Applied Mathematics and Computation 217 (2011) 8179–8195

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Positive operator based iterative algorithms for solving Lyapunov equations for Itô stochastic systems with Markovian jumps ☆

Zhao-Yan Li [a], Bin Zhou [b,*], James Lam [c], Yong Wang [a]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China
[b] Center for Control Theory and Guidance Technology, Harbin Institute of Technology, P.O. Box 416, Harbin 150001, PR China
[c] Department of Mechanical Engineering, University of Hong Kong, Hong Kong

ARTICLE INFO

Keywords:
Iteration
Jump system
Lyapunov equation
Markov process
Mean square stability
Positive operator
Stochastic system

ABSTRACT

This paper studies the iterative solutions of Lyapunov matrix equations associated with Itô stochastic systems having Markovian jump parameters. For the discrete-time case, when the associated stochastic system is mean square stable, two iterative algorithms with one in direct form and the other one in implicit form are established. The convergence of the implicit iteration is proved by the properties of some positive operators associated with the stochastic system. For the continuous-time case, a transformation is first performed so that it is transformed into an equivalent discrete-time Lyapunov equation. Then the iterative solution can be obtained by applying the iterative algorithm developed for discrete-time Lyapunov equation. Similar to the discrete-time case, an implicit iteration is also proposed for the continuous case. For both discrete-time and continuous-time Lyapunov equations, the convergence rates of the established algorithms are analyzed and compared. Numerical examples are worked out to validate the effectiveness of the proposed algorithms.

1. Introduction

Stochastic systems including Markovian jump system as a special case has attracted a lot of researchers in the past several decades since many practical systems can be modeled by stochastic systems with Markovian jumping parameters, for example, network control systems [23], a model with random abrupt changes, sudden environmental changes and abrupt variations of the operating point and so on [20]. A set of control problems that have been well studied in deterministic setting are extensively extended to stochastic setting. These problems include stability and stabilization [8,11], detectability and observability [13], estimation [5], time-delayed control [19], robust control [26], and filtering [21,24,25].

Amongst some of the most important problems in control theory, stability and stabilization in both deterministic setting and stochastic setting have received much more attention than other problems. For deterministic system, it is well known that the stability of a system is equivalent to the existence of a positive definite solution of the associated Lyapunov equation. This elegant result has been extended to the stochastic setting by virtue of (stochastic) Lyapunov direct approach. For example, for the

☆ This work is supported in part by the National Natural Science Foundation of China under grant numbers 60904007, 61074111 and 10771044, the China Postdoctoral Science Foundation under grant number 20100480059, the Foundation for Innovative Research Group of the National Natural Science Foundation of China under grant 601021002, the Development Program for Outstanding Young Teachers at Harbin Institute of Technology under grant number HITQNJS.2009.054, the Heilongjiang Postdoctoral Foundation of China under Grant No. LRB10-194, HKU CRCG 201007176243, and by GRE HKU 7138/10E.

* Corresponding author.
E-mail addresses: zhaoyanlee@gmail.com, lizhaoyan@hit.edu.cn (Z.-Y. Li), binzhoulee@163.com, binzhou@hit.edu.cn (B. Zhou), james.lam@hku.hk (J. Lam), mathwy@hit.edu.cn (Y. Wang).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.01.031

Itô stochastic system with Markovian jumps, it is shown that the mean square stability of the system is equivalent to the existence of positive definite solution of some coupled Lyapunov equations (see, e.g., [27]). Lyapunov equation is not only fundamental in stability analysis but also in stabilizing controller design in both deterministic setting and stochastic setting. Hence, finding solution to this class of equations, especially, in the case that the associated system is (mean square) stable, is extremely important. However, to the best knowledge of the authors, a few results for Lyapunov equations associated with stochastic system are available in the literature. For example, for the coupled Lyapunov equation associated with discrete-time Markovian jump linear system which is mean square stable, reference [1] presents an iterative approach by assuming the zero initial condition and Schur stability of each subsystem. However, it is shown in [22] that these two assumptions are not necessary for the convergence of the proposed iterative algorithm. Moreover, without assuming the mean square stability of the associated stochastic system, references [22,29] propose two alternative algorithms by using implicit iteration and gradient based iteration, respectively. We notice that linear matrix equation is a hot topic in control theory and has received much attention in the past several decades (see, for example, [2,35,37,3,4,6,7,12,14,33,15,16,18,34,36,32,30] and the references therein). However, as far as we know, no method is available for Lyapunov equation associated with continuous-time stochastic systems.

In this paper, we will study the numerical solution of the Lyapunov equations associated with Itô stochastic systems with Markovian jumps which include the Markovian jump system as a special case. For the discrete-time Lyapunov equation, when the associated stochastic system is mean square stable, we give a direct iteration and an implicit iteration to compute the numerical solutions, whose convergence are verified by virtue of properties of some positive operators. A necessary and sufficient condition for the convergence of the implicit iteration is also established based on the spectral radius of some auxiliary matrix. For the continuous-time Lyapunov equation, a transformation is first performed so that it is transformed into an equivalent discrete-time Lyapunov equation. An iterative solution can then be obtained by applying the iterative algorithm developed for the discrete-time Lyapunov equation. Similar to the discrete-time case, we also give an implicit iteration for the continuous-time equation. For both discrete-time and continuous-time Lyapunov equations, the convergence rates of their corresponding algorithms are analyzed and compared. Numerical examples are worked out to validate the effectiveness of the proposed algorithms.

The rest of this paper is organized as follows: Section 2 presents some notations and preliminary results that will used in the paper. The discrete-time stochastic Lyapunov equation and the continuous-time Lyapunov equation are then respectively studied in Sections 3 and 4. Numerical examples are reported in Sections 5 and 6 concludes the paper.

