cdots, a_N)$. Moreover, we can readily compute the determinant of A in the form

$$|A| = \theta^N (1 - q^N),\ q = \frac{(\theta-1)(-1)^k}{\theta}.$$

We can see that the matrix is always invertible for all values of k whenever $\theta \neq \frac{1}{2}$. This establishes existence and uniqueness of E, and thus $P_h^\star u$.

In what follows, we shall derive the optimal approximation error estimates (2.10). It is well known (see [13]) that the inverse of a nonsingular circulant matrix is also circulant. So, we can compute the inverse of A in the form of $A^{-1} = \beta_N \text{circ}(1, q, q^2, \cdots, q^{N-1})$, where

$$\beta_N = \frac{1}{\theta(1-q^N)}.$$

Clearly, for $\theta > \frac{1}{2}$,

$$\lim_{N\to\infty} \beta_N = \frac{1}{\theta}, \qquad |q| < 1.$$

Consequently, β_N is bounded, namely, there exists a positive constant C such that for all N, $\beta_N \le C$. Therefore, $\alpha_{j,k}$ can be solved explicitly from (2.14)

$$\alpha_{j,k} = \beta_N(1-\theta)\sum_{m=1}^{N} d_{j,m}\eta_{m+1}, \quad j = 1, \ldots, N,$$

where $\{d_{j,m}\}_{m=1}^N$ $(j = 1, \ldots, N)$ are the entries in the jth row of the circulant matrix $\text{circ}(1, q, q^2, \cdots, q^{N-1})$. Since

$$|\eta_{j+1}| \le \|u - P_h^- u\|_{L^\infty(I_{j+1})} \le Ch^{k+1}\|u\|_{W^{k+1,\infty}(I_{j+1})} \le Ch^{k+1}\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)},$$

we arrive at a bound for $\alpha_{j,k}$ $(j = 1, \ldots, N)$ as follows:

$$\begin{aligned}
|\alpha_{j,k}| &\le \beta_N|1-\theta|Ch^{k+1}\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}\left(1 + |q| + |q|^2 + \cdots + |q|^{N-1}\right) \\
&\le C\frac{|1-\theta|}{1-|q|}h^{k+1}\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)} \\
&:= C(\theta)h^{k+1}\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}. \qquad (2.15)
\end{aligned}$$

In the case $p = \infty$, (2.15) yields a bound for $\|E\|_{L^\infty(I_j)}$:

$$\|E\|_{L^\infty(I_j)} = \|\alpha_{j,k}P_{j,k}\|_{L^\infty(I_j)} = |\alpha_{j,k}|\|P_k\|_{L^\infty([-1,1])} \le C(\theta)h^{k+1}\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}.$$

Then, the error estimate (2.10a) follows by the approximation property of the projection P_h^-, (2.8), and the triangle inequality.

In the case $p = 2$, (2.10b) is a direct consequence of (2.10a). The proof of this lemma is thus completed. □

Remark 2.7. The error estimate (2.10) is global because of the coupling of the information from different cells (2.9b). Moreover, the bound is optimal, and they hold true for all values of $k \ge 0$ and arbitrary nonuniform regular meshes. In addition, (2.10b) trivially implies that

$$\|u - P_h^\star u\|_{L^2(I)} \le C(\theta)h^{k+1}\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}. \qquad (2.16)$$

In what follows, for the one-dimensional linear problems discussed in this section, we would like to introduce the DG discretization operator $\mathcal{D}$ as in [18, 24]: for each cell $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$,

$$\mathcal{D}_{I_j}(w, v; \hat{w}) = -\int_{I_j} wv_x dx + \hat{w}v^-|_{j+\frac{1}{2}} - \hat{w}v^+|_{j-\frac{1}{2}}, \qquad (2.17)$$

and we denote

$$\mathcal{D}(w, v; \hat{w}) = \sum_{j=1}^{N} \mathcal{D}_{I_j}(w, v; \hat{w}).$$

Using the definitions of the operator $\mathcal{D}$ and the projection $P_h^\star$, we have the following lemma.

Lemma 2.8. *For periodic boundary conditions and $v_h \in V_h^k$, we have*

$$\mathcal{D}(u - P_h^\star u, v_h; u - \widehat{P_h^\star u}) = 0, \tag{2.18}$$

where $P_h^\star$ is the projection defined by (2.9)*.*

Note that for a given function u, the projection $P_h^\star u$ is a unique polynomial in P^k, and hence $P_h^\star u_h = u_h$. Therefore, $e := u - u_h = u - P_h^\star u + P_h^\star e$. To estimate $\|u(T) - u_h(T)\|_{L^2(I)}$, it remains to estimate $\|P_h^\star e(T)\|_{L^2(I)}$, which is contained in the following lemma.

Lemma 2.9. *Under the same conditions as in Theorem* 2.2*, we have*

$$\|P_h^\star e(T)\|_{L^2(I)} \le C(1+T)h^{k+1}, \tag{2.19}$$

where C depends solely on θ and $\|u_t\|_{W^{k+1,\infty}(\mathcal{I}_h)}$.

Proof. Using the DG discretization operator $\mathcal{D}$, the DG scheme (2.2) can be written as

$$\int_{I_j} (u_h)_t v_h dx + \mathcal{D}_{I_j}(u_h, v_h; \hat{u}_h) = 0,$$

for any $v_h \in V_h^k$ and $j = 1, \dots, N$. Since the exact solution u also satisfies the weak formulation, we thus have the Galerkin orthogonality

$$\int_{I_j} (u - u_h)_t v_h dx + \mathcal{D}_{I_j}(u - u_h, v_h; u - \hat{u}_h) = 0,$$

which holds for any $v_h \in V_h^k$ and $j = 1, \dots, N$. Now, we take $v_h = P_h^\star e \in V_h^k$ in the above identity and sum over all j to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|P_h^\star e\|_{L^2(I)}^2 + \int_I (u - P_h^\star u)_t P_h^\star e dx + \mathcal{D}(u - P_h^\star u, P_h^\star e; u - \widehat{P_h^\star u})\\ + \mathcal{D}(P_h^\star e, P_h^\star e; \widehat{P_h^\star e}) = 0.\end{aligned} \tag{2.20}$$

By Lemma 2.8,

$$\mathcal{D}(u - P_h^\star u, P_h^\star e; u - \widehat{P_h^\star u}) = 0.$$

By the same argument as that in the proof of L^2-stability,

$$\mathcal{D}(P_h^\star e, P_h^\star e; \widehat{P_h^\star e}) = (\theta - \frac{1}{2}) \sum_{j=1}^N [\![P_h^\star e]\!]_{j+\frac{1}{2}}^2.$$

Inserting the above two estimates into (2.20) and taking into account $\theta > \frac{1}{2}$, we get, after a straightforward application of the Cauchy–Schwarz inequality

$$\frac{1}{2}\frac{d}{dt}\|P_h^\star e\|_{L^2(I)}^2 \le \|(u - P_h^\star u)_t\|_{L^2(I)} \|P_h^\star e\|_{L^2(I)},$$

which is,

$$\frac{d}{dt}\|P_h^\star e\|_{L^2(I)} \le \|(u - P_h^\star u)_t\|_{L^2(I)} \le C(\theta) h^{k+1} \|u_t\|_{W^{k+1,\infty}(\mathcal{I}_h)},$$

where we have used the fact that $(P_h^\star u)_t = P_h^\star u_t$ and the approximation error estimate (2.16). The estimate (2.19) follows immediately by integrating the above inequality with respect to time between 0 and T and by using the initial error estimate (2.5). This completes the proof of Lemma 2.9. □

We are now ready to get the final error estimate (2.6) by combining (2.16) and (2.19). This completes the proof of Theorem 2.2 when the periodic boundary condition is considered.

2.4.2. *The inflow boundary condition case.* When the inflow boundary condition $u(0,t) = g(t)$ is considered, to perform error analysis, we need to introduce another projection $\tilde{P}_h$, which is defined as the element of V_h^k that satisfies

$$\int_{I_j} \tilde{P}_h u(x) v_h dx = \int_{I_j} u(x) v_h dx \qquad \forall v_h \in P^{k-1}(I_j), \tag{2.21a}$$

$$\widehat{\tilde{P}_h u} = \hat{u} \qquad \text{at } x_{j+\frac{1}{2}},\ j = 1, \dots, N-1, \tag{2.21b}$$

$$(\tilde{P}_h u)^- = u^- \qquad \text{at } x_{j+\frac{1}{2}},\ j = N \tag{2.21c}$$

with $\theta > \frac{1}{2}$. Again, when $\theta = 1$, the projection $\tilde{P}_h$ reduces to the standard P_h^- projection.

Existence and the optimal approximation property of the projection $\tilde{P}_h$ are established in the following lemma.

Lemma 2.10. *Assume that u is sufficiently smooth, i.e., $u \in W^{k+1,\infty}(\mathcal{I}_h)$. Then, $\tilde{P}_h u$ exists and is unique. Moreover, there holds the following approximation properties*

$$\|u - \tilde{P}_h u\|_{L^\infty(I_j)} \le C h^{k+1} \|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}, \tag{2.22a}$$

$$\|u - \tilde{P}_h u\|_{L^2(I_j)} \le C h^{k+\frac{3}{2}} \|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}, \tag{2.22b}$$

where $C = C(\theta)$ is independent of the cell I_j and the mesh size h.

Proof. For $u \in H^1(\mathcal{I}_h)$, let $P_h^- u$ and $\tilde{P}_h u$ be defined by (2.7) and (2.21), respectively. Denote $\tilde{P}_h u - u = \tilde{P}_h u - P_h^- u + P_h^- u - u := \tilde{E} + P_h^- u - u$. If we can prove the existence and uniqueness of $\tilde{E}$, then $\tilde{P}_h u = \tilde{E} + P_h^- u$ will exist and is unique. To do that, we combine (2.7) and (2.21) to obtain

$$\int_{I_j} \tilde{E} v_h dx = 0 \qquad \forall v_h \in P^{k-1}(I_j), \tag{2.23a}$$

$$\widehat{\tilde{E}} = (1-\theta)(u - P_h^- u)^+ \qquad \text{at } x_{j+\frac{1}{2}},\ j = 1, \dots, N-1, \tag{2.23b}$$

$$\tilde{E}^-_{j+\frac{1}{2}} = 0 \qquad \text{at } x_{j+\frac{1}{2}},\ j = N. \tag{2.23c}$$

In contrast to (2.11), the conditions (2.23) are no longer globally coupled. To be more specific, denote by $\tilde{E}_j$ the restriction of $\tilde{E}$ to I_j and let

$$\tilde{E}_j(x) = \sum_{l=0}^{k} \tilde{\alpha}_{j,l} P_{j,l}(x) = \sum_{l=0}^{k} \tilde{\alpha}_{j,l} P_l(\xi).$$

The equality (2.23a) and the orthogonality of the Legendre polynomials yield that

$$\tilde{\alpha}_{j,l} = 0, \quad l = 0, \dots, k-1,\ j = 1, \dots, N.$$

Hence, $\tilde{E}_j(x) = \tilde{\alpha}_{j,k} P_k(\xi)$. It follows from the equality (2.23c) that,

$$\tilde{\alpha}_{N,k} = 0.$$

Thus, the $N \times N$ linear system (2.23b) and (2.23c) can be decoupled and solved explicitly with $\tilde{\alpha}_{N,k} = 0$ as a starting point, i.e., for $j = 1, \ldots, N-1$,

$$\tilde{\alpha}_{j,k} = \frac{1-\theta}{\theta} \sum_{m=1}^{N-j} q^{m-1} \eta_{j+m},$$

where q and η_j have been defined in subsection 2.4.1. Next, by an analysis similar to that in the proof of (2.15), we have that, for $j = 1, \ldots, N$

$$|\tilde{\alpha}_{j,k}| \le C(\theta) h^{k+1} \|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}, \tag{2.24}$$

and hence the error estimate (2.22) follows immediately. This completes the proof of Lemma 2.10. □

We are now ready to prove Theorem 2.2 when the inflow boundary condition is considered. Note that for a given function u, $\tilde{P}_h u$ is a unique polynomial in P^k, and thus $\tilde{P}_h u_h = u_h$. Therefore, $e := u - u_h = u - \tilde{P}_h u + \tilde{P}_h e$.

By Galerkin orthogonality, we arrive at the following identity:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{P}_h e\|^2_{L^2(I)} + \int_I (u - \tilde{P}_h u)_t \tilde{P}_h e dx + \mathcal{D}(u - \tilde{P}_h u, \tilde{P}_h e; \widehat{u - \tilde{P}_h u}) \\ + \mathcal{D}(\tilde{P}_h e, \tilde{P}_h e; \widehat{\tilde{P}_h e}) = 0. \end{aligned} \tag{2.25}$$

On the one hand,

$$\mathcal{D}(u - \tilde{P}_h u, \tilde{P}_h e; \widehat{u - \tilde{P}_h u}) = 0.$$

On the other hand, since $(\tilde{P}_h e)^-_{\frac{1}{2}} = 0$,

$$\mathcal{D}(\tilde{P}_h e, \tilde{P}_h e; \widehat{\tilde{P}_h e}) = (\theta - \frac{1}{2}) \sum_{j=1}^{N-1} [\![\tilde{P}_h e]\!]^2_{j+\frac{1}{2}} + \frac{1}{2}((\tilde{P}_h e)^-_{N+\frac{1}{2}})^2 + \frac{1}{2}((\tilde{P}_h e)^+_{\frac{1}{2}})^2.$$

Inserting the above two estimates into (2.25) and using the approximation error estimate (2.22), we get

$$\frac{d}{dt}\|\tilde{P}_h e\|_{L^2(I)} \le C(\theta) h^{k+1} \|u_t\|_{W^{k+1,\infty}(\mathcal{I}_h)}.$$

To complete the proof of Theorem 2.2 for the inflow boundary condition case, we only need to integrate the above inequality over $[0, T]$ and use the triangle inequality.

3. The DG method for the multidimensional case

In this section, we consider the semidiscrete DG method using upwind-biased fluxes for the multidimensional linear conservation laws (1.1). Without loss of generality, we describe our DG scheme and prove optimal a priori error estimates in two dimensions ($d = 2$); all the arguments we present in our analysis depend on the tensor product structure of the mesh and can be easily extended to the more general cases $d > 2$. Hence, from now on, we shall restrict ourselves mainly to the following two-dimensional problem

$$u_t + u_x + u_y = 0, \qquad (x, y, t) \in \Omega \times (0, T], \tag{3.1a}$$

$$u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \Omega, \tag{3.1b}$$

which corresponds to the case $d = 2$ with $\boldsymbol{x} = (x, y)$ and $c_1 = c_2 = 1$ in (1.1). For the sake of simplicity, we consider only the periodic boundary conditions; the initial-boundary value problems can be studied along the same lines of the one-dimensional case.

3.1. **Notation and definitions in the two-dimensional case.**

3.1.1. *The meshes.* Let $\Omega_h = \{K\}$ denote a tessellation of Ω with shape-regular rectangular elements K, and set $\partial\Omega_h = \{\partial K : K \in \Omega_h\}$. For each $K \in \Omega_h$, we denote by h_K the diameter of K and set, as usual, $h = \max_{K\in\Omega_h} h_K$. The finite element space associated with the mesh Ω_h is of the form

$$Z_h := \{v \in L^2(\Omega) : v|_K \in Q^k(K) \quad \forall K \in \Omega_h\},$$

where $Q^k(K)$ is the space of tensor product of polynomials of degrees at most k in each variable defined on K.

We would like to adopt the following notation, which are standard in DG analysis. For an arbitrary rectangular element $K := I_i \times J_j$ with $I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $J_j = (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$, we denote $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$, $y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}})$, $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, and $h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$. Similar to the one-dimensional case, $(u_h)^-_{i+\frac{1}{2},y}$, $(u_h)^+_{i+\frac{1}{2},y}$, $(u_h)^-_{x,j+\frac{1}{2}}$, and $(u_h)^+_{x,j+\frac{1}{2}}$ are well defined. Moreover, we use $[\![u_h]\!]_{i+\frac{1}{2},y} = (u_h)^+_{i+\frac{1}{2},y} - (u_h)^-_{i+\frac{1}{2},y}$ and $\{\!\{u_h\}\!\}_{i+\frac{1}{2},y} = \frac{1}{2}((u_h)^+_{i+\frac{1}{2},y} + (u_h)^-_{i+\frac{1}{2},y})$ to denote the jump and the mean value of u_h on the vertical edge $(x_{i+\frac{1}{2}}, y)$ when $y \in J_j$. Analogously, on the horizontal edges, we can define $(u_h)^-_{x,j+\frac{1}{2}}$, $(u_h)^+_{x,j+\frac{1}{2}}$, $[\![u_h]\!]_{x,j+\frac{1}{2}}$ and $\{\!\{u_h\}\!\}_{x,j+\frac{1}{2}}$ when $x \in I_i$.

3.1.2. *Function spaces and norms.* Denote by $\|\cdot\|_{L^2(K)}$ and $\|\cdot\|_{L^2(\partial K)}$ the standard L^2 norms on K and ∂K. For any integer $l \geq 0$, the standard Sobolev l norm on K is denoted by $\|\cdot\|_{H^l(K)}$. Furthermore, the norms of the broken Sobolev spaces $W^{l,p}(\Omega_h) = \{v \in L^2(\Omega) : v|_K \in W^{l,p}(K) \quad \forall K \in \Omega_h\}$ with $p = 2, \infty$ are given by

$$\|v\|_{H^l(\Omega_h)} = \left(\sum_{K\in\Omega_h} \|v\|^2_{H^l(K)}\right)^{\frac{1}{2}}, \qquad \|v\|_{W^{l,\infty}(\Omega_h)} = \max_{K\in\Omega_h} \|v\|_{W^{l,\infty}(K)}.$$

In the case $l = 0$, we denote $\|v\|_{L^2(\Omega_h)} = \|v\|_{H^0(\Omega_h)}$.

3.2. **The DG method and stability analysis.**

3.2.1. *The DG scheme.* The approximation of the semidiscrete DG scheme to solve (3.1) is as follows. Find, for any time $t \in (0, T]$, the unique function $u_h = u_h(t) \in Z_h$ such that

$$\begin{aligned} &\int_K (u_h)_t v_h d\boldsymbol{x} - \int_K u_h (v_h)_x d\boldsymbol{x} + \int_{J_j} (\hat{u}_h v_h)_{i+\frac{1}{2},y} dy - \int_{J_j} (\hat{u}_h v_h)_{i-\frac{1}{2},y} dy \\ &\qquad - \int_K u_h (v_h)_y d\boldsymbol{x} + \int_{I_i} (\hat{u}_h v_h)_{x,j+\frac{1}{2}} dx - \int_{I_i} (\hat{u}_h v_h)_{x,j-\frac{1}{2}} dx = 0 \end{aligned} \tag{3.2}$$

holds for all $v_h \in Z_h$ and all $K \in \Omega_h$, where $\int_K (\cdot) d\boldsymbol{x}$ stands for $\int_{I_i}\int_{J_j} (\cdot) dy dx$. Here, $(\hat{u}_h)_{i+\frac{1}{2},y}$ and $(\hat{u}_h)_{x,j+\frac{1}{2}}$ are chosen to be upwind-biased numerical fluxes, i.e.,

$$(\hat{u}_h)_{i+\frac{1}{2},y} \equiv \hat{u}_h(x_{i+\frac{1}{2}}, y) = \theta_1 u_h(x^-_{i+\frac{1}{2}}, y) + (1-\theta_1) u_h(x^+_{i+\frac{1}{2}}, y), \tag{3.3a}$$

$$(\hat{u}_h)_{x,j+\frac{1}{2}} \equiv \hat{u}_h(x, y_{j+\frac{1}{2}}) = \theta_2 u_h(x, y^-_{j+\frac{1}{2}}) + (1-\theta_2) u_h(x, y^+_{j+\frac{1}{2}}) \tag{3.3b}$$

with $\theta_1, \theta_2 > \frac{1}{2}$. We would like to emphasize that for the boundary integral terms in (3.2), if v_h is not single-valued on inter-element faces, we take its value from inside of K and restrict on ∂K. For the initial condition, we simply take $u_h(0) = \mathbb{P}_h u_0$ and we have

$$\|u_0 - \mathbb{P}_h u_0\|_{L^2(\Omega_h)} \le Ch^{k+1}\|u_0\|_{H^{k+1}(\Omega_h)}, \tag{3.4}$$

where $\mathbb{P}_h$ is the L^2 projection into Z_h.

3.2.2. *Stability analysis.* The DG scheme using the upwind-biased numerical fluxes for the two-dimensional linear conservation laws satisfies the following L^2-stability.

Proposition 3.1 (stability). *The solution of the semidiscrete DG method defined by* (3.2) *with the numerical fluxes* (3.3) *satisfies*

$$\|u_h(T)\|_{L^2(\Omega_h)} \le \|u_h(0)\|_{L^2(\Omega_h)}.$$

Proof. Taking $v_h = u_h$ in the DG scheme (3.2), summing over all K and using upwind-biased numerical fluxes (3.3), we get, for the periodic boundary conditions

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2_{L^2(\Omega_h)} + (\theta_1 - \frac{1}{2})\sum_{j=1}^{N_2}\int_{J_j}\sum_{i=1}^{N_1}[\![u_h]\!]^2_{i+\frac{1}{2},y}dy + (\theta_2 - \frac{1}{2})\sum_{i=1}^{N_1}\int_{I_i}\sum_{j=1}^{N_2}[\![u_h]\!]^2_{x,j+\frac{1}{2}}dx = 0,$$

where we have used the fact that $\sum_{K\in\Omega_h}(\cdot) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(\cdot)$ for Cartesian meshes. Since $\theta_1, \theta_2 > \frac{1}{2}$, we get

$$\frac{d}{dt}\|u_h\|^2_{L^2(\Omega_h)} \le 0,$$

from which the L^2-stability result follows immediately. □

3.3. **A priori error estimates.** Let us now state the a priori error estimates for the two-dimensional case, whose proof will be given in the next subsection.

Theorem 3.2 (error estimate). *Assume that the exact solution u of* (3.1) *is sufficiently smooth with bounded derivatives, i.e., $\|u\|_{W^{2k+3,\infty}(\Omega_h)}$ and $\|u_t\|_{W^{k+1,\infty}(\Omega_h)}$ are bounded uniformly for any time $t \in [0,\ T]$. Let u_h be the numerical solution of the semidiscrete DG scheme* (3.2) *when the upwind-biased numerical fluxes* (3.3) *with $\theta_1, \theta_2 > \frac{1}{2}$ are used. For two-dimensional Cartesian meshes, if the finite element space Z_h of piecewise tensor product polynomials of degree $k \ge 0$ is used, then for $T > 0$ there holds the error estimate*

$$\|u(T) - u_h(T)\|_{L^2(\Omega_h)} \le C(1+T)h^{k+1}, \tag{3.5}$$

where C depends on θ_1, θ_2, k, $\|u\|_{W^{2k+3,\infty}(\Omega_h)}$, and $\|u_t\|_{W^{k+1,\infty}(\Omega_h)}$, but is independent of h.

3.4. **Proof of the error estimates.** In this section we prove Theorem 3.2 stated in the previous section. To do that, we proceed as follows. First, in subsection 3.4.1, we establish the existence as well as uniqueness of a suitably defined special projection $\Pi_h^\star$. Moreover, the optimal approximation properties of $\Pi_h^\star$ are derived. We proceed in subsection 3.4.2 by performing an energy analysis, from which the optimal convergence result follows. Finally, we complete the proof of Theorem 3.2 in subsection 3.4.3 by proving a superconvergence result that is used in our analysis.

3.4.1. *Estimate of* $\|u - \Pi_h^\star u\|_{L^2(\Omega_h)}$. Prior to giving the definition of the special projection $\Pi_h^\star$, we would like to recall the projection Π_h^- introduced by Cockburn et al. in their study of optimal error estimates of the LDG method for elliptic problems on Cartesian meshes in [9]. The projection Π_h^- is defined to be the tensor product of the projections in one dimension. Specifically, on a rectangular element K, for $u \in C^0(\overline{K})$, we have

$$\Pi_h^- u = P_{h_x}^- \otimes P_{h_y}^- u$$

where the sub-subscripts x and y indicate that the one-dimensional projection P_h^- defined by (2.7) is applied with respect to the corresponding variable.

For clarity, we shall list explicitly the formulations for Π_h^-. On a rectangular element $K := I_i \times J_j = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$, we have, for all $v_h \in Q^{k-1}(K)$, that

$$\int_K \Pi_h^- u(x,y) v_h(x,y) d\boldsymbol{x} = \int_K u(x,y) v_h(x,y) d\boldsymbol{x}, \tag{3.6a}$$

$$\int_{J_j} \Pi_h^- u(x_{i+\frac{1}{2}}^-, y) v_h(x_{i+\frac{1}{2}}^-, y) dy = \int_{J_j} u(x_{i+\frac{1}{2}}^-, y) v_h(x_{i+\frac{1}{2}}^-, y) dy, \tag{3.6b}$$

$$\int_{I_i} \Pi_h^- u(x, y_{j+\frac{1}{2}}^-) v_h(x, y_{j+\frac{1}{2}}^-) dx = \int_{I_i} u(x, y_{j+\frac{1}{2}}^-) v_h(x, y_{j+\frac{1}{2}}^-) dx, \tag{3.6c}$$

$$\Pi_h^- u(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-) = u(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-). \tag{3.6d}$$

The projection defined above has the following approximation error estimates. Let $u \in H^{k+1}(K)$, then

$$\|u - \Pi_h^- u\|_{L^2(K)} + h_K \|u - \Pi_h^- u\|_{H^1(K)} \le C h_K^{k+1} \|u\|_{H^{k+1}(K)}. \tag{3.7a}$$

Moreover, for $u \in W^{k+1,\infty}(K)$,

$$\|u - \Pi_h^- u\|_{L^\infty(\partial K)} \le C h_K^{k+1} \|u\|_{W^{k+1,\infty}(K)}. \tag{3.7b}$$

For more details of approximation properties for the projection Π_h^-, see [9].

We are now ready to present the definition of the special projection $\Pi_h^\star$, which is defined to be the tensor product of the projection $P_h^\star$ in one dimension. That is, for $u \in W^{1,\infty}(\Omega_h)$, we define

$$\Pi_h^\star u = P_{h_x}^\star \otimes P_{h_y}^\star u, \tag{3.8}$$

where the sub-subscripts x and y indicate that the one-dimensional projection $P_h^\star$ defined by (2.9) is applied with respect to the corresponding variable. To be more specific, $\Pi_h^\star u$ is a unique polynomial in Q^k that satisfies

$$\int_K \Pi_h^\star u(x,y) v_h(x,y) d\boldsymbol{x} = \int_K u(x,y) v_h(x,y) d\boldsymbol{x}, \tag{3.9a}$$

$$\int_{J_j} \widehat{\Pi_h^\star u}(x_{i+\frac{1}{2}}, y) v_h(x_{i+\frac{1}{2}}^-, y) dy = \int_{J_j} \widehat{u}(x_{i+\frac{1}{2}}, y) v_h(x_{i+\frac{1}{2}}^-, y) dy, \tag{3.9b}$$

$$\int_{I_i} \widehat{\Pi_h^\star u}(x, y_{j+\frac{1}{2}}) v_h(x, y_{j+\frac{1}{2}}^-) dx = \int_{I_i} \widehat{u}(x, y_{j+\frac{1}{2}}) v_h(x, y_{j+\frac{1}{2}}^-) dx, \tag{3.9c}$$

$$\widetilde{\Pi_h^\star u}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = \widetilde{u}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) \tag{3.9d}$$

for all $v_h \in Q^{k-1}(K)$ and $K \in \Omega_h$. Here and in what follows, for any $w \in W^{1,\infty}(\Omega_h)$,

$$\begin{aligned}
\widehat{w}(x_{i+\frac{1}{2}}, y) &= \theta_1 w(x^-_{i+\frac{1}{2}}, y) + (1-\theta_1) w(x^+_{i+\frac{1}{2}}, y),\\
\widehat{w}(x, y_{j+\frac{1}{2}}) &= \theta_2 w(x, y^-_{j+\frac{1}{2}}) + (1-\theta_2) w(x, y^+_{j+\frac{1}{2}}),\\
\widetilde{w}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) &= \theta_1\theta_2 w(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) + \theta_1(1-\theta_2) w(x^-_{i+\frac{1}{2}}, y^+_{j+\frac{1}{2}})\\
&\quad + \theta_2(1-\theta_1) w(x^+_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) + (1-\theta_1)(1-\theta_2) w(x^+_{i+\frac{1}{2}}, y^+_{j+\frac{1}{2}})
\end{aligned}$$

with $\theta_1, \theta_2 > \frac{1}{2}$. In particular, when $\theta_1 = \theta_2 = 1$, the projection $\Pi_h^\star$ reduces to the projection Π_h^-.