2. Notations and preliminaries

We let $\mathbf{R}^{n\times m}$, $\mathbf{C}^{n\times m}$ represent, respectively, the $n\times m$ dimension real and complex matrices, $\mathbf{S}^{n\times n}$ represent the n dimension symmetry matrices, A^{T} represent its transpose, $\otimes$ represent the Kronecker product, $\mathbf{C}^{-}=\{\lambda:\mathrm{Re}\{\lambda\}<0,\lambda\in\mathbf{C}\}$ and $\mathbf{C}^{\odot}=\{\lambda:|\lambda|<1,\lambda\in\mathbf{C}\}$. Let $\mathcal{R}^{n\times n}(\mathcal{S}^{n\times n})$ denote the linear space made up of all ordered N-tuple of real matrices (real symmetric matrices), that is, $C=(C_1,C_2,\ldots,C_N)$ with $C_i\in\mathbf{R}^{n\times n}(C_i\in\mathbf{S}^{n\times n})$, where $N\geqslant 1$ is a given integer. We denote $\mathcal{R}_{+}^{n\times n}=\{C=(C_1,C_2,\ldots,C_N)\in\mathcal{R}^{n\times n}:C_i>0,i\in\mathcal{N}\}$ where $\mathcal{N}=\{1,2,\ldots,N\}$. For any matrix $X=[x_1\quad x_2\quad\cdots\quad x_n]\in\mathbf{C}^{m\times n}$, the stretching function is defined as

$$\mathrm{vec}(\cdot)=\begin{bmatrix}x_1^{\mathrm{T}} & x_2^{\mathrm{T}} & \cdots & x_n^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}:\mathbf{C}^{m\times n}\to\mathbf{C}^{mn}.$$

For any $P=(P_1,P_2,\ldots,P_N)\in\mathcal{R}^{n\times n}$, define the operator $\varphi(\cdot):\mathcal{R}^{n\times n}\to\mathbf{R}^{n^2N}$ as follows:

$$\varphi(P)\triangleq\begin{bmatrix}\mathrm{vec}^{\mathrm{T}}(P_1) & \mathrm{vec}^{\mathrm{T}}(P_2) & \cdots & \mathrm{vec}^{\mathrm{T}}(P_N)\end{bmatrix}^{\mathrm{T}}=\mathrm{vec}([P_1\quad P_2\quad\cdots\quad P_N]).$$

We use $\mathrm{diag}\{A_1,A_2,\ldots,A_s\}$ to denote a diagonal matrix whose diagonal elements are A_i, $i=1,2,\ldots,s$. Finally, we let $\sigma(\cdot)$ and $\rho(\cdot)$ represent, respectively, the spectrum and spectral radius of a matrix or a linear operator.

The positive operator introduced below takes very important functions in this paper.

Definition 1. Let $\mathcal{W}$ be some finite-dimensional real vector space, ordered by a closed, solid, pointed convex cone $\mathcal{W}_{+}$. A linear operator $\mathcal{L}:\mathcal{W}\to\mathcal{W}$ is called positive (denoted by $\mathcal{L}>0$) if $\mathcal{L}(\mathcal{W}_{+})\subset\mathcal{W}_{+}$.

Lemma 1 [17]. *Let $\mathcal{L}:\mathcal{W}\to\mathcal{W}$ be positive and $\rho(\mathcal{L})$ be the spectral radius of $\mathcal{L}$, namely, $\rho(\mathcal{L})=\max\{|\sigma(\mathcal{L})|\}$. Then the following statements are equivalent:*

1. *$\rho(\mathcal{L})<1$;*
2. *There exists an $X>0$ ($X<0$ means that $X\in\mathrm{int}\,\mathcal{W}_{+}$) such that $\mathcal{L}(X)-X<0$;*
3. *For any $Y>0$, there exists an $X>0$ such that $\mathcal{L}(X)-X=-Y$.*

Finally, we introduce the standard concept of asymptotic exponential convergence rate of linear iteration.

Definition 2. Consider the following iteration

$$x(k+1)=Mx(k)+g,\quad g\in\mathbf{R}^{n},\tag{1}$$

where M and g are respectively constant matrix and vector. Then the number $R(M) = -\ln \rho(M)$ is called the asymptotic exponential convergence rate of the iteration in (1).

Basically, the above definition means that the smaller the spectral radius $\rho(M)$, the larger the asymptotic exponential convergence rate is, namely, iteration (1) converges faster.

3. Discrete-time Lyapunov equation

We consider the discrete-time stochastic system described by Itô difference equation with Markovian jumps:

$$x(k+1) = A_0(\theta(k))x(k) + \sum_{s=1}^{r} A_s(\theta(k))x(k)\omega_s(k), \tag{2}$$

where $x(k) \in \mathbf{R}^n$ is the system state, $\omega_s(k) \in \mathbf{R}, s \in \{1,2,\ldots,r\}$ are sequences of real random variables defined on a complete probability space $\{\Omega, \mathcal{F}, \mu\}$ and are independent wide sense stationary, second-order processes with $\mathbf{E}\{\omega_i(k)\} = 0$ and $\mathbf{E}\{\omega_i(t)\omega_j(s)\} = \delta_{ij}, i,j \in \{1,r\}$, and $\{\theta(k), k \geqslant 0\}$ is a discrete-time homogeneous Markovian chain with finite state space $\mathcal{N}$, transition probability matrix $\Pi = [\pi_{ij}]_{N\times N}$ and initial distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$.

Definition 3. The discrete-time Itô stochastic linear system (2) is said to be mean square stable if

$$\lim_{k\to\infty} \mathbf{E}\left\{\|x(k,\theta_0,x_0)\|^2\right\} = 0, \quad \forall x_0 \in \mathbf{R}^n.$$

Moreover, we say shortly that $A = (A_0, A_1, \ldots, A_r)$ is mean square stable.