Existence and the optimal approximation property of the *global* projection $\Pi_h^\star$ are established in the following lemma.

Lemma 3.3. *Assume that u is sufficiently smooth and periodic. Then, there exists a unique $\Pi_h^\star$ satisfying the conditions (3.9). Moreover, there holds the following approximation properties*

$$\|u - \Pi_h^\star u\|_{L^2(K)} + h\|u - \Pi_h^\star u\|_{H^1(K)} \le C h^{k+2} \|u\|_{W^{k+1,\infty}(\Omega_h)}, \tag{3.10}$$

where $C = C(\theta_1, \theta_2, k)$ is independent of the element K and the mesh size h.

Proof. Let $\Pi_h^- u$ and $\Pi_h^\star u$ be defined by (3.6) and (3.9), respectively. Denote $\Pi_h^\star u - u = \Pi_h^\star u - \Pi_h^- u + \Pi_h^- u - u := \mathcal{E} + \Pi_h^- u - u$. If we can prove the existence and uniqueness of $\mathcal{E}$, then $\Pi_h^\star u = \mathcal{E} + \Pi_h^- u$ will exist and is unique. To do that, we start by combining (3.6) and (3.9) to obtain, for all $v_h \in Q^{k-1}(K)$ and $K \in \Omega_h$,

$$\int_K \mathcal{E}(x,y) v_h(x,y) d\boldsymbol{x} = 0, \tag{3.11a}$$

$$\int_{J_j} \hat{\mathcal{E}}(x_{i+\frac{1}{2}}, y) v_h(x^-_{i+\frac{1}{2}}, y) dy = (1-\theta_1) \int_{J_j} (u - \Pi_h^- u)(x^+_{i+\frac{1}{2}}, y) v_h(x^-_{i+\frac{1}{2}}, y) dy, \tag{3.11b}$$

$$\int_{I_i} \hat{\mathcal{E}}(x, y_{j+\frac{1}{2}}) v_h(x, y^-_{j+\frac{1}{2}}) dx = (1-\theta_2) \int_{I_i} (u - \Pi_h^- u)(x, y^+_{j+\frac{1}{2}}) v_h(x, y^-_{j+\frac{1}{2}}) dx, \tag{3.11c}$$

$$\tilde{\mathcal{E}}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = \overline{u - \Pi_h^- u}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}), \tag{3.11d}$$

where

$$\begin{aligned}
\overline{u - \Pi_h^- u}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) &= \theta_1(1-\theta_2)(u - \Pi_h^- u)(x^-_{i+\frac{1}{2}}, y^+_{j+\frac{1}{2}})\\
&\quad + \theta_2(1-\theta_1)(u - \Pi_h^- u)(x^+_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}})\\
&\quad + (1-\theta_1)(1-\theta_2)(u - \Pi_h^- u)(x^+_{i+\frac{1}{2}}, y^+_{j+\frac{1}{2}}).
\end{aligned}$$

Next, let $P_l(\mu)$ be the lth-order Legendre polynomials that are orthogonal on $[-1,\ 1]$ with $\mu = \frac{2(x-x_i)}{h_i^x}$. Thus, the restriction of $\mathcal{E}$ to each element $K = I_i \times J_j$, denoted by $\mathcal{E}_{i,j}$, can be expressed in the form

$$\mathcal{E}_{i,j}(x,y) = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k} \alpha_{i,j,l_1,l_2} P_{i,l_1}(x) P_{j,l_2}(y) = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k} \alpha_{i,j,l_1,l_2} P_{l_1}(\mu) P_{l_2}(\nu)$$

with $\nu = \frac{2(y-y_j)}{h_j^y}$.

In what follows, we shall prove existence and uniqueness of $\Pi_h^\star$ as well as its optimal approximation properties in five steps.

Step 1. The equality (3.11a) and the orthogonality of the Legendre polynomials yield that

$$\alpha_{i,j,l_1,l_2} = 0,$$

for $l_1, l_2 = 0, \dots, k-1$ and $i = 1, \cdots, N_1, j = 1, \cdots, N_2$. Hence,

$$\begin{aligned}\mathcal{E}_{i,j}(x,y) &= \alpha_{i,j,k,0}P_k(\mu)P_0(\nu) + \alpha_{i,j,k,1}P_k(\mu)P_1(\nu) + \cdots + \alpha_{i,j,k,k-1}P_k(\mu)P_{k-1}(\nu)\\ &\quad + \alpha_{i,j,0,k}P_0(\mu)P_k(\nu) + \alpha_{i,j,1,k}P_1(\mu)P_k(\nu) + \cdots + \alpha_{i,j,k-1,k}P_{k-1}(\mu)P_k(\nu)\\ &\quad + \alpha_{i,j,k,k}P_k(\mu)P_k(\nu),\end{aligned} \tag{3.12}$$

where each line on the right-hand side will be denoted by S_1, S_2 and S_3. In the subsequent three steps, we will work on the coefficients involved in S_1, S_2 and S_3 separately.

Step 2. Let us first deal with α_{i,j,k,l_2} ($l_2 = 0, \cdots, k-1$) involved in S_1. It follows from the equality (3.11b) and the orthogonality of the Legendre polynomials that, for $l_2 = 0, \cdots, k-1$ and $i = 1, \cdots, N_1, j = 1, \cdots, N_2$,

$$\theta_1 \alpha_{i,j,k,l_2} + (1-\theta_1)(-1)^k \alpha_{i+1,j,k,l_2} = \frac{2l_2+1}{2}(1-\theta_1)\mathtt{b}_{i+1}^{(j)}, \tag{3.13}$$

where

$$\mathtt{b}_{i+1}^{(j)} = \int_{-1}^{1} (u - \Pi_h^- u)(x_{i+\frac{1}{2}}^+, y(\nu))P_{l_2}(\nu)d\nu$$

with $y(\nu) = y_j + \frac{h_j^y}{2}\nu$. It is easy to show that, for $i = 1, \cdots, N_1, j = 1, \cdots, N_2$,

$$\begin{aligned}\left|\mathtt{b}_{i+1}^{(j)}\right| &\le 2\|u - \Pi_h^- u\|_{L^\infty(\partial K_R)}\|P_{l_2}\|_{L^\infty([-1,1])}\\ &\le Ch^{k+1}\|u\|_{W^{k+1,\infty}(K_R)} \le Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)},\end{aligned}$$

where $K_R = I_{i+1} \times J_j$ and we have used the approximation error estimate (3.7b). Thus, for each $l_2 = 0, \cdots, k-1$, the linear system (3.13) can be written in the matrix form

$$A_1 \alpha_{k,l_2} = \frac{2l_2+1}{2}(1-\theta_1)\mathtt{b}, \tag{3.14}$$

where $A_1 = \mathrm{circ}(\theta_1, (1-\theta_1)(-1)^k, 0, \cdots, 0)$ is an $N_1 \times N_1$ circulant matrix, and

$$\alpha_{k,l_2} = \left(\alpha_{k,l_2}^{(1)}, \alpha_{k,l_2}^{(2)}, \cdots, \alpha_{k,l_2}^{(N_2)}\right), \qquad \mathtt{b} = \left(\mathtt{b}^{(1)}, \mathtt{b}^{(2)}, \cdots, \mathtt{b}^{(N_2)}\right),$$

with

$$\alpha_{k,l_2}^{(j)} = \begin{pmatrix} \alpha_{1,j,k,l_2} \\ \alpha_{2,j,k,l_2} \\ \vdots \\ \alpha_{N_1,j,k,l_2} \end{pmatrix}, \qquad \mathtt{b}^{(j)} = \begin{pmatrix} \mathtt{b}_2^{(j)} \\ \mathtt{b}_3^{(j)} \\ \vdots \\ \mathtt{b}_{N_1+1}^{(j)} \end{pmatrix}, \ j = 1, \cdots, N_2.$$

Similar to the one-dimensional case, we conclude that the coefficient matrix A_1 is always invertible whenever $\theta_1 \ne \frac{1}{2}$ and that $A_1^{-1} = \beta_{N_1}\mathrm{circ}(1, q_1, q_1^2, \cdots, q_1^{N_1-1})$, where

$$\beta_{N_1} = \frac{1}{\theta_1(1-q_1^{N_1})}, \ q_1 = \frac{(\theta_1-1)(-1)^k}{\theta_1}.$$

Clearly, for $\theta_1 > \frac{1}{2}$,

$$\lim_{N_1\to\infty} \beta_{N_1} = \frac{1}{\theta_1}, \qquad |q_1| < 1.$$

Hence, there exists a positive constant C such that $\beta_{N_1} \le C$. Now, for each $l_2 = 0, \cdots, k-1$, we can solve α_{i,j,k,l_2} explicitly from (3.14)

$$\alpha_{i,j,k,l_2} = \frac{2l_2+1}{2}(1-\theta_1)\beta_{N_1}\sum_{m=1}^{N_1} r_{i,m}\mathsf{b}_{m+1}^{(j)}, \quad i = 1, \cdots, N_1,\ j = 1, \cdots, N_2,$$

where $\{r_{i,m}\}_{m=1}^{N_1}$ $(i = 1, \cdots, N_1)$ are the entries in the ith row of the circulant matrix $\mathrm{circ}(1, q_1, q_1^2, \cdots, q_1^{N_1-1})$. Consequently, we arrive at a bound for α_{i,j,k,l_2} as follows

$$\begin{aligned} |\alpha_{i,j,k,l_2}| &\le \frac{2k-1}{2}|1-\theta_1|Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}\left(1+|q_1|+|q_1|^2+\cdots+|q_1|^{N_1-1}\right) \\ &\le \frac{2k-1}{2}\frac{|1-\theta_1|}{1-|q_1|}Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)} \\ &:= C(\theta_1,k)h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}, \end{aligned} \tag{3.15}$$

which holds for all $l_2 = 0, \cdots, k-1$ and $i = 1, \cdots, N_1, j = 1, \cdots, N_2$.

Step 3. Next we work on $\alpha_{i,j,l_1,k}$ $(l_1 = 0, \cdots, k-1)$ involved in S_2. It follows from the equality (3.11c) and the orthogonality of the Legendre polynomials that, for $l_1 = 0, \cdots, k-1$ and $i = 1, \cdots, N_1, j = 1, \cdots, N_2$,

$$\theta_2\alpha_{i,j,l_1,k} + (1-\theta_2)(-1)^k\alpha_{i,j+1,l_1,k} = \frac{2l_1+1}{2}(1-\theta_2)\mathsf{c}_{j+1}^{(i)}, \tag{3.16}$$

where

$$\mathsf{c}_{j+1}^{(i)} = \int_{-1}^{1}(u-\Pi_h^- u)(x(\mu), y_{j+\frac{1}{2}}^+)P_{l_1}(\mu)d\mu$$

with $x(\mu) = x_i + \frac{h_i^x}{2}\mu$. It is easy to show that, for $i = 1, \cdots, N_1, j = 1, \cdots, N_2$,

$$\begin{aligned} \left|\mathsf{c}_{j+1}^{(i)}\right| &\le 2\|u-\Pi_h^- u\|_{L^\infty(\partial K_A)}\|P_{l_1}\|_{L^\infty([-1,1])} \\ &\le Ch^{k+1}\|u\|_{W^{k+1,\infty}(K_A)} \le Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}, \end{aligned}$$

where $K_A = I_i \times J_{j+1}$ and we have used the approximation error estimate (3.7b). Thus, for each $l_1 = 0, \cdots, k-1$, the linear system (3.16) can be written in the matrix form

$$A_2\alpha_{l_1,k} = \frac{2l_1+1}{2}(1-\theta_2)\mathsf{c}, \tag{3.17}$$

where $A_2 = \mathrm{circ}(\theta_2, (1-\theta_2)(-1)^k, 0, \cdots, 0)$ is an $N_2 \times N_2$ circulant matrix, and

$$\alpha_{l_1,k} = \left(\alpha_{l_1,k}^{(1)}, \alpha_{l_1,k}^{(2)}, \cdots, \alpha_{l_1,k}^{(N_1)}\right), \qquad \mathsf{c} = \left(\mathsf{c}^{(1)}, \mathsf{c}^{(2)}, \cdots, \mathsf{c}^{(N_1)}\right),$$

with

$$\alpha_{l_1,k}^{(i)} = \begin{pmatrix} \alpha_{i,1,l_1,k} \\ \alpha_{i,2,l_1,k} \\ \vdots \\ \alpha_{i,N_2,l_1,k} \end{pmatrix}, \qquad \mathsf{c}^{(i)} = \begin{pmatrix} \mathsf{c}_2^{(i)} \\ \mathsf{c}_3^{(i)} \\ \vdots \\ \mathsf{c}_{N_2+1}^{(i)} \end{pmatrix}, \ i = 1, \cdots, N_1.$$

Similar to the one-dimensional case, we conclude that the coefficient matrix A_2 is always invertible whenever $\theta_2 \neq \frac{1}{2}$ and that $A_2^{-1} = \beta_{N_2}\mathrm{circ}(1, q_2, q_2^2, \cdots, q_2^{N_2-1})$, where

$$\beta_{N_2} = \frac{1}{\theta_2(1-q_2^{N_2})}, \; q_2 = \frac{(\theta_2-1)(-1)^k}{\theta_2}.$$

Clearly, for $\theta_2 > \frac{1}{2}$,

$$\lim_{N_2\to\infty} \beta_{N_2} = \frac{1}{\theta_2}, \qquad |q_2| < 1.$$

Hence, there exists a positive constant C such that $\beta_{N_2} \leq C$. Now, for each $l_1 = 0, \cdots, k-1$, we can solve $\alpha_{i,j,l_1,k}$ explicitly from (3.17)

$$\alpha_{i,j,l_1,k} = \frac{2l_1+1}{2}(1-\theta_2)\beta_{N_2}\sum_{m=1}^{N_2} s_{j,m}\mathtt{c}_{m+1}^{(i)}, \quad j = 1, \cdots, N_2, \; i = 1, \cdots, N_1,$$

where $\{s_{j,m}\}_{m=1}^{N_2}$ $(j = 1, \cdots, N_2)$ are the entries in the jth row of the circulant matrix $\mathrm{circ}(1, q_2, q_2^2, \cdots, q_2^{N_2-1})$. Consequently, we arrive at a bound for $\alpha_{i,j,l_1,k}$ as follows:

$$\begin{aligned} |\alpha_{i,j,l_1,k}| &\leq \frac{2k-1}{2}|1-\theta_2|Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}\left(1+|q_2|+|q_2|^2+\cdots+|q_2|^{N_2-1}\right) \\ &\leq \frac{2k-1}{2}\frac{|1-\theta_2|}{1-|q_2|}Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)} \\ &:= C(\theta_2,k)h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}, \end{aligned} \tag{3.18}$$

which holds for all $l_1 = 0, \cdots, k-1$ and $i = 1, \cdots, N_1, j = 1, \cdots, N_2$.

Step 4. Let us now handle $\alpha_{i,j,k,k}$ involved in S_3. The equality (3.11d) yields that, after some algebraic manipulations, for $i = 1, \cdots, N_1, j = 1, \cdots, N_2$,

$$\begin{aligned} &\theta_1\theta_2\alpha_{i,j,k,k} + \theta_1(1-\theta_2)(-1)^k\alpha_{i,j+1,k,k} + \theta_2(1-\theta_1)(-1)^k\alpha_{i+1,j,k,k} \\ &\qquad + (1-\theta_1)(1-\theta_2)(-1)^{2k}\alpha_{i+1,j+1,k,k} = \mathtt{d}_{i,j}, \end{aligned} \tag{3.19}$$

where $\mathtt{d}_{i,j}$ is a combination of some terms that are of order h^{k+1} by using the estimates (3.15), (3.18) and (3.7b). That is, for $i = 1, \cdots, N_1, j = 1, \cdots, N_2$, we have

$$|\mathtt{d}_{i,j}| \leq C(\theta_1, \theta_2, k)h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}.$$

If we now denote

$$\alpha_{k,k} = \begin{pmatrix} \alpha_{k,k}^{(1)} \\ \alpha_{k,k}^{(2)} \\ \vdots \\ \alpha_{k,k}^{(N_1)} \end{pmatrix}, \qquad \mathtt{d} = \begin{pmatrix} \mathtt{d}^{(1)} \\ \mathtt{d}^{(2)} \\ \vdots \\ \mathtt{d}^{(N_1)} \end{pmatrix}$$

with

$$\alpha_{k,k}^{(i)} = \begin{pmatrix} \alpha_{i,1,k,k} \\ \alpha_{i,2,k,k} \\ \vdots \\ \alpha_{i,N_2,k,k} \end{pmatrix}, \qquad \mathtt{d}^{(i)} = \begin{pmatrix} \mathtt{d}_{i,1} \\ \mathtt{d}_{i,2} \\ \vdots \\ \mathtt{d}_{i,N_2} \end{pmatrix}, \quad i = 1, \cdots, N_1,$$

then the system (3.19) can be written in the matrix form

$$A_1 \otimes A_2\alpha_{k,k} = \mathtt{d}, \tag{3.20}$$

where A_1 and A_2, respectively, are the coefficient matrices for the corresponding one-dimensional variable, and $\otimes$ is the Kronecker product of two matrices. Clearly, both A_1 and A_2 are invertible. Moreover,

$$(A_1 \otimes A_2)^{-1} = A_1^{-1} \otimes A_2^{-1}.$$

Therefore, by an analysis similar to that in the proof of (3.15) or (3.18), we get

$$\begin{aligned} |\alpha_{i,j,k,k}| &\leq \frac{1}{1-|q_1|}\frac{1}{1-|q_2|}Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)} \\ &:= C(\theta_1,\theta_2,k)h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}, \end{aligned} \tag{3.21}$$

which holds for all $i = 1, \cdots, N_1, j = 1, \cdots, N_2$.

From the above analysis, we conclude that $\mathcal{E}$ is uniquely defined on each element. This establishes the existence and uniqueness of $\Pi_h^\star$.

Step 5. We are now ready to derive the optimal approximation error estimate (3.10). Let us start by proving

$$\|u - \Pi_h^\star u\|_{L^2(K)} \leq C(\theta_1,\theta_2,k)h^{k+2}\|u\|_{W^{k+1,\infty}(\Omega_h)}. \tag{3.22}$$

To do that, we collect the estimates (3.15), (3.18) and (3.21) into (3.12) to obtain

$$\|\mathcal{E}\|_{L^\infty(K)} \leq C(\theta_1,\theta_2,k)h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)},$$

which implies that

$$\|\mathcal{E}\|_{L^2(K)} \leq h\|\mathcal{E}\|_{L^\infty(K)} \leq C(\theta_1,\theta_2,k)h^{k+2}\|u\|_{W^{k+1,\infty}(\Omega_h)}.$$

Thus, (3.22) follows from the approximation error estimate of the projection Π_h^-

$$\|u - \Pi_h^- u\|_{L^2(K)} \leq Ch^{k+1}\|u\|_{H^{k+1}(K)} \leq Ch^{k+2}\|u\|_{W^{k+1,\infty}(K)}$$

and the triangle inequality. Next, since $\|P'_{i,l_1}\|_{L^\infty(I_i)} \leq Ch^{-1}$ and $\|P'_{j,l_2}\|_{L^\infty(J_j)} \leq Ch^{-1}$, and hence

$$\|\mathcal{E}\|_{W^{1,\infty}(K)} \leq C(\theta_1,\theta_2,k)h^{k}\|u\|_{W^{k+1,\infty}(\Omega_h)},$$

the estimate

$$\|u - \Pi_h^\star u\|_{H^1(K)} \leq Ch^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}$$

can be proved analogously. This completes the proof of Lemma 3.3. □

Thus, the result of Lemma 3.3, (3.10), produces the estimate of $\|u - \Pi_h^\star u\|_{L^2(\Omega_h)}$

$$\|u - \Pi_h^\star u\|_{L^2(\Omega_h)} \leq C(\theta_1,\theta_2,k)h^{k+1}\|u\|_{W^{k+1,\infty}(\Omega_h)}. \tag{3.23}$$

3.4.2. *Estimate of* $\|\Pi_h^\star e\|_{L^2(\Omega_h)}$. Note that for a given function u, $\Pi_h^\star u$ is a unique polynomial in Q^k, and hence $\Pi_h^\star u_h = u_h$. Thus, $e := u - u_h = u - \Pi_h^\star u + \Pi_h^\star e$. To estimate $\|u(T) - u_h(T)\|_{L^2(\Omega_h)}$, it remains to estimate $\|\Pi_h^\star e(T)\|_{L^2(\Omega_h)}$, which is given in the following lemma.

Lemma 3.4. *Assume that* $u \in W^{2k+3,\infty}(\Omega_h)$ *and* $u_t \in W^{k+1,\infty}(\Omega_h)$. *Then we have*

$$\|\Pi_h^\star e(T)\|_{L^2(\Omega_h)} \leq C(1+T)h^{k+1}, \tag{3.24}$$

where C depends solely on θ_1, θ_2, k, $\|u\|_{W^{2k+3,\infty}(\Omega_h)}$ and $\|u_t\|_{W^{k+1,\infty}(\Omega_h)}$.

Proof. We begin by noting that the exact solution u of the problem (3.1) also satisfies the weak formulation (3.2), so we have the Galerkin orthogonality

$$\int_K e_t v_h d\boldsymbol{x} - \int_K e(v_h)_x d\boldsymbol{x} + \int_{J_j} ((u-\hat{u}_h)v_h)_{i+\frac{1}{2},y} dy - \int_{J_j} ((u-\hat{u}_h)v_h)_{i-\frac{1}{2},y} dy \\ - \int_K e(v_h)_y d\boldsymbol{x} + \int_{I_i} ((u-\hat{u}_h)v_h)_{x,j+\frac{1}{2}} dx - \int_{I_i} ((u-\hat{u}_h)v_h)_{x,j-\frac{1}{2}} dx = 0$$

for any $v_h \in Z_h$ and $K \in \Omega_h$. Next, we take $v_h = \Pi_h^\star e$ in the above identity, sum over all the element K and use periodic boundary conditions to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Pi_h^\star e\|^2_{L^2(\Omega_h)} + \int_{\Omega_h} (u-\Pi_h^\star u)_t \Pi_h^\star e d\boldsymbol{x} + \sum_{K\in\Omega_h} W_K = \sum_{K\in\Omega_h} G_K(u, \Pi_h^\star e),$$

where

$$W_K = -\int_K \Pi_h^\star e(\Pi_h^\star e)_x d\boldsymbol{x} + \int_{J_j} (\widehat{\Pi_h^\star e}\Pi_h^\star e)_{i+\frac{1}{2},y} dy - \int_{J_j} (\widehat{\Pi_h^\star e}\Pi_h^\star e)_{i-\frac{1}{2},y} dy \\ - \int_K \Pi_h^\star e(\Pi_h^\star e)_y d\boldsymbol{x} + \int_{I_i} (\widehat{\Pi_h^\star e}\Pi_h^\star e)_{x,j+\frac{1}{2}} dx - \int_{I_i} (\widehat{\Pi_h^\star e}\Pi_h^\star e)_{x,j-\frac{1}{2}} dx,$$

$$G_K(u,\Pi_h^\star e) = \int_K (u-\Pi_h^\star u)(\Pi_h^\star e)_x d\boldsymbol{x} - \int_{J_j} ((u-P_{h_y}^\star u)(\Pi_h^\star e)^-)_{i+\frac{1}{2},y} dy \\ + \int_{J_j} ((u-P_{h_y}^\star u)(\Pi_h^\star e)^+)_{i-\frac{1}{2},y} dy + \int_K (u-\Pi_h^\star u)(\Pi_h^\star e)_y d\boldsymbol{x} \\ - \int_{I_i} ((u-P_{h_x}^\star u)(\Pi_h^\star e)^-)_{x,j+\frac{1}{2}} dx + \int_{I_i} ((u-P_{h_x}^\star u)(\Pi_h^\star e)^+)_{x,j-\frac{1}{2}} dx. \tag{3.25}$$

Here, we have used the fact that on the vertical edges $(x_{i+\frac{1}{2}}, y)$, $\Pi_h^\star u = P_{h_y}^\star u$ and that on the horizontal edges $(x, y_{j+\frac{1}{2}})$, $\Pi_h^\star u = P_{h_x}^\star u$, by the definition of $\Pi_h^\star$, (3.8).

By the same argument as that in the proof of L^2-stability, we have that

$$\sum_{K\in\Omega_h} W_K = (\theta_1 - \frac{1}{2})\sum_{j=1}^{N_2}\int_{J_j}\sum_{i=1}^{N_1}[\![\Pi_h^\star e]\!]^2_{i+\frac{1}{2},y} dy + (\theta_2 - \frac{1}{2})\sum_{i=1}^{N_1}\int_{I_i}\sum_{j=1}^{N_2}[\![\Pi_h^\star e]\!]^2_{x,j+\frac{1}{2}} dx.$$

From the properties of the projection $\Pi_h^\star$, (3.9), we can see that $\Pi_h^\star$ cannot eliminate terms involving $u - \Pi_h^\star u$ in G_K. However, the projection $\Pi_h^\star$ defined on Cartesian meshes has the following superconvergence result, whose proof is deferred to subsection 3.4.3.

Lemma 3.5. *Let $G_K(u, \Pi_h^\star e)$ be defined by* (3.25)*. Then we have*

$$|G_K(u,\Pi_h^\star e)| \le Ch^{k+2}\|u\|_{W^{2k+3,\infty}(\Omega_h)}\|\Pi_h^\star e\|_{L^2(K)}.$$

Collecting the above estimates for $\sum_{K\in\Omega_h} W_K$ and $G_K(u, \Pi_h^\star e)$, we get, after suitable applications of the Cauchy–Schwarz inequality

$$\frac{1}{2}\frac{d}{dt}\|\Pi_h^\star e\|^2_{L^2(\Omega_h)} \le \|(u-\Pi_h^\star u)_t\|_{L^2(\Omega_h)}\|\Pi_h^\star e\|_{L^2(\Omega_h)} \\ + Ch^{k+1}\|u\|_{W^{2k+3,\infty}(\Omega_h)}\|\Pi_h^\star e\|_{L^2(\Omega_h)} \\ \le Ch^{k+1}\|\Pi_h^\star e\|_{L^2(\Omega_h)},$$

where in the last step we have used the fact that $(\Pi_h^\star u)_t = \Pi_h^\star u_t$ and the approximation error estimate (3.23). The estimate (3.24) now follows by using the initial error estimate (3.4). This finishes the proof of Lemma 3.4. ☐

To prove Theorem 3.2, we need only to use the triangle inequality and apply Lemmas 3.3 and 3.4 for $\|u - \Pi_h^\star u\|_{L^2(\Omega_h)}$ and $\|\Pi_h^\star e\|_{L^2(\Omega_h)}$.

3.4.3. *Proof of Lemma* 3.5. Let us first rewrite $G_K(u, \Pi_h^\star e)$ into

$$G_K(u, \Pi_h^\star e) = G_K^1(u, \Pi_h^\star e) + G_K^2(u, \Pi_h^\star e),$$

where

$$\begin{aligned} G_K^1(u, \Pi_h^\star e) = &\int_K (u - \Pi_h^\star u)(\Pi_h^\star e)_x d\boldsymbol{x} - \int_{J_j} ((u - P_{h_y}^\star u)(\Pi_h^\star e)^-)_{i+\frac{1}{2},y} dy \\ &+ \int_{J_j} ((u - P_{h_y}^\star u)(\Pi_h^\star e)^+)_{i-\frac{1}{2},y} dy, \\ G_K^2(u, \Pi_h^\star e) = &\int_K (u - \Pi_h^\star u)(\Pi_h^\star e)_y d\boldsymbol{x} - \int_{I_i} ((u - P_{h_x}^\star u)(\Pi_h^\star e)^-)_{x,j+\frac{1}{2}} dx \\ &+ \int_{I_i} ((u - P_{h_x}^\star u)(\Pi_h^\star e)^+)_{x,j-\frac{1}{2}} dx. \end{aligned}$$

Since the analysis for G_K^1 and G_K^2 are analogous, we only present here the detailed proofs for G_K^1.