Regarding mean square stability of system (2), the following result is standard.

Lemma 2. *The discrete-time Itô stochastic linear system (2) is mean square stable if and only if the discrete-time stochastic Lyapunov matrix equation*

$$\sum_{s=0}^{r} A_{si}^{\mathrm{T}}\left(\sum_{j=1}^{N}\pi_{ij}P_j\right)A_{si} - P_i = -Q_i, \quad i \in \mathcal{N}, \tag{3}$$

has a unique solution $P = (P_1, P_2, \ldots, P_N) \in \mathcal{R}_+^{n\times n}$ for any given $Q = (Q_1, Q_2, \ldots, Q_N) \in \mathcal{R}_+^{n\times n}$.

It is thus important to consider numerical solutions of the matrix equation (3). However, a few result is available in the literature for this problem. For example, according to Remark 1 in [22], the equations in (3) are a special case of the coupled Sylvester matrix equations studied in [6] by introducing auxiliary variables. Therefore, the number of variables would be doubled, which increases the computational cost. Recently, we proposed in [28] alternative gradient based iterative solutions of general coupled matrix equations which include (3) as special cases. However, since the method in [28] does not take full advantage of the special structures of Eq. (3), the convergence performances is not satisfactory as can be seen in our numerical examples given in Section 5. In this section, we will present two classes of iterative algorithms for solving these equations by taking full usage of the special structures of them.

3.1. Direct iteration

In this subsection, we present a natural and simple iterative algorithm for solving Eq. (3).

Theorem 1. *Assume that the discrete-time Itô stochastic linear system (2) is mean square stable. Let $P^* = \left(P_1^*, P_2^*, \ldots, P_N^*\right) \in \mathcal{R}_+^{n\times n}$ be the unique solution of Eq. (3). Define iteration*

$$P_i(k+1) = \sum_{s=0}^{r} A_{si}^{\mathrm{T}}\left(\sum_{j=1}^{N}\pi_{ij}P_j(k)\right)A_{si} + Q_i, \quad i \in \mathcal{N}. \tag{4}$$

Then $\{P(k)\}_{k=0}^{\infty}$ converges and $\lim_{k\to\infty}P(k) = P^*$.

Proof. Define operator $\mathcal{J} : \mathcal{R}^{n\times n} \to \mathcal{R}^{n\times n}$ as

$$\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_N), \quad \mathcal{J}_i(P) = \sum_{s=1}^{r} A_{si}^{\mathrm{T}}\left(\sum_{j=1}^{N}\pi_{ij}P_j\right)A_{si}, \ i \in \mathcal{N}.$$

Obviously, $\mathcal{J}$ is a linear positive operator. Then iteration (4) can be written as

$$P(k+1) = \mathcal{J}(P(k)) + Q, \tag{5}$$

where $Q = (Q_1, Q_2, \ldots, Q_N)$. Because P^* is the unique positive definite solution of Eq. (3), so $P^* = \mathcal{J}(P^*) + Q, \ Q > 0$. By Lemma 1, we know that $\rho(\mathcal{J}) < 1$. Then iteration (5) converges for any initial condition $P(0) \in \mathcal{C}^{n\times n}$. Set $\lim_{k\to\infty}P(k) = P^\dagger$. Taking limit on both sides of (5) and using $P^* = \mathcal{J}(P^*) + Q$ produce

$$P^{\dagger} = \mathcal{J}(P^{\dagger}) + Q = \mathcal{J}(P^{\dagger}) + P^{*} - \mathcal{J}(P^{*}).$$

Let $\Delta P = P^{\dagger} - P^{*}$. Then it follows that $\Delta P = \mathcal{J}(\Delta P)$. Because $\rho(\mathcal{J}) < 1$, so $\Delta P = \mathcal{J}(\Delta P)$ has a unique solution 0, that is, $P^{\dagger} = P^{*}$. The proof is completed. □

Regarding the convergence rate of iteration (4), we have the following result.

Proposition 1. *The asymptotic exponential convergence rate of iteration (4) is* $-\ln \rho(J)$, *where*

$$J = \left(\sum_{s=0}^{r} \Upsilon_s\right)(\Pi \otimes I_{n^2}), \tag{6}$$

with

$$\Upsilon_s = \mathrm{diag}\left\{A_{s1}^{\mathrm{T}} \otimes A_{s1}^{\mathrm{T}}, A_{s2}^{\mathrm{T}} \otimes A_{s2}^{\mathrm{T}}, \ldots, A_{sN}^{\mathrm{T}} \otimes A_{sN}^{\mathrm{T}}\right\}.$$

Proof. Taking $\varphi(\cdot)$ on both sides of (4) and denoting $p(k) = \varphi(P(k))$ gives

$$p(k+1) = Jp(k) + \varphi(Q),$$

where $J = [J_{ij}]_{Nn^2 \times Nn^2}$ with $J_{ij}^{\mathrm{T}} = \sum_{s=0}^{r} \pi_{ij} A_{si} \otimes A_{si} \in \mathbf{R}^{n^2 \times n^2}$. Direct manipulation shows that J can be simplified as (6). The result then follows from Definition 2. □