On an arbitrary element $K = I_i \times J_j$, by the definition of the projection $\Pi_h^\star$, (3.8), and the Cartesian structure of the mesh, we follow the same arguments in the proof of the superconvergence result in [9] to obtain

$$G_K^1(u, \Pi_h^\star e) = 0 \qquad \forall u \in P^{k+1}(K), \quad \Pi_h^\star e \in Q^k(K). \tag{3.26}$$

Next, on each element K, consider the Taylor expansion of u around (x_i, y_j),

$$u = Tu + R_{k+2},$$

where

$$\begin{aligned} Tu &= \sum_{l=0}^{k+1} \sum_{m=0}^{l} \frac{1}{(l-m)!m!} \frac{\partial^l u(x_i, y_j)}{\partial x^{l-m} \partial y^m} (x - x_i)^{l-m} (y - y_j)^m, \\ R_{k+2} &= (k+2) \sum_{m=0}^{k+2} \frac{(x - x_i)^{k+2-m} (y - y_j)^m}{(k+2-m)!m!} \int_0^1 (1-s)^{k+1} \frac{\partial^{k+2} u(x_i^s, y_j^s)}{\partial x^{k+2-m} \partial y^m} ds \end{aligned}$$

with $x_i^s = x_i + s(x - x_i), y_j^s = y_j + s(y - y_j)$. Clearly, $Tu \in P^{k+1}(K)$. Note that the operator $\Pi_h^\star$ is linear and thus $\Pi_h^\star u = \Pi_h^\star Tu + \Pi_h^\star R_{k+2}$. So, by (3.26), we get

$$G_K^1(u, \Pi_h^\star e) = T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 &= \int_K (R_{k+2} - \Pi_h^\star R_{k+2})(\Pi_h^\star e)_x d\boldsymbol{x}, \\ T_2 &= -\int_{J_j} ((R_{k+2} - \Pi_h^\star R_{k+2})(\Pi_h^\star e)^-)_{i+\frac{1}{2},y} dy, \\ T_3 &= \int_{J_j} ((R_{k+2} - \Pi_h^\star R_{k+2})(\Pi_h^\star e)^+)_{i-\frac{1}{2},y} dy, \end{aligned}$$

which will be estimated one by one below.

From the approximation properties of the projection $\Pi_h^\star$, Lemma 3.3, we have that

$$\|R_{k+2} - \Pi_h^\star R_{k+2}\|_{L^2(K)} \le Ch^{k+2} \|R_{k+2}\|_{W^{k+1,\infty}(\Omega_h)},$$

where $C = C(\theta_1, \theta_2, k)$. It is easy to show that

$$\|R_{k+2}\|_{W^{k+1,\infty}(\Omega_h)} = \max_{K\in\Omega_h} \|R_{k+2}\|_{W^{k+1,\infty}(K)} \le Ch\|u\|_{W^{2k+3,\infty}(\Omega_h)}.$$

Combining the above two estimates, we arrive at

$$\|R_{k+2} - \Pi_h^\star R_{k+2}\|_{L^2(K)} \le Ch^{k+3}\|u\|_{W^{2k+3,\infty}(\Omega_h)}. \tag{3.27}$$

Analogously, we have that

$$\|R_{k+2} - \Pi_h^\star R_{k+2}\|_{H^1(K)} \le Ch^{k+2}\|u\|_{W^{2k+3,\infty}(\Omega_h)}. \tag{3.28}$$

It follows from the Cauchy–Schwarz inequality, the estimate (3.27), and the inverse inequality that

$$\begin{aligned}|T_1| &\le \|R_{k+2} - \Pi_h^\star R_{k+2}\|_{L^2(K)}\|(\Pi_h^\star e)_x\|_{L^2(K)}\\ &\le Ch^{k+2}\|u\|_{W^{2k+3,\infty}(\Omega_h)}\|\Pi_h^\star e\|_{L^2(K)}.\end{aligned}$$

To estimate T_2, we begin by combining the estimates (3.27) and (3.28), and using the trace inequality to get

$$\|R_{k+2} - \Pi_h^\star R_{k+2}\|_{L^2(\partial K)} \le Ch^{k+\frac{5}{2}}\|u\|_{W^{2k+3,\infty}(\Omega_h)}.$$

Next, by the Cauchy–Schwarz inequality and the inverse inequality, we arrive at

$$|T_2| \le \|R_{k+2} - \Pi_h^\star R_{k+2}\|_{L^2(\partial K)}\|\Pi_h^\star e\|_{L^2(\partial K)} \le Ch^{k+2}\|u\|_{W^{2k+3,\infty}(\Omega_h)}\|\Pi_h^\star e\|_{L^2(K)}.$$

Analogously, we have that

$$|T_3| \le Ch^{k+2}\|u\|_{W^{2k+3,\infty}(\Omega_h)}\|\Pi_h^\star e\|_{L^2(K)}.$$

The estimate for G_K^1 now follows by collecting the results for T_i $(i = 1, 2, 3)$, obtained above. The proof of Lemma 3.5 is thus completed.

4. Fully discrete analysis

In this section, we shall perform stability analysis and optimal error estimates for the RKDG method for linear hyperbolic equations. For simplicity, we restrict ourselves to the one-dimensional linear conservation laws

$$u_t + \beta u_x = 0, \qquad (x,t) \in [0, 2\pi] \times (0, T], \tag{4.1a}$$

$$u(x, 0) = u_0(x), \quad x \in R, \tag{4.1b}$$

(β is the given constant) with periodic boundary conditions and the time discretization is the TVDRK3 method [21]. Using an energy technique, we prove the L^2-norm stability and optimal a priori error estimate under the standard temporal-spatial CFL condition $\tau \le \gamma h$, where τ is the time step and γ is independent of h and τ. We would like to point out that our analysis rely on a similar idea as that in [26, 27]. However, one of the main differences is that the newly designed projection $P_h^\star$ is employed here, which not only eliminates boundary terms involving projection error, but possesses optimal interpolation properties. This entails us to obtain optimal convergence results, where, for example, in [3] quasi-optimal convergence rate of the form $\mathcal{O}(h^{k+1/2} + \tau^3)$ is obtained for general fluxes.

4.1. **Fully discrete DG scheme.** For equation (4.1), we would like to introduce the DG spatial operator $\mathcal{H}$, which is bilinear. For any functions w and v in $H^1(\mathcal{I}_h)$, on each element I_j we define

$$\mathcal{H}_j(w,v) = -\beta\mathcal{D}_{I_j}(w,v;\hat{w}),$$

where $\mathcal{D}_{I_j}(w,v;\hat{w})$ has been given in (2.17). Using this notation, the semidiscrete DG scheme solving (4.1) reads

$$\int_{I_j} (u_h)_t v_h dx = \mathcal{H}_j(u_h, v_h)$$

for any $v_h \in V_h$ and $j = 1,\ldots,N$.

For a given time step τ, the solution of the scheme is denoted by $u_h^n(x) = u_h(x, n\tau)$. Applying TVDRK3 method to the semidiscrete DG scheme, we will obtain the fully discrete RKDG method. First we set the initial value $u_h^0 = \mathbb{P}_h u_0(x)$. Then for each $n \geq 0$, the approximate solution from the time $n\tau$ to the next time $(n+1)\tau$ is defined as follows: find $u_h^{n,1}, u_h^{n,2}$, and u_h^{n+1} in the finite element space V_h such that, for any $v_h \equiv v_h(x) \in V_h$ and $j = 1,\ldots,N$, on each element I_j it holds that

$$\int_{I_j} u_h^{n,1} v_h dx = \int_{I_j} u_h^n v_h dx + \tau\mathcal{H}_j(u_h^n, v_h), \tag{4.2a}$$

$$\int_{I_j} u_h^{n,2} v_h dx = \frac{3}{4}\int_{I_j} u_h^n v_h dx + \frac{1}{4}\int_{I_j} u_h^{n,1} v_h dx + \frac{\tau}{4}\mathcal{H}_j(u_h^{n,1}, v_h), \tag{4.2b}$$

$$\int_{I_j} u_h^{n+1} v_h dx = \frac{1}{3}\int_{I_j} u_h^n v_h dx + \frac{2}{3}\int_{I_j} u_h^{n,2} v_h dx + \frac{2\tau}{3}\mathcal{H}_j(u_h^{n,2}, v_h). \tag{4.2c}$$

To facilitate our analysis, we denote by $(\cdot,\cdot)_{I_j}$ and $(\cdot,\cdot)$ the scalar inner product on $L^2(I_j)$ and $L^2(I)$, respectively, and by $\|\cdot\|_{I_j}$, $\|\cdot\|$ the associated norms.

To end this subsection, we list some inverse properties of the finite element space V_h. For any $v_h \in V_h$, there exist positive constants μ_1, μ_2 independent of v_h and h, such that

$$\text{(i) } \|(v_h)_x\| \leq \mu_1 h^{-1}\|v_h\|, \quad \text{(ii) } \|v_h\|_{\Gamma_h} \leq \mu_2 h^{-1/2}\|v_h\|.$$

Here and below, Γ_h is the union of all cell boundary points, and for any $w \in H^1(\mathcal{I}_h)$, the L^2-norm on Γ_h is defined by

$$\|w\|_{\Gamma_h} = \left(\sum_{j=1}^{N}\left((w_{j+\frac{1}{2}}^-)^2 + (w_{j+\frac{1}{2}}^+)^2\right)\right)^{1/2}.$$

4.2. **Stability result.** In this subsection, using the arguments in [26, 27], we are able to show the L^2-norm stability for the fully discrete RKDG method with TVDRK3 time-marching.

For equation (4.1), the DG spatial operator and periodic boundary conditions yield

$$\mathcal{H}(w,v) = \sum_{j=1}^{N}\mathcal{H}_j(w,v) = \sum_{j=1}^{N}\left(\int_{I_j}\beta w v_x dx + \beta\hat{w}_{j+\frac{1}{2}}[\![v]\!]_{j+\frac{1}{2}}\right).$$

Here, at each element boundary point

$$\hat{w} = \begin{cases} \theta w^- + (1-\theta)w^+, & \text{if } \beta > 0, \\ \theta w^+ + (1-\theta)w^-, & \text{if } \beta < 0 \end{cases}$$

with $\theta > \frac{1}{2}$. Since the stability analysis follows from the arguments similar to that in [26] and depends on some elementary properties of the bilinear operator $\mathcal{H}$, we list the results only and omit the detailed proofs to save space. In what follows, we denote

$$\mu = \max\{\mu_1, (\mu_2)^2\}, \; \delta = \sqrt{\theta^2 + (1-\theta)^2}, \; \mathbb{a} = \frac{(1+\sqrt{2}\delta)^2}{3}, \; \mathbb{b} = \theta - \frac{1}{2}. \tag{4.3}$$

Lemma 4.1. *For any $w, v \in V_h$, we have $|\mathcal{H}(w,v)| \le (1+\sqrt{2}\delta)|\beta|\mu h^{-1}\|w\|\|v\|$.*

Lemma 4.2. *For any $w, v \in H^1(\mathcal{I}_h)$, we have*

$$\mathcal{H}(w,v) + \mathcal{H}(v,w) = -2\mathbb{b}\sum_{j=1}^{N} |\beta| [\![w]\!]_{j+\frac{1}{2}} \cdot [\![v]\!]_{j+\frac{1}{2}}, \tag{4.4a}$$

$$\mathcal{H}(w,w) = -\mathbb{b}\sum_{j=1}^{N} |\beta| [\![w]\!]^2_{j+\frac{1}{2}}. \tag{4.4b}$$

Using the above lemmas, we will obtain the L^2-norm stability result for the RKDG method with TVDRK3 time discretization. For more details, see [22, 26].

Theorem 4.3 (stability)**.** *The numerical solution u_h of the fully discrete scheme* (4.2) *with the explicit TVDRK3 time discretization satisfies the following strong stability*

$$\|u_h^{n+1}\| \le \|u_h^n\| \tag{4.5}$$

under the CFL condition

$$\mu|\beta|\tau h^{-1} \le \frac{1}{1+\sqrt{2}\delta}, \tag{4.6}$$

where μ and δ are constants defined in (4.3)*.*

Remark 4.4. Following an online version of [26] in [27], we could have easily provided a more explicit and technical proof of the stability result (4.5), under a slightly different CFL condition

$$\mu|\beta|\tau h^{-1} \le \frac{1}{\sqrt{\mathbb{b}^2 + \mathbb{a}} + \mathbb{b}}, \tag{4.7}$$

where $\mathbb{a}$ and $\mathbb{b}$ are positive constants defined in (4.3). Analysis of the above CFL condition for stability is also reflected in the derivation of optimal error estimates. Indeed, in subsection 4.3, under the same CFL condition (4.7), optimal a priori error estimate of the form $\mathcal{O}(h^{k+1} + \tau^3)$ is obtained. Several approximate values of $\frac{1}{1+\sqrt{2}\delta}$ and $\frac{1}{\sqrt{\mathbb{b}^2+\mathbb{a}}+\mathbb{b}}$ for different θ are given in Table 4.1. From the table, we can see that for small values of θ, say, $\theta \le \frac{3}{2}$, the CFL condition (4.7) is better than (4.6), while for large values of θ, say $\theta \ge 2$, the CFL condition (4.6) is better than (4.7).

TABLE 4.1. Approximate values of $\frac{1}{1+\sqrt{2\delta}}$ and $\frac{1}{\sqrt{\mathbb{b}^2+\mathbb{a}}+\mathbb{b}}$ for different θ.

	$\theta=\frac{1}{2}$	$\theta=\frac{3}{5}$	$\theta=\frac{3}{4}$	$\theta=1$	$\theta=\frac{3}{2}$	$\theta=2$	$\theta=3$	$\theta=5$
$\frac{1}{1+\sqrt{2\delta}}$	0.500	0.495	0.472	0.414	0.309	0.240	0.164	0.099
$\frac{1}{\sqrt{\mathbb{b}^2+\mathbb{a}}+\mathbb{b}}$	0.866	0.787	0.667	0.505	0.321	0.231	0.147	0.084

4.3. **Optimal a priori error estimate.** In this subsection, we state and prove optimal a priori error estimate for the fully discrete RKDG scheme with the explicit TVDRK3 time discretization. The main work is to eliminate cell boundary terms involving projection error and to deal with the accumulation of the error from the time discretization.

Theorem 4.5 (error estimate). *Assume that the exact solution u of* (4.1) *is sufficiently smooth with bounded derivatives, i.e.,* $\|u\|_{W^{k+1,\infty}(\mathcal{I}_h)}$, $\|u_t\|_{W^{k+1,\infty}(\mathcal{I}_h)}$ *and* $\|u_{tttt}\|$ *are bounded uniformly for any time* $t\in[0,T]$. *Let* u_h *be the numerical solution of the fully discrete RKDG scheme* (4.2) *with the explicit TVDRK3 time marching when the upwind-biased numerical flux is used. For regular triangulations of* $I=[0,2\pi]$, *if the finite element space* V_h *of piecewise polynomials with arbitrary degree* $k\geq 0$ *is used, then the error estimate*

$$\max_{n\tau\leq T}\|u(t^n)-u_h^n\|\leq C(h^{k+1}+\tau^3) \tag{4.8}$$

holds under the standard CFL condition (4.7). *Here C is a positive constant independent of h, τ, and u_h.*

4.3.1. *The error equation and the energy equality.* Following [26], reference functions are introduced. Let $u^{(0)}(x,t)=u(x,t)$ and

$$u^{(1)}(x,t)=u^{(0)}(x,t)-\tau\beta\partial_x u^{(0)}(x,t), \tag{4.9a}$$

$$u^{(2)}(x,t)=\frac{3}{4}u^{(0)}(x,t)+\frac{1}{4}u^{(1)}(x,t)-\frac{1}{4}\tau\beta\partial_x u^{(1)}(x,t). \tag{4.9b}$$

Denote $u^{n,\sharp}=u^{(\sharp)}(x,t^n)$ for any time level n and $\sharp=0,1,2$. Then, for any n and inner stage $\sharp=0,1,2$, we would like to split the error $e^{n,\sharp}=u^{n,\sharp}-u_h^{n,\sharp}$ ($u^{n,0}=u^n$, $u_h^{n,0}=u_h^n$) into two parts, that is, $e^{n,\sharp}=\xi^{n,\sharp}-\eta^{n,\sharp}$, with

$$\xi^{n,\sharp}=P_h^\star u^{n,\sharp}-u_h^{n,\sharp},\quad \eta^{n,\sharp}=P_h^\star u^{n,\sharp}-u^{n,\sharp},$$

where $P_h^\star$ is the *globally* defined projection in (2.9).

By Lemma 2.6, for smooth enough u, we have

$$\|\eta^{n,\sharp}\|\leq C_1 h^{k+1},\ \forall n: n\tau\leq T. \tag{4.10a}$$

Denote by $\mathbb{d}^n=d_0\eta^n+d_1\eta^{n,1}+d_2\eta^{n,2}+d_3\eta^{n+1}$ a linear combination of errors in different stages, where $d_i, i=0,1,2,3$, are any four constants satisfying $d_0+d_1+d_2+d_3=0$. Since the projection operator $P_h^\star$ is linear in time, thus we have

$$\|\mathbb{d}^n\|\leq C_2 h^{k+1}\tau,\ \forall n: n\tau\leq T, \tag{4.10b}$$

when u_t is smooth enough. Here, C_1 and C_2 are positive constants independent of n, h, and τ.

To obtain the error equation for fully discrete scheme, we need to present the local truncation error in time for the reference functions [26], which is,

$$u(x,t+\tau) = \frac{1}{3}u^{(0)}(x,t) + \frac{2}{3}u^{(2)}(x,t) - \frac{2}{3}\tau\beta\partial_x u^{(2)}(x,t) + \mathcal{T}(x,t), \tag{4.11}$$

where $\mathcal{T}(x,t)$ is the local truncation error in time and $\|\mathcal{T}(x,t)\| = \mathcal{O}(\tau^4)$ uniformly for any time $t \in [0,T]$. Denote $e^{n+1} = \xi^{n+1} - \eta^{n+1}$ with $\xi^{n+1} = P_h^\star u^{n+1} - u_h^{n+1}$, $\eta^{n+1} = P_h^\star u^{n+1} - u^{n+1}$.

We are now ready to get the error equation by subtracting the DG scheme (4.2) about the numerical solution from a similar weak formulation based on (4.9) and (4.11) with $t = t^n$ about the exact solution that for any $v_h \in V_h$ and $j = 1, \ldots, N$,

$$(\xi^{n,1}, v_h)_{I_j} = (\xi^n, v_h)_{I_j} + \tau\mathcal{J}_j(v_h), \tag{4.12a}$$

$$(\xi^{n,2}, v_h)_{I_j} = \frac{3}{4}(\xi^n, v_h)_{I_j} + \frac{1}{4}(\xi^{n,1}, v_h)_{I_j} + \frac{\tau}{4}\mathcal{K}_j(v_h), \tag{4.12b}$$

$$(\xi^{n+1}, v_h)_{I_j} = \frac{1}{3}(\xi^n, v_h)_{I_j} + \frac{2}{3}(\xi^{n,2}, v_h)_{I_j} + \frac{2\tau}{3}\mathcal{L}_j(v_h), \tag{4.12c}$$

where

$$\mathcal{J}_j(v_h) = \frac{1}{\tau}(\eta^{n,1} - \eta^n, v_h)_{I_j} + \mathcal{H}_j(e^n, v_h), \tag{4.13a}$$

$$\mathcal{K}_j(v_h) = \frac{1}{\tau}(4\eta^{n,2} - 3\eta^n - \eta^{n,1}, v_h)_{I_j} + \mathcal{H}_j(e^{n,1}, v_h), \tag{4.13b}$$

$$\mathcal{L}_j(v_h) = \frac{1}{2\tau}(3\eta^{n+1} - \eta^n - 2\eta^{n,2} + 3\mathcal{T}(x,t^n), v_h)_{I_j} + \mathcal{H}_j(e^{n,2}, v_h). \tag{4.13c}$$

The three integrals on the right-hand side of (4.13) will be denoted by $\mathcal{J}_j^{\text{tm}}(v_h)$, $\mathcal{K}_j^{\text{tm}}(v_h)$, and $\mathcal{L}_j^{\text{tm}}(v_h)$, respectively. Also, similar to $\mathcal{H}$, the removal of the subscript j from $\mathcal{J}_j$, $\mathcal{J}_j^{\text{tm}}$, etc., indicates summation over j.

Taking $v_h = \xi^n, 4\xi^{n,1}$ and $6\xi^{n,2}$ in the error equations (4.12a), (4.12b), and (4.12c), respectively, we arrive at the energy equality for ξ^n in the form

$$3\|\xi^{n+1}\|^2 - 3\|\xi^n\|^2 = S_1 + S_2, \tag{4.14}$$

where

$$S_1 = \tau\left(\mathcal{J}(\xi^n) + \mathcal{K}(\xi^{n,1}) + 4\mathcal{L}(\xi^{n,2})\right),$$
$$S_2 = \|2\xi^{n,2} - \xi^{n,1} - \xi^n\|^2 + 3(\xi^{n+1} - \xi^n, \xi^{n+1} - 2\xi^{n,2} + \xi^n).$$

4.3.2. *Relationships between $\xi^{n,\sharp}$ in different time stages.* Following [26], we introduce the following notation

$$\mathbb{G}_1^n = \xi^{n,1} - \xi^n, \quad \mathbb{G}_2^n = 2\xi^{n,2} - \xi^{n,1} - \xi^n, \quad \mathbb{G}_3^n = \xi^{n+1} - 2\xi^{n,2} + \xi^n,$$

and show in the following lemma their relationships, which are useful for the estimate of S_2. The proof is straightforward by some combinations of the error equation, (4.12), and by the fact that $\mathcal{H}(\eta^{n,\sharp}, v_h) = 0$ due to the definition of the projection $P_h^\star$, (2.9), so it is omitted here.

Lemma 4.6. *For the fully discrete RKDG method* (4.2), *we have*

(4.15a) $$(\mathbb{G}_1^n, v_h) = \tau\left(\mathcal{H}(\xi^n, v_h) + \mathcal{J}^{\rm tm}(v_h)\right),$$

(4.15b) $$(\mathbb{G}_2^n, v_h) = \frac{\tau}{2}\left(\mathcal{H}(\mathbb{G}_1^n, v_h) + \mathcal{K}^{\rm tm}(v_h) - \mathcal{J}^{\rm tm}(v_h)\right),$$

(4.15c) $$(\mathbb{G}_3^n, v_h) = \frac{\tau}{3}\left(\mathcal{H}(\mathbb{G}_2^n, v_h) + 2\mathcal{L}^{\rm tm}(v_h) - \mathcal{K}^{\rm tm}(v_h) - \mathcal{J}^{\rm tm}(v_h)\right),$$

for any $v_h \in V_h$.

The next lemma shows the relationships between $\|\xi^n\|$, $\|\xi^{n,1}\|$, and $\|\xi^{n,2}\|$, which are helpful to control the error at intermediate stages.

Lemma 4.7. *If the time step satisfies* $\tau = \mathcal{O}(h)$, *then we have*

(4.16a) $$\|\xi^{n,1}\|^2 \le C\|\xi^n\|^2 + Ch^{2k+2},$$

(4.16b) $$\|\xi^{n,2}\|^2 \le C\|\xi^n\|^2 + C\|\xi^{n,1}\|^2 + Ch^{2k+2},$$

where C *is a positive constant independent of* n, h, τ, *and* u_h.

Proof. To prove (4.16a), we take $v_h = \xi^{n,1}$ in the error equation (4.12a) and get

(4.17) $$\|\xi^{n,1}\|^2 = (\xi^n, \xi^{n,1}) + \tau\left(\mathcal{J}^{\rm tm}(\xi^{n,1}) + \mathcal{H}(\xi^n, \xi^{n,1})\right),$$

since $\mathcal{H}(\eta^n, \xi^{n,1}) = 0$. Note that

$$|\mathcal{J}^{\rm tm}(\xi^{n,1})| \le Ch^{k+1}\|\xi^{n,1}\|$$

by the approximation property (4.10b), and

$$|\mathcal{H}(\xi^n, \xi^{n,1})| \le (1 + \sqrt{2}\delta)|\beta|\mu h^{-1}\|\xi^n\|\|\xi^{n,1}\|$$

by Lemma 4.1. Substituting above two estimates into (4.17) and taking into account $\tau = \mathcal{O}(h) < 1$, we arrive at

$$\|\xi^{n,1}\| \le C\|\xi^n\| + Ch^{k+1},$$

which implies (4.16a).

Analogously, (4.16b) can be obtained by taking $v_h = \xi^{n,2}$ in (4.12b), and details are omitted. ☐

4.3.3. *Estimates for* S_1 *and* S_2. Let us begin by presenting the estimate for S_1, which follows by the approximation property (4.10b), and the negative semidefiniteness of $\mathcal{H}$, (4.4b).

Lemma 4.8. *We have that*

(4.18a) $$\tau\mathcal{J}(\xi^n) \le C\tau(\|\xi^n\|^2 + h^{2k+2}) - \mathbb{b}\tau\sum_{j=1}^{N}|\beta|[\![\xi^n]\!]^2_{j+\frac{1}{2}},$$

(4.18b) $$\tau\mathcal{K}(\xi^{n,1}) \le C\tau(\|\xi^{n,1}\|^2 + h^{2k+2}) - \mathbb{b}\tau\sum_{j=1}^{N}|\beta|[\![\xi^{n,1}]\!]^2_{j+\frac{1}{2}},$$

(4.18c) $$\tau\mathcal{L}(\xi^{n,2}) \le C\tau(\|\xi^{n,2}\|^2 + h^{2k+2}) + C\tau^7 - \mathbb{b}\tau\sum_{j=1}^{N}|\beta|[\![\xi^{n,2}]\!]^2_{j+\frac{1}{2}},$$

where C *is independent of* n, h, τ, *and* u_h.

Next, we move on to the estimate for S_2 resulting from the time discretization, in which the temporal-spatial condition is needed.

Lemma 4.9. *Assume that the temporal-spatial condition* (4.7) *holds. Then we have*

$$S_2 \le C\tau\left(\|\xi^n\|^2 + \|\xi^{n,1}\|^2 + \|\xi^{n,2}\|^2 + h^{2k+2}\right) + C\tau^7 + \mathbb{b}\tau \sum_{j=1}^{N} |\beta| \left([\![\xi^n]\!]^2_{j+\frac12} + [\![\xi^{n,1}]\!]^2_{j+\frac12}\right), \tag{4.19}$$

where C is independent of n, h, τ, and u_h.

Proof. Note that $\xi^{n+1} - \xi^n = \mathbb{G}_1^n + \mathbb{G}_2^n + \mathbb{G}_3^n$, and thus

$$S_2 = (\mathbb{G}_2^n, \mathbb{G}_2^n) + 3(\mathbb{G}_1^n, \mathbb{G}_3^n) + 3(\mathbb{G}_2^n, \mathbb{G}_3^n) + 3(\mathbb{G}_3^n, \mathbb{G}_3^n) := \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4,$$

which will be estimated separately.