3.2. Implicit iteration

Notice that we can rewrite the Lyapunov matrix equation (3) as follows:

$$\left(\sqrt{\pi_{ii}}A_{0i}^{\mathrm{T}}\right)P_i(\sqrt{\pi_{ii}}A_{0i}) - (1+\gamma_i)P_i = -A_{0i}^{\mathrm{T}}\left(\sum_{j=1,j\neq i}^{N} \pi_{ij}P_j\right)A_{0i} - \gamma_i P_i - \sum_{s=1}^{r} A_{si}^{\mathrm{T}}\left(\sum_{j=1}^{N} \pi_{ij}P_j\right)A_{si} - Q_i, \quad i \in \mathcal{N}.$$

where $\gamma_i,\ i \in \mathcal{N}$, are any scalars. Based on this expression and motivated by the work [22], we can present the following implicit iteration for solving Eq. (3):

$$\left(\sqrt{\pi_{ii}}A_{0i}^{\mathrm{T}}\right)P_i(k+1)(\sqrt{\pi_{ii}}A_{0i}) - (1+\gamma_i)P_i(k+1) = -A_{0i}^{\mathrm{T}}\left(\sum_{j=1,j\neq i}^{N} \pi_{ij}P_j(k)\right)A_{0i} - \gamma_i P_i(k) - \sum_{s=1}^{r} A_{si}^{\mathrm{T}}\left(\sum_{j=1}^{N} \pi_{ij}P_j(k)\right)A_{si} - Q_i, \quad i \in \mathcal{N}. \tag{7}$$

Notice that at every step of the above iteration, N normal discrete-time Lyapunov equations in the form of

$$A^{\mathrm{T}}XA - X = -Z, \quad Z > 0 \tag{8}$$

are solved. Fortunately, there are many numerically stable algorithms for solving such kind of equations. One of the most effective algorithms is the well-known Hessenberg-Schur form based approach [10] which has been imbedded in the Matlab function `dyap`.

Regarding the convergence of iteration (7), we can prove the following result.

Theorem 2. *If the discrete-time Itô stochastic linear system (2) is mean square stable, then for any* $\gamma_i \geqslant 0,\ i \in \mathcal{N}$, *the iteration in (7) converges to the unique solution* $P^{*} = (P_1^{*}, P_2^{*}, \ldots, P_N^{*}) \in \mathcal{R}_{+}^{n\times n}$ *of the discrete-time Lyapunov equation (3) for any initial condition* $P(0) = (P_1(0), P_2(0), \ldots, P_N(0))$.

Proof. Define the operator $\mathcal{R}$ as

$$\mathcal{R}(P) = (\mathcal{R}_1(P), \mathcal{R}_2(P), \ldots, \mathcal{R}_N(P)),$$
$$\mathcal{R}_i(P) = \left(\sqrt{\pi_{ii}}A_{0i}^{\mathrm{T}}\right)P_i(\sqrt{\pi_{ii}}A_{0i}) - (1+\gamma_i)P_i, \quad i \in \mathcal{N},$$

and the operator $\mathcal{S}$ as

$$\mathcal{S}(P) = (\mathcal{S}_1(P), \mathcal{S}_2(P), \ldots, \mathcal{S}_N(P)),$$
$$\mathcal{S}_i(P) = A_{0i}^{\mathrm{T}}\left(\sum_{j=1,j\neq i}^{N} \pi_{ij}P_j\right)A_{0i} + \sum_{s=1}^{r} A_{si}^{\mathrm{T}}\left(\sum_{j=1}^{N} \pi_{ij}P_j\right)A_{si} + \gamma_i P_i, \quad i \in \mathcal{N},$$

where $P = (P_1, P_2, \ldots, P_N)$. Then the discrete-time stochastic Lyapunov matrix equation (3) can be written as

$$\mathcal{R}(P) + \mathcal{S}(P) = -Q, \tag{9}$$

where $Q = (Q_1, Q_2, \ldots, Q_N)$. Or equivalently,

$$\mathcal{R}(P) = -\mathcal{S}(P) - Q \triangleq -X.$$

As the discrete-time Itô stochastic linear system (2) is mean square stable and $Q > 0$, we know that the discrete-time stochastic Lyapunov matrix equation (3) has a unique solution $P > 0$, which implies that $X > 0$ as $\mathcal{S}$ is a positive operator. That is to say,

$$\mathcal{R}_i(P) = \left(\sqrt{\pi_{ii}}A_{0i}^{\mathrm{T}}\right)P_i(\sqrt{\pi_{ii}}A_{0i}) - (1+\gamma_i)P_i = -X_i < 0, \quad \forall i \in \mathcal{N}.$$

By Lyapunov stability theorem, the above equations indicate that $\sqrt{\frac{\pi_{ii}}{(1+\gamma_i)}}A_{0i},\ i \in \mathcal{N}$, are Schur stable. Hence

$$\sigma\left(\sqrt{\frac{\pi_{ii}}{1+\gamma_i}}A_{0i}\right) \subset \mathbf{C}^{\odot},$$

and it follows that

$$\sigma(\mathcal{R}_i) = \left\{\lambda\mu - 1 : \lambda, \mu \in \sigma\left(\sqrt{\frac{\pi_{ii}}{1+\gamma_i}}A_{0i}\right)\right\} \subset \mathbf{C}^{-}.$$

By observing the special structure of the operator $\mathcal{R}$, we conclude that

$$\sigma(\mathcal{R}) = \bigcup_{i=1}^{N}\sigma(\mathcal{R}_i) \subset \mathbf{C}^{-}.$$

Hence $\mathcal{R}$ is invertible and we can rewrite (9) as $\mathcal{R}(\mathcal{I} + \mathcal{R}^{-1}\mathcal{S})(P) = -Q$, or

$$(\mathcal{I} - \mathcal{T})(P) = \mathcal{R}^{-1}(-Q), \tag{10}$$

where $\mathcal{I}$ is the identity operator and $\mathcal{T} = -\mathcal{R}^{-1}\mathcal{S}$. Let

$$\mathcal{R}^{-1}(-Q) = -\mathcal{R}^{-1}(Q) = Y = (Y_1, Y_2, \ldots, Y_N), \tag{11}$$

namely,

$$\mathcal{R}_i(Y) = \left(\sqrt{\pi_{ii}}A_{0i}^{\mathrm{T}}\right)Y_i(\sqrt{\pi_{ii}}A_{0i}) - (1+\gamma_i)Y_i = -Q_i < 0, \quad \forall i \in \mathcal{N}.$$