To do that, let us begin by estimating $\Theta_1 + \Theta_2$. By taking different test functions in (4.15b) and (4.15c) of Lemma 4.6, and using (4.4a) of Lemma 4.2, we get

$$\begin{aligned}
\Theta_1 + \Theta_2 &= -(\mathbb{G}_2^n, \mathbb{G}_2^n) + 2(\mathbb{G}_2^n, \mathbb{G}_2^n) + 3(\mathbb{G}_1^n, \mathbb{G}_3^n)\\
&= -\|\mathbb{G}_2^n\|^2 + \tau\left(\mathcal{H}(\mathbb{G}_1^n, \mathbb{G}_2^n) + \mathcal{H}(\mathbb{G}_2^n, \mathbb{G}_1^n)\right)\\
&\quad + \tau\left(\mathcal{K}^{\rm tm}(\mathbb{G}_2^n) - \mathcal{J}^{\rm tm}(\mathbb{G}_2^n) + 2\mathcal{L}^{\rm tm}(\mathbb{G}_1^n) - \mathcal{K}^{\rm tm}(\mathbb{G}_1^n) - \mathcal{J}^{\rm tm}(\mathbb{G}_1^n)\right)\\
&= -\|\mathbb{G}_2^n\|^2 - 2\mathbb{b}\tau \sum_{j=1}^{N} |\beta| [\![\mathbb{G}_1^n]\!]_{j+\frac12} [\![\mathbb{G}_2^n]\!]_{j+\frac12}\\
&\quad + \tau\left(\mathcal{K}^{\rm tm}(\mathbb{G}_2^n) - \mathcal{J}^{\rm tm}(\mathbb{G}_2^n) + 2\mathcal{L}^{\rm tm}(\mathbb{G}_1^n) - \mathcal{K}^{\rm tm}(\mathbb{G}_1^n) - \mathcal{J}^{\rm tm}(\mathbb{G}_1^n)\right).
\end{aligned}$$

Denote $\lambda_{\max} = \mu|\beta|\tau h^{-1}$. It follows from Young's inequality and the approximation property (4.10b) that

$$\begin{aligned}
\Theta_1 + \Theta_2 &\le -\|\mathbb{G}_2^n\|^2 + 2\mathbb{b}\tau \sum_{j=1}^{N} |\beta| \left(\frac{[\![\mathbb{G}_1^n]\!]^2_{j+\frac12}}{4} + [\![\mathbb{G}_2^n]\!]^2_{j+\frac12}\right)\\
&\quad + C\tau\left(h^{2k+2} + \tau^6 + \|\mathbb{G}_1^n\|^2 + \|\mathbb{G}_2^n\|^2\right)\\
&\le -\|\mathbb{G}_2^n\|^2 + \mathbb{b}\tau \sum_{j=1}^{N} |\beta| \left([\![\xi^n]\!]^2_{j+\frac12} + [\![\xi^{n,1}]\!]^2_{j+\frac12}\right) + 2\mathbb{b}\tau \sum_{j=1}^{N} |\beta| [\![\mathbb{G}_2^n]\!]^2_{j+\frac12}\\
&\quad + C\tau\left(h^{2k+2} + \tau^6 + \|\mathbb{G}_1^n\|^2 + \|\mathbb{G}_2^n\|^2\right).
\end{aligned} \tag{4.20a}$$

The identities (4.15c) and (4.4b) together with the approximate property (4.10b) give us the estimate for Θ_3 as follows.

$$\begin{aligned}
\Theta_3 &= \tau\mathcal{H}(\mathbb{G}_2^n, \mathbb{G}_2^n) + \tau\left(2\mathcal{L}^{\rm tm}(\mathbb{G}_2^n) - \mathcal{K}^{\rm tm}(\mathbb{G}_2^n) - \mathcal{J}^{\rm tm}(\mathbb{G}_2^n)\right)\\
&\le -\mathbb{b}\tau \sum_{j=1}^{N} |\beta| [\![\mathbb{G}_2^n]\!]^2_{j+\frac12} + C\tau\left(h^{2k+2} + \tau^6 + \|\mathbb{G}_2^n\|^2\right).
\end{aligned} \tag{4.20b}$$

To estimate Θ_4, we first use (4.15c) and Lemma 4.1 to obtain

$$\begin{aligned}
\|\mathbb{G}_3^n\|^2 = (\mathbb{G}_3^n, \mathbb{G}_3^n) &= \frac{\tau}{3}\mathcal{H}(\mathbb{G}_2^n, \mathbb{G}_3^n) + \frac{\tau}{3}(2\mathcal{L}^{\rm tm}(\mathbb{G}_3^n) - \mathcal{K}^{\rm tm}(\mathbb{G}_3^n) - \mathcal{J}^{\rm tm}(\mathbb{G}_3^n))\\
&\le \frac{\tau}{3}(1 + \sqrt{2}\delta)|\beta|\mu h^{-1}\|\mathbb{G}_2^n\|\|\mathbb{G}_3^n\| + C\tau(h^{k+1} + \tau^3)\|\mathbb{G}_3^n\|,
\end{aligned}$$

which implies

$$\Theta_4 = 3\|\mathbb{G}_3^n\|^2 \le \mathbb{a}\lambda_{\max}^2\|\mathbb{G}_2^n\|^2 + C\tau(h^{2k+2} + \tau^6), \tag{4.20c}$$

where $\mathbb{a} = \frac{(1+\sqrt{2}\delta)^2}{3}$ has been given in (4.3). We now collect the estimates for Θ_i $(i = 1, 2, 3, 4)$ in (4.20) and take into account $\|\mathbb{G}_1^n\|^2 + \|\mathbb{G}_2^n\|^2 \le C(\|\xi^n\|^2 + \|\xi^{n,1}\|^2 + \|\xi^{n,2}\|^2)$ to get

$$S_2 \le -(1 - 2\mathbb{b}\lambda_{\max} - \mathbb{a}\lambda_{\max}^2)\|\mathbb{G}_2^n\|^2 + \mathbb{b}\tau \sum_{j=1}^{N} |\beta| \left([\![\xi^n]\!]_{j+\frac{1}{2}}^2 + [\![\xi^{n,1}]\!]_{j+\frac{1}{2}}^2 \right)$$
$$+ C\tau \left(\|\xi^n\|^2 + \|\xi^{n,1}\|^2 + \|\xi^{n,2}\|^2 + h^{2k+2} \right) + C\tau^7.$$

Note that the temporal-spatial condition (4.7) implies $1 - 2\mathbb{b}\lambda_{\max} - \mathbb{a}\lambda_{\max}^2 \ge 0$ and thus the proof of Lemma 4.9 is completed. □

4.3.4. *Proof of Theorem* 4.5. We now collect the estimates for S_1, S_2 in Lemmas 4.8 and 4.9 into the energy identity (4.14) and employ the relationships in Lemma 4.7, we finally obtain for $(n + 1)\tau \le T$,

$$3\|\xi^{n+1}\|^2 - 3\|\xi^n\|^2 \le C\|\xi^n\|^2\tau + Ch^{2k+2}\tau + C\tau^7,$$

where C is a positive constant independent of n, h, and τ. Then, a direct application of the discrete Gronwall inequality yields

$$\|\xi^n\|^2 \le Ch^{2k+2} + C\tau^6, \ n\tau \le T.$$

Therefore, the optimal error estimate for the fully discrete RKDG scheme with explicit TVDRK3 time discretization follows by combining the above estimate with the approximation property (4.10a). This completes the proof of Theorem 4.5.

5. Numerical experiments

The purpose of this section is to numerically validate the a priori error estimates of the DG method using upwind-biased fluxes for linear conservation laws. In most of the numerical experiments below, in order to reduce the time errors, we use the strong stability preserving ninth order (SSP9) time discretization [14] in time and take $\Delta t = CFL\ h$ for one- and two-dimensional linear problems with periodic boundary conditions, and we use TVDRK3 time-marching and take $\Delta t = CFL\ h^2$ for other cases. One exception is the computation for Table 5.2, which is used to verify the sharpness of Theorem 4.5, hence we use the TVDRK3 time discretization with standard CFL condition $\Delta t = CFL\ h$.

Example 5.1. To test the validity of optimal error estimates of DG methods (2.2) with numerical fluxes (2.3), we solve the linear conservation law

$$\begin{cases} u_t + u_x = 0, \\ u(x, 0) = \sin(x) \end{cases} \tag{5.1}$$

with periodic boundary conditions. The exact solution to this problem is

$$u(x, t) = \sin(x - t). \tag{5.2}$$

Table 5.1 lists the numerical errors and their orders with different values of θ at $T = 1$. We test this example using P^k polynomials with $0 \le k \le 4$ on a nonuniform mesh which is a 10% random perturbation of the uniform mesh. From the table we conclude that, for all values of $\theta > \frac{1}{2}$, one can always observe $(k + 1)$th order of accuracy, indicating that the error estimates obtained in Theorem 2.2 are sharp. Moreover, in the case of the same P^k elements and the same meshes, it seems that for even values of k, smaller θ would produce better approximations and that for odd values of k, larger θ would produce better approximations as far as the magnitude of errors is concerned.

TABLE 5.1. The errors $\|u - u_h\|_{L^2(I)}$ and orders for Example 5.1 using P^k polynomials with different θ on a random mesh of N cells. SSP9 time discretization. $T = 1$.

		$\theta = 0.75$		$\theta = 1$		$\theta = 2$	
	N	L^2 error	Order	L^2 error	Order	L^2 error	Order
P^0	20	8.50E-02	–	1.22E-01	–	2.72E-01	–
	40	4.29E-02	1.00	6.27E-02	0.97	1.52E-01	0.86
	80	2.17E-02	1.01	3.19E-02	1.01	8.03E-02	0.95
	160	1.08E-02	1.01	1.60E-02	0.99	4.13E-02	0.96
P^1	20	6.77E-03	–	4.30E-03	–	3.00E-03	–
	40	1.78E-03	1.96	1.07E-03	2.03	7.35E-04	2.07
	80	4.55E-04	2.03	2.74E-04	2.04	1.95E-04	1.97
	160	1.14E-04	2.00	6.84E-05	2.00	4.85E-05	2.01
P^2	20	8.84E-05	–	1.13E-04	–	2.33E-04	–
	40	1.12E-05	3.04	1.42E-05	3.05	3.16E-05	2.93
	80	1.44E-06	3.05	1.80E-06	3.07	4.07E-06	3.05
	160	1.72E-07	3.07	2.20E-07	3.03	5.07E-07	3.00
P^3	20	3.40E-06	–	2.25E-06	–	1.73E-06	–
	40	2.15E-07	4.05	1.36E-07	4.12	1.03E-07	4.15
	80	1.41E-08	4.07	9.00E-09	4.05	7.50E-09	3.90
	160	8.79E-10	4.00	5.61E-10	4.00	4.61E-10	4.02
P^4	20	3.05E-08	–	3.61E-08	–	7.20E-08	–
	40	9.78E-10	5.05	1.17E-09	5.04	2.44E-09	4.97
	80	3.31E-11	5.04	3.79E-11	5.11	7.97E-11	5.10
	160	9.43E-13	5.13	1.12E-12	5.08	2.43E-12	5.04

In order to verify the sharpness of Theorem 4.5, we also consider TVDRK3 time discretization for Example 5.1 and take $\Delta t = CFL\ h$. We use different CFL numbers for different θ and k, and use more refined meshes whenever necessary to show clean orders. Table 5.2 lists the numerical errors and their orders with different θ at $T = 1$. From the table we conclude that the error is indeed $\mathcal{O}(\tau^3 + h^{k+1})$, which is $\mathcal{O}(h^{k+1})$ for $k \le 2$ and $\mathcal{O}(\tau^3) = \mathcal{O}(h^3)$ when $k \ge 3$.

Example 5.2. In this example, we consider the problem (5.1) with exact solution (5.2) and the inflow boundary condition

$$u(0, t) = g(t), \tag{5.3}$$

where $g(t)$ corresponds to the data from the exact solution.

TABLE 5.2. The errors $\|u - u_h\|_{L^2(I)}$ and orders for Example 5.1 using P^k polynomials with different θ on a random mesh of N cells. TVDRK3 time discretization. $T = 1$.

		$\theta = 0.75$		$\theta = 1$		$\theta = 1.50$	
	N	L^2 error	Order	L^2 error	Order	L^2 error	Order
		$CFL = 1.75$		$CFL = 1.00$		$CFL = 0.50$	
P^0	20	8.77E-02	–	1.22E-01	–	2.00E-01	–
	40	4.35E-02	1.03	6.28E-02	0.98	1.08E-01	0.91
	80	2.18E-02	1.03	3.19E-02	1.01	5.59E-02	0.98
	160	1.08E-02	1.01	1.60E-02	0.99	2.84E-02	0.98
		$CFL = 0.50$		$CFL = 0.34$		$CFL = 0.17$	
P^1	20	6.79E-03	–	4.32E-03	–	3.23E-03	–
	40	1.78E-03	1.96	1.07E-03	2.04	7.95E-04	2.06
	80	4.55E-04	2.03	2.74E-04	2.04	2.07E-04	2.01
	160	1.14E-04	2.00	6.84E-05	2.00	5.15E-05	2.00
		$CFL = 0.25$		$CFL = 0.17$		$CFL = 0.08$	
P^2	20	9.07E-05	–	1.13E-04	–	1.75E-04	–
	40	1.14E-05	3.04	1.42E-05	3.04	2.26E-05	3.01
	80	1.49E-06	3.04	1.80E-06	3.07	2.88E-06	3.07
	160	1.77E-07	3.07	2.20E-07	3.03	3.56E-07	3.01
		$CFL = 0.15$		$CFL = 0.10$		$CFL = 0.05$	
P^3	160	9.32E-09	–	2.80E-09	–	5.77E-10	–
	320	1.26E-09	3.01	3.74E-10	3.02	5.55E-11	3.51
	640	1.57E-10	3.00	4.64E-11	3.00	6.09E-12	3.18
	1280	1.96E-11	3.00	5.79E-12	3.00	7.34E-13	3.05
		$CFL = 0.10$		$CFL = 0.07$		$CFL = 0.03$	
P^4	20	1.25E-06	–	4.33E-07	–	6.42E-08	–
	40	1.64E-07	2.99	5.62E-08	3.00	4.78E-09	3.81
	80	2.19E-08	2.99	7.52E-09	3.00	5.96E-10	3.10
	160	2.74E-09	3.00	9.43E-10	3.00	7.42E-11	3.01

We test this example using P^k polynomials with $0 \le k \le 4$ on a nonuniform mesh which is a 10% random perturbation of the uniform mesh. The numerical errors and their orders obtained by using different values of θ at $T = 1$ are listed in Table 5.3. From the table, we observe almost the same results as that in Table 5.1; that is, the conclusions also hold true for the initial-boundary value problems.

Example 5.3. To illustrate that the optimal error estimate still holds for the linear variable coefficient equations, we solve

$$\begin{cases} u_t + (a(x)u)_x = b(x,t), \\ u(x,0) = \sin(x) \end{cases} \tag{5.4}$$

TABLE 5.3. The errors $\|u - u_h\|_{L^2(I)}$ and orders for Example 5.2 using P^k polynomials with different θ on a random mesh of N cells. $T = 1$.