As $\rho\left(\sqrt{\frac{\pi_{ii}}{1+\gamma_i}}A_{0i}\right) < 1$, we get from the well-known Lyapunov stability theorem that $Y_i > 0$, namely, $Y > 0$. Therefore, by Definition 1, it follows from (11) that $-\mathcal{R}^{-1}$ is a positive operator. As $\mathcal{S}$ is also a positive operator, we know that $\mathcal{T} = -\mathcal{R}^{-1}\mathcal{S}$ must be a positive operator too. Rearrange (10) as

$$\mathcal{T}(P) - P = -Y.$$

As $P > 0$ and $Y > 0$, we get from Lemma 1 that

$$\rho(\mathcal{T}) < 1. \tag{12}$$

Now, by using the operators $\mathcal{R}$ and $\mathcal{S}$, the iteration in (7) can be rewritten as

$$\mathcal{R}(P(k+1)) = -\mathcal{S}(P(k)) - Q, \quad P(0) \in \mathcal{C}^{n\times n},$$

namely,

$$P(k+1) = -\mathcal{R}^{-1}\mathcal{S}(P(k)) - \mathcal{R}^{-1}(Q) = \mathcal{T}(P(k)) - \mathcal{R}^{-1}(Q).$$

The convergence of the algorithm then follows from (12). Finally, when it converges to P^*, then P^* must satisfy

$$\mathcal{R}(P^*) = -\mathcal{S}(P^*) - Q,$$

which indicates that P^* is the unique solution of Eq. (3). The proof is completed. □

A couple of remarks regarding the implicit iteration in (7) are given in order.

Remark 1. Similar to iteration (7), we can also construct the following iteration

$$\left(\sqrt{\pi_{ii}}A_{mi}^{\mathrm{T}}\right)P_i(k+1)(\sqrt{\pi_{ii}}A_{mi}) - (1+\gamma_i)P_i(k+1) = -A_{mi}^{\mathrm{T}}\left(\sum_{j=1,j\neq i}^{N}\pi_{ij}P_j(k)\right)A_{mi} - \gamma_i P_i(k) - \sum_{s=0,s\neq m}^{r}A_{si}^{\mathrm{T}}\left(\sum_{j=1}^{N}\pi_{ij}P_j(k)\right)A_{si} - Q_i,$$

where $i \in \mathcal{N},\ \forall m \in \{1, 2, \ldots, r\}$. In this case, Theorem 2 is also true.

Remark 2. Numerical experience indicates that implicit iteration (7) converges faster than the direct iteration (4) in most cases. However, since the accuracy of implicit iteration (7) relies heavily on the accuracy of solutions of the discrete-time

Lyapunov equation (8), direct iteration (4) can yield more accuracy solutions than implicit iteration (7). These characteristics can be observed from the numerical examples in Section 5.

Remark 3. Though the free parameters $\gamma_i,\ i \in \mathcal{N}$, can be arbitrarily chosen in iteration (7), different values of them will lead to different convergence rate of the iteration. Numerical experience shows that, under the condition of mean square stability of the discrete-time Itô stochastic linear system (2), the smaller the values of $\gamma_i,\ i \in \mathcal{N}$, the faster the iteration converges. For this reason, we can simply choose

$$\gamma_i = 0, \quad \forall i \in \mathcal{N}.$$

Remark 4. In Theorem 2, the condition that the discrete-time Itô stochastic linear system (2) is mean square stable is sufficient but not necessary for the convergence of iteration (7). For example, we assume $N = 1,\ s = 1$ and $n = 1$. Then the stochastic Lyapunov equation (3) becomes

$$A_0^{\mathrm{T}} P A_0 + A_1^{\mathrm{T}} P A_1 - P = -Q,$$

where $A_1 \neq 0$. The iteration in (7) then reads

$$P(k+1) = -\frac{\gamma + A_1^2}{A_0^2 - 1 - \gamma} P(k) - \frac{1}{A_0^2 - 1 - \gamma} Q, \quad \gamma \geqslant 0,$$

which converges if and only if

$$\rho\left(\frac{A_1^2 + \gamma}{A_0^2 - 1 - \gamma}\right) = \left|\frac{A_1^2 + \gamma}{A_0^2 - 1 - \gamma}\right| < 1. \tag{13}$$

On the other hand, the discrete-time Itô stochastic linear system (2) is mean square stable if and only if $\sigma\left(A_0^2 + A_1^2\right) \subset \mathbf{C}^{\odot}$, i.e.

$$A_0^2 + A_1^2 < 1. \tag{14}$$

Obviously, (14) implies (13) but the converse is not true.

Motivated by Remark 4, we give the following necessary and sufficient condition for the convergence of iteration (7). To this end, we first denote

$$M = -\mathrm{diag}\{M_1, M_2, \ldots, M_N\},$$

with $M_i = \pi_{ii} A_{0i}^{\mathrm{T}} \otimes A_{0i}^{\mathrm{T}} - (1 + \gamma_i) I_{n^2},\ i \in \mathcal{N}$ and

$$T = M^{-1} W, \tag{15}$$

where $W = [W_{ij}]_{Nn^2 \times Nn^2}$, with

$$W_{ii} = \pi_{ii} \sum_{s=1}^{r} A_{si}^{\mathrm{T}} \otimes A_{si}^{\mathrm{T}} + \gamma_i I_{n^2},$$

$$W_{ij} = \pi_{ij} \sum_{s=0}^{r} A_{si}^{\mathrm{T}} \otimes A_{si}^{\mathrm{T}}, \quad \forall i \neq j,\ i,j \in \mathcal{N}.$$

Since at each step in iteration (7), a Lyapunov equation in the form of (8) should be solved, we should find conditions to guarantee that these Lyapunov equations have unique solutions. A necessary and sufficient condition is that M_i is nonsingular, or equivalently, the following assumption is true:

Assumption 1. The matrix M is invertible.

The above assumption is not restrictive since we can always find $\gamma_i,\ i \in \mathcal{N}$, such that it is satisfied.

Theorem 3. *Assume that the discrete-time stochastic Lyapunov equation (3) has a unique solution $P^* = (P_1^*, P_2^*, \ldots, P_N^*) \in \mathcal{R}_+^{n\times n}$. Then for any initial condition $P(0) = (P_1(0), P_2(0), \ldots, P_N(0)) \in \mathcal{C}^{n\times n}$, the iteration in (7) converges to P^* if and only if $\rho(T) < 1$. Moreover, the asymptotic exponential convergence rate of iteration (7) is $-\ln \rho(T)$.*

Proof. Taking $\mathrm{vec}(\cdot)$ on both sides of (7) and using the fact that $M_i,\ i \in \mathcal{N}$, are nonsingular, we get

$$p_i(k+1) = -M_i^{-1} \sum_{j=1, j\neq i}^{N} \pi_{ij} \left(A_{0i}^{\mathrm{T}} \otimes A_{0i}\right) p_j(k) - \gamma_i M_i^{-1} p_i - M_i^{-1} \sum_{s=1}^{r} \sum_{j=1}^{N} \pi_{ij} \left(A_{si}^{\mathrm{T}} \otimes A_{si}\right) p(k) - M_i^{-1} q_i,$$

where $p_i(k) = \mathrm{vec}(P_i(k))$ and $q_i = \mathrm{vec}(Q_i)$. The above iterations can be written in the following compact form

$$p(k+1) = M^{-1}Wp(k) + M^{-1}q = Tp(k) + M^{-1}q, \tag{16}$$

by denoting $p(k) = \varphi(P(k))$ and $q = \varphi(Q)$. Clearly, the above iteration converges if and only if $\rho(T) < 1$. Moreover, if it converges to a constant denoted by $P^{\dagger}$, then $P^{\dagger}$ must satisfy Eq. (3). The proof is finished by our assumption that Eq. (3) has a unique solution. □

In [22], the authors considered the following matrix equation

$$A_i^{\mathrm{T}}\left(\sum_{j=1}^{N}\pi_{ij}P_j\right)A_i - P_i + S_i = 0, \quad i \in \mathcal{N}, \tag{17}$$

where $S_i \geqslant 0,\ i \in \mathcal{N}$, are given matrices, associated with the discrete-time Markovian jump linear system

$$x(k+1) = A(\theta(k))x(k), \tag{18}$$

where $\{\theta(k), k \geqslant 0\}$ is a discrete-time homogeneous Markovian chain as defined for system (2) and $A_i = A(\theta(k) = i)$. The following implicit iteration was proposed there to obtain numerical solution of Eq. (17):

$$\pi_{ii}A_i^{\mathrm{T}}P_i(k+1)A_i - P_i(k+1) + A_i^{\mathrm{T}}\left(\sum_{j=1, j\neq i}^{N}\pi_{ij}P_j(k)\right)A_i + S_i = 0, \quad i \in \mathcal{N}. \tag{19}$$

We notice that the above iteration is a special case of (7) by setting $s = 0$ and $\gamma_i = 0,\ i \in \mathcal{N}$. Then based on Theorem 2, the following corollary can be obtained immediately regarding the convergence of iteration (19).

Corollary 1. *If the discrete-time Markovian jump linear system (18) is mean square stable, then the iteration in (19) converges to the unique solution $P^* = (P_1^*, P_2^*, \ldots, P_N^*) \in \mathcal{R}_+^{n\times n}$ of Eq. (17) for any initial condition $P(0) = (P_1(0), P_2(0), \ldots, P_N(0)) \in \mathcal{C}^{n\times n}$.*

To judge the convergence of iteration (19), the main results in [22] (namely, Theorem 1) which can be regarded as a special case of Theorem 3 relies on eigenvalues of an auxiliary matrix of dimensions n^2N. Different from that result, our result in Corollary 1 is very neat and simple.

4. Continuous-time Lyapunov equation

In this section we consider the Itô differential equation with Markovian jumps,

$$\mathrm{d}x(t) = A_0(\theta(t))x(t)\mathrm{d}t + \sum_{k=1}^{r}A_k(\theta(t))x(t)\mathrm{d}\omega_k(t), \tag{20}$$

where $x(t) \in \mathbf{R}^n$ is the system state, $\omega_k(t) \in \mathbf{R},\ k \in \{1, r\}$ are sequences of real random variables defined on a complete probability space $\{\Omega, \mathcal{F}, \mu\}$ and are independent wide sense stationary, second-order processes with $\mathbf{E}\{\omega_i(t)\} = 0$ and $\mathbf{E}\{\omega_i(t)\omega_j(s)\} = \delta_{ij},\ i,j \in \{1, r\}$, and $\{\theta(t), t \geqslant 0\}$ is a continuous-time discrete-state Markovian process taking value in finite set $\mathcal{N}$ with transition probability rate matrix $\Pi = [\pi_{ij}]_{N\times N}$ satisfying

$$\begin{cases} \pi_{ij} \geqslant 0, & i \neq j, \\ \pi_{ij} < 0, & i = j, \\ \sum_{j=1}^{N}\pi_{ij} = 0. \end{cases}$$

Let the initial condition for the system (20) be $x(0) = x_0$ and $\theta(0) = \theta_0$.

Definition 4. The continuous-time Markovian jump Itô stochastic system (20) is asymptotically mean square stable (AMSS) if for any $x_0 \in \mathbf{R}^n$ and $\theta_0 \in \mathcal{N}$, there holds

$$\lim_{t\to\infty}\mathbf{E}\{\|x(t)\|^2\} = 0,$$

where $x(t) = x(t, x_0, \theta_0)$ is a sample solution of the system.