		$\theta = 0.75$		$\theta = 1$		$\theta = 2$	
	N	L^2 error	Order	L^2 error	Order	L^2 error	Order
P^0	20	8.97E-02	–	1.25E-01	–	2.57E-01	–
	40	4.49E-02	1.01	6.57E-02	0.95	1.52E-01	0.77
	80	2.27E-02	1.02	3.38E-02	0.99	8.35E-02	0.89
	160	1.13E-02	1.01	1.71E-02	0.98	4.40E-02	0.93
P^1	20	6.63E-03	–	4.30E-03	–	3.13E-03	–
	40	1.75E-03	1.95	1.07E-03	2.03	7.63E-04	2.07
	80	4.51E-04	2.02	2.74E-04	2.04	1.99E-04	2.00
	160	1.13E-04	1.99	6.84E-05	2.00	4.87E-05	2.03
P^2	20	8.93E-05	–	1.13E-04	–	2.24E-04	–
	40	1.12E-05	3.04	1.42E-05	3.05	3.10E-05	2.90
	80	1.45E-06	3.05	1.80E-06	3.07	4.04E-06	3.04
	160	1.72E-07	3.07	2.20E-07	3.03	5.05E-07	3.00
P^3	20	3.30E-06	–	2.24E-06	–	1.79E-06	–
	40	2.12E-07	4.03	1.36E-07	4.11	1.07E-07	4.14
	80	1.39E-08	4.05	9.00E-09	4.05	7.63E-09	3.93
	160	8.74E-10	3.99	5.61E-10	4.00	4.62E-10	4.05
P^4	20	5.80E-08	–	5.91E-08	–	7.79E-08	–
	40	1.33E-09	5.54	1.46E-09	5.44	2.49E-09	5.05
	80	3.87E-11	5.27	4.25E-11	5.27	8.05E-11	5.11
	160	1.01E-12	5.26	1.17E-12	5.18	2.43E-12	5.05

with periodic boundary conditions, where

$$a(x) = \sin(x),$$
$$b(x,t) = (\sin(x) + 1)\cos(x+t) + \cos(x)\sin(x+t).$$

The exact solution to this problem is

$$u(x,t) = \sin(x+t). \tag{5.5}$$

Note that $a(x)$ changes sign across cell boundaries. If $a(x_{j+\frac{1}{2}})$ is positive, then at $x_{j+\frac{1}{2}}$, we take $\hat{u}_h = \theta u_h^- + (1-\theta)u_h^+$; otherwise, we choose $\hat{u}_h = \theta u_h^+ + (1-\theta)u_h^-$. We test this example using P^k polynomials with $0 \le k \le 4$ on a nonuniform mesh which is a 10% random perturbation of the uniform mesh. The results in Table 5.4 show that the rate of convergence of the error, $\|u - u_h\|_{L^2(I)}$, achieves the expected $(k+1)$th order of accuracy. This example demonstrates that the conclusions also hold true for the variable coefficient problems.

TABLE 5.4. The errors $\|u - u_h\|_{L^2(I)}$ and orders for Example 5.3 using P^k polynomials with different θ on a random mesh of N cells. $T = 1$.

		$\theta = 0.75$		$\theta = 1$		$\theta = 2$	
	N	L^2 error	Order	L^2 error	Order	L^2 error	Order
P^0	20	8.02E-02	–	1.07E-01	–	1.97E-01	–
	40	4.09E-02	0.99	5.70E-02	0.92	1.21E-01	0.71
	80	2.11E-02	0.99	2.99E-02	0.96	6.97E-02	0.83
	160	1.06E-02	0.99	1.53E-02	0.97	3.73E-02	0.90
P^1	20	6.61E-03	–	4.23E-03	–	3.06E-03	–
	40	1.69E-03	2.00	1.06E-03	2.03	7.67E-04	2.03
	80	4.37E-04	2.02	2.70E-04	2.04	1.96E-04	2.03
	160	1.11E-04	1.98	6.73E-05	2.00	4.75E-05	2.05
P^2	20	9.12E-05	–	1.23E-04	–	2.59E-04	–
	40	1.12E-05	3.08	1.47E-05	3.12	3.42E-05	2.97
	80	1.48E-06	3.01	1.86E-06	3.08	4.31E-06	3.09
	160	1.78E-07	3.06	2.26E-07	3.04	5.28E-07	3.03
P^3	20	3.68E-06	–	2.31E-06	–	1.79E-06	–
	40	2.21E-07	4.12	1.43E-07	4.08	1.17E-07	4.00
	80	1.40E-08	4.11	9.09E-09	4.11	7.68E-09	4.06
	160	8.67E-10	4.02	5.51E-10	4.04	4.41E-10	4.12
P^4	20	4.42E-08	–	5.29E-08	–	9.68E-08	–
	40	1.08E-09	5.45	1.30E-09	5.45	2.85E-09	5.17
	80	3.54E-11	5.09	4.03E-11	5.17	8.85E-11	5.17
	160	1.01E-12	5.13	1.19E-12	5.08	2.63E-12	5.07

Example 5.4. In this example, we consider the following two-dimensional problem

$$\begin{cases} u_t + u_x + u_y = 0, \\ u(x, y, 0) = \sin(x + y) \end{cases} \tag{5.6}$$

with periodic boundary conditions. The exact solution is

$$u(x, y, t) = \sin(x + y - 2t).$$

We test this example using Q^k polynomials with $0 \le k \le 2$ on a nonuniform mesh which is a 30% random perturbation of the uniform mesh. Different combinations of (θ_1, θ_2) are considered in our computations. The results in Table 5.5 show that the order of convergence of the error, $\|u - u_h\|_{L^2(\Omega_h)}$, achieves the expected $(k + 1)$th order of accuracy, indicating that the error estimates given in Theorem 3.2 are sharp.

TABLE 5.5. The errors $\|u - u_h\|_{L^2(\Omega_h)}$ and orders for Example 5.4 using Q^k polynomials with different (θ_1, θ_2) on a random mesh of $N_1 \times N_2$ cells. $T = 1$.

		$(\theta_1,\theta_2)=(0.75,0.75)$		$(\theta_1,\theta_2)=(0.75,2)$		$(\theta_1,\theta_2)=(2,2)$	
	$N_1 \times N_2$	L^2 error	Order	L^2 error	Order	L^2 error	Order
Q^0	10×10	2.81E-01	–	4.91E-01	–	6.04E-01	–
	20×20	1.54E-01	0.91	3.16E-01	0.67	4.39E-01	0.49
	40×40	8.43E-02	0.91	1.81E-01	0.83	2.72E-01	0.72
	80×80	4.14E-02	1.14	9.64E-02	1.02	1.52E-01	0.94
Q^1	10×10	3.67E-02	–	3.18E-02	–	2.50E-02	–
	20×20	1.08E-02	1.86	8.97E-03	1.93	6.57E-03	2.04
	40×40	2.89E-03	1.98	2.49E-03	1.92	1.89E-03	1.86
	80×80	7.16E-04	2.25	5.93E-04	2.32	4.26E-04	2.41
Q^2	10×10	1.56E-03	–	2.05E-03	–	2.41E-03	–
	20×20	2.16E-04	3.02	3.20E-04	2.83	4.00E-04	2.74
	40×40	3.07E-05	2.92	4.52E-05	2.93	5.62E-05	2.94
	80×80	3.05E-06	3.50	5.52E-06	3.39	6.95E-06	3.37

6. Concluding remarks

In this paper, an analysis of the L^2-stability and optimal error estimates to DG methods using upwind-biased numerical fluxes applied to linear conservation laws is carried out. Optimal a priori error estimates are obtained in one dimension and in multidimensions for Cartesian meshes. Our analysis is valid for arbitrary nonuniform regular meshes and for polynomials of degree $k \geq 0$, no matter the periodic boundary condition or the inflow boundary condition is concerned. The main technical difficulties are the construction and analysis of some suitable projections $P_h^\star$ and $\tilde{P}_h$ corresponding to different boundary conditions for the one-dimensional case and $\Pi_h^\star$ for the multidimensional case. The sharpness of our theoretical results is confirmed by a series of numerical experiments. Extensions of this work to nonlinear equations and high order wave equations are challenging and this will be carried out in the future.

References

[1] J. L. Bona, H. Chen, O. Karakashian, and Y. Xing, *Conservative, discontinuous Galerkin-methods for the generalized Korteweg-de Vries equation*, Math. Comp. **82** (2013), no. 283, 1401–1432, DOI 10.1090/S0025-5718-2013-02661-0. MR3042569

[2] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954 (2008m:65001)

[3] E. Burman, A. Ern, and M. A. Fernández, *Explicit Runge-Kutta schemes and finite elements with symmetric stabilization for first-order linear PDE systems*, SIAM J. Numer. Anal. **48** (2010), no. 6, 2019–2042, DOI 10.1137/090757940. MR2740540 (2012a:65231)

[4] P. Castillo, B. Cockburn, D. Schötzau, and C. Schwab, *Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection-diffusion problems*, Math. Comp. **71** (2002), no. 238, 455–478, DOI 10.1090/S0025-5718-01-01317-5. MR1885610 (2003e:65214)

[5] Y. Cheng and C.-W. Shu, *A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives*, Math. Comp. **77** (2008), no. 262, 699–730, DOI 10.1090/S0025-5718-07-02045-5. MR2373176 (2008m:65252)

[6] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4. MR0520174 (58 #25001)

[7] B. Cockburn, B. Dong, and J. Guzmán, *Optimal convergence of the original DG method for the transport-reaction equation on special meshes*, SIAM J. Numer. Anal. **46** (2008), no. 3, 1250–1265, DOI 10.1137/060677215. MR2390992 (2009c:65299)

[8] B. Cockburn, S. Hou, and C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. IV. The multidimensional case*, Math. Comp. **54** (1990), no. 190, 545–581, DOI 10.2307/2008501. MR1010597 (90k:65162)

[9] B. Cockburn, G. Kanschat, I. Perugia, and D. Schötzau, *Superconvergence of the local discontinuous Galerkin method for elliptic problems on Cartesian grids*, SIAM J. Numer. Anal. **39** (2001), no. 1, 264–285 (electronic), DOI 10.1137/S0036142900371544. MR1860725 (2002g:65140)

[10] B. Cockburn, S. Y. Lin, and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. III. One-dimensional systems*, J. Comput. Phys. **84** (1989), no. 1, 90–113, DOI 10.1016/0021-9991(89)90183-6. MR1015355 (90k:65161)

[11] B. Cockburn and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework*, Math. Comp. **52** (1989), no. 186, 411–435, DOI 10.2307/2008474. MR983311 (90k:65160)

[12] B. Cockburn and C.-W. Shu, *The Runge-Kutta discontinuous Galerkin method for conservation laws. V. Multidimensional systems*, J. Comput. Phys. **141** (1998), no. 2, 199–224, DOI 10.1006/jcph.1998.5892. MR1619652 (99c:65181)

[13] P. J. Davis, *Circulant Matrices*, John Wiley & Sons, New York-Chichester-Brisbane, 1979. A Wiley-Interscience Publication; Pure and Applied Mathematics. MR543191 (81a:15003)

[14] S. Gottlieb, C.-W. Shu, and E. Tadmor, *Strong stability-preserving high-order time discretization methods*, SIAM Rev. **43** (2001), no. 1, 89–112 (electronic), DOI 10.1137/S003614450036757X. MR1854647 (2002f:65132)

[15] M. J. Grote, A. Schneebeli, and D. Schötzau, *Discontinuous Galerkin finite element method for the wave equation*, SIAM J. Numer. Anal. **44** (2006), no. 6, 2408–2431 (electronic), DOI 10.1137/05063194X. MR2272600 (2007k:65149)

[16] M. J. Grote and D. Schötzau, *Optimal error estimates for the fully discrete interior penalty DG method for the wave equation*, J. Sci. Comput. **40** (2009), no. 1-3, 257–272, DOI 10.1007/s10915-008-9247-z. MR2511734 (2010k:65200)

[17] P. Lasaint and P.-A. Raviart, *On a finite element method for solving the neutron transport equation*, Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974), Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974, pp. 89–123. Publication No. 33. MR0658142 (58 #31918)

[18] X. Meng, C.-W. Shu, and B. Wu, *Superconvergence of the local discontinuous Galerkin method for linear fourth-order time-dependent problems in one space dimension*, IMA J. Numer. Anal. **32** (2012), no. 4, 1294–1328, DOI 10.1093/imanum/drr047. MR2991829

[19] W. H. Reed and T. R. Hill, *Triangular mesh methods for the neutron transport equation*, Tech. Report LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, NM, 1973.

[20] G. R. Richter, *An optimal-order error estimate for the discontinuous Galerkin method*, Math. Comp. **50** (1988), no. 181, 75–88, DOI 10.2307/2007915. MR917819 (88j:65197)

[21] C.-W. Shu and S. Osher, *Efficient implementation of essentially nonoscillatory shock-capturing schemes*, J. Comput. Phys. **77** (1988), no. 2, 439–471, DOI 10.1016/0021-9991(88)90177-5. MR954915 (89g:65113)

[22] E. Tadmor, *From semidiscrete to fully discrete: stability of Runge-Kutta schemes by the energy method. II*, Collected Lectures on the Preservation of Stability Under Discretization (Fort Collins, CO, 2001), SIAM, Philadelphia, PA, 2002, pp. 25–49. MR2026662

[23] Y. Xing, C.-S. Chou, and C.-W. Shu, *Energy conserving local discontinuous Galerkin methods for wave propagation problems*, Inverse Probl. Imaging **7** (2013), no. 3, 967–986, DOI 10.3934/ipi.2013.7.967. MR3105364

[24] Y. Xu and C.-W. Shu, *Optimal error estimates of the semidiscrete local discontinuous Galerkin methods for high order wave equations*, SIAM J. Numer. Anal. **50** (2012), no. 1, 79–104, DOI 10.1137/11082258X. MR2888305

[25] Q. Zhang and C.-W. Shu, *Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws*, SIAM J. Numer. Anal. **42** (2004), no. 2, 641–666 (electronic), DOI 10.1137/S0036142902404182. MR2084230 (2005h:65149)

[26] Q. Zhang and C.-W. Shu, *Stability analysis and a priori error estimates of the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws*, SIAM J. Numer. Anal. **48** (2010), no. 3, 1038–1063, DOI 10.1137/090771363. MR2669400 (2011g:65205)

[27] Q. Zhang and C.-W. Shu, *Stability analysis and a priori error estimates to the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws*, Technical report 2009-28, Brown University, Providence, RI, 2009; available online at `http://www.brown.edu/research/projects/scientific-computing/sites/brown.edu.research.projects.scientific-computing/files/uploads/Stability%20analysis%20and%20a%20priori%20error%20estimates%20to%20the%20third%20order%20explicit%20Runge-Kutta.pdf`

Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China
E-mail address: xiongmeng@hit.edu.cn

Division of Applied Mathematics, Brown University, Providence, RI 02912
E-mail address: shu@dam.brown.edu

Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China
E-mail address: mathwby@hit.edu.cn