It is well known that the mean square stability of the continuous-time Markovian jump Itô stochastic system (20) is closely related with the following continuous-time stochastic Lyapunov equation

$$A_{0i}^{\mathrm{T}}P_i + P_iA_{0i} + \sum_{k=1}^{r}A_{ki}^{\mathrm{T}}P_iA_{ki} + \sum_{j=1}^{N}\pi_{ij}P_j = -Q_i, \quad i \in \mathcal{N}, \tag{21}$$

as the following lemma shows.

Lemma 3 [27]. *The continuous-time Markovian jump Itô stochastic system (20) is mean square stable if and only if the continuous-time stochastic Lyapunov equation (21) has a unique solution* $P = (P_1, P_2, \ldots, P_N) \in \mathcal{R}_+^{n\times n}$ *for any given* $Q = (Q_1, Q_2, \ldots, Q_N) \in \mathcal{R}_+^{n\times n}$.

Lemma 3 presents an elegant characterization for the mean square stability of the continuous-time Markovian jump Itô stochastic system (20) via solutions of the stochastic Lyapunov equation (21). However, similar to the discrete-time case, solving such equation is not an easy task, especially, if the dimensions of the system and/or the number of r and N are large enough. In this section, we will present iterative algorithm to solve this class of equations.

4.1. Direct iteration based on transformation

To present our first result, we need an auxiliary result. Let $0 < \alpha_i \notin \sigma\left(A_{0i} + \frac{\pi_{ii}}{2} I_n\right),\ \forall i \in \mathcal{N}$, be a set of scalars and

$$\mathcal{A}_{0i} = \left(\alpha_i I_n + A_{0i} + \frac{\pi_{ii}}{2} I_n\right) T_i, \quad \mathcal{A}_{ki} = \sqrt{2\alpha_i} A_{ki} T_i, \quad \mathcal{B}_i = \sqrt{2\alpha_i} T_i,$$

where

$$T_i = \left(\alpha_i I_n - \frac{\pi_{ii}}{2} I_n - A_{0i}\right)^{-1}, \quad \forall i \in \mathcal{N}.$$

Define operator $\mathcal{H} : \mathcal{R}^{n\times n} \to \mathcal{R}^{n\times n}$ as

$$\begin{aligned} &\mathcal{H}(P) = (\mathcal{H}_1(P), \mathcal{H}_2(P), \ldots, \mathcal{H}_N(P)), \\ &\mathcal{H}_i(P) = \sum_{s=0}^{r} \mathcal{A}_{si}^{\mathrm{T}} P_i \mathcal{A}_{si} + \mathcal{B}_i^{\mathrm{T}} \left(\sum_{j=1, j\neq i}^{N} \pi_{ij} P_j\right) \mathcal{B}_i, \quad \forall i \in \mathcal{N}, \end{aligned} \tag{22}$$

where $P = (P_1, P_2, \ldots, P_N)$.

Lemma 4. $P = (P_1, P_2, \ldots, P_N) \in \mathcal{R}_+^{n\times n}$ *is a solution of Eq. (21) if and only if it is the solution of equation*

$$P = \mathcal{H}(P) + \mathcal{Q}, \tag{23}$$

where

$$\mathcal{Q} = \left(\mathcal{B}_1^{\mathrm{T}} Q_1 \mathcal{B}_1, \mathcal{B}_2^{\mathrm{T}} Q_2 \mathcal{B}_2, \ldots, \mathcal{B}_N^{\mathrm{T}} Q_N \mathcal{B}_N\right). \tag{24}$$

Proof. Set

$$C_{0i} = A_{0i} + \frac{\pi_{ii}}{2} I_n, \quad Q_i' = \sum_{j=1, j\neq i}^{N} \pi_{ij} P_j + Q_i.$$

Let P be a solution of (21). Then for any $i \in \mathcal{N}$,

$$\begin{aligned} (\alpha_i I_n - C_{0i})^{\mathrm{T}} P_i (\alpha_i I_n - C_{0i}) &= \alpha_i^2 P_i - \alpha_i C_{0i}^{\mathrm{T}} P_i - \alpha_i P_i C_{0i} + C_{0i}^{\mathrm{T}} P_i C_{0i} \\ &= \alpha_i^2 P_i + C_{0i}^{\mathrm{T}} P_i C_{0i} + \alpha_i P_i C_{0i} + \alpha_i C_{0i}^{\mathrm{T}} P_i - \alpha_i \left(P_i C_{0i} + C_{0i}^{\mathrm{T}} P_i\right) + \alpha_i \left(Q_i' + \sum_{s=1}^{r} A_{si}^{\mathrm{T}} P_i A_{si}\right) \\ &= (\alpha_i I_n + C_{0i})^{\mathrm{T}} P_i (\alpha_i I_n + C_{0i}) + 2\alpha_i \left(Q_i' + \sum_{s=1}^{r} A_{si}^{\mathrm{T}} P_i A_{si}\right). \end{aligned}$$

Since $\alpha_i \notin \sigma\left(A_{0i} + \frac{\pi_{ii}}{2} I_n\right),\ \forall i \in \mathcal{N}$, we get from the above equation that

$$\begin{aligned} P_i &= (\alpha_i I - C_{0i})^{-\mathrm{T}} (\alpha_i I + C_{0i})^{\mathrm{T}} P_i (\alpha_i I + C_{0i}) (\alpha_i I - C_{0i})^{-1} + 2\alpha_i (\alpha_i I - C_{0i})^{-\mathrm{T}} \left(Q_i' + \sum_{s=1}^{r} A_{si}^{\mathrm{T}} P_i A_{si}\right) (\alpha_i I - C_{0i})^{-1} \\ &= \mathcal{A}_{0i}^{\mathrm{T}} P_i \mathcal{A}_{0i} + \sum_{s=1}^{r} \mathcal{A}_{si}^{\mathrm{T}} P_i \mathcal{A}_{si} + \mathcal{B}_i^{\mathrm{T}} \left(\sum_{j=1, j\neq i}^{N} \pi_{ij} P_j\right) \mathcal{B}_i + \mathcal{B}_i^{\mathrm{T}} Q_i \mathcal{B}_i = \mathcal{H}_i(P) + \mathcal{B}_i^{\mathrm{T}} Q_i \mathcal{B}_i, \end{aligned}$$

which implies that P also satisfies (23). Since the above process is reversible, the proof is completed. □

Based on the above lemma, we can present the following result regarding iterative solution of Eq. (21).

Theorem 4. *Assume that the continuous-time Markovian jump Itô stochastic system (20) is mean square stable and* $P^* = (P_1^*, P_2^*, \ldots, P_N^*) \in \mathcal{R}_+^{n\times n}$ *is the unique solution of Eq. (21). Let*

$$P(k+1) = \mathcal{H}(P(k)) + \mathcal{Q}, \quad P(0) \in \mathcal{C}^{n\times n}, \tag{25}$$

where $\mathcal{H}$ and $\mathcal{Q}$ are defined in (22) and (24), respectively. Then $\{P(k)\}_{k=0}^{\infty}$ converges and $\lim_{k\to\infty} P(k) = P^*$.

Proof. If $P^* \in \mathcal{R}_+^{n\times n}$ is the solution of Eq. (21). Then it follows from Lemma 4 that P^* satisfies

$$\mathcal{H}(P^*) - P^* = -\mathcal{Q}.$$

Notice that $\mathcal{H}$ is a positive linear operator. Then it follows from Lemma 1 that

$$\rho(\mathcal{H}) < 1.$$

Then similar to the proof of Theorem 1, we can show that the iteration in (25) converges to the unique solution P^*. The proof is finished. □

Obviously, when $\alpha_i,\ i \in \mathcal{N}$, take different values, the asymptotic exponential convergence rate of iteration (25) is different. One may ask whether there exist $\alpha_i,\ i \in \mathcal{N}$, such that the asymptotic exponential convergence rate of the iteration is maximized. The following result answers this question partially.

Proposition 2. *There exists at least a set of $\alpha_i,\ i \in \mathcal{N}$, such that the asymptotic exponential convergence rate of the iteration in (25) is maximized.*

Proof. By taking $\varphi(\cdot)$ on both sides of (25), we obtain

$$p(k+1) = Hp(k) + \mathcal{B}q,$$

where $p(k) = \varphi(P(k)),\ q = \varphi(Q)$

$$H = \begin{bmatrix} \sum_{s=0}^{r} \mathcal{A}_{s1} \otimes \mathcal{A}_{s1} & \pi_{12}\mathcal{B}_1 \otimes \mathcal{B}_1 & \cdots & \pi_{1N}\mathcal{B}_1 \otimes \mathcal{B}_1 \\ \pi_{21}\mathcal{B}_2 \otimes \mathcal{B}_2 & \sum_{s=0}^{r} \mathcal{A}_{s2} \otimes \mathcal{A}_{s2} & \cdots & \pi_{2N}\mathcal{B}_2 \otimes \mathcal{B}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1}\mathcal{B}_N \otimes \mathcal{B}_N & \pi_{N2}\mathcal{B}_N \otimes \mathcal{B}_N & \cdots & \sum_{s=0}^{r} \mathcal{A}_{sN} \otimes \mathcal{A}_{sN} \end{bmatrix}, \tag{26}$$

and

$$\mathcal{B} = \mathrm{diag}(\mathcal{B}_1 \otimes \mathcal{B}_1, \mathcal{B}_2 \otimes \mathcal{B}_2, \ldots, \mathcal{B}_N \otimes \mathcal{B}_N).$$

Hence, clearly, the asymptotic exponential convergence rate of the iteration is $-\ln(\rho(H))$. Denote

$$\mathcal{E}_i(A) = \left(A_{0i} + \frac{\pi_{ii}}{2} I_n\right) \otimes I_n + I_n \otimes \left(A_{0i} + \frac{\pi_{ii}}{2} I_n\right) + \sum_{s=1}^{r} A_{si} \otimes A_{si}.$$

Then, for any $i \in \mathcal{N}$, we can compute

$$\begin{aligned}
\sum_{s=0}^{r} \mathcal{A}_{si} \otimes \mathcal{A}_{si} &= \mathcal{A}_{0i} \otimes \mathcal{A}_{0i} + \sum_{k=1}^{r} \mathcal{A}_{ki} \otimes \mathcal{A}_{ki} = (\alpha_i I_n + C_{0i})(\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n + C_{0i})(\alpha_i I_n - C_{0i})^{-1} \\
&\quad + \sum_{k=1}^{r} \sqrt{2\alpha_i} A_{ki} (\alpha_0 I_n - C_{0i})^{-1} \otimes \sqrt{2\alpha_i} A_{ki} (\alpha_0 I_n - C_{0i})^{-1} \\
&= \left((\alpha_i I_n + C_{0i}) \otimes (\alpha_i I_n + C_{0i}) + 2\alpha_i \sum_{s=1}^{r} A_{si} \otimes A_{si} \right) (\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1} \\
&= ((\alpha_i I_n + C_{0i}) \otimes (\alpha_i I_n + C_{0i}) + 2\alpha_i(\mathcal{E}_i(A) - C_{0i} \otimes I_n - I_n \otimes C_{0i})) \cdot (\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1} \\
&= \left(\alpha_i^2 I_{n^2} - \alpha_i C_{0i} \otimes I_n - \alpha_i I_n \otimes C_{0i} + C_{0i} \otimes C_{0i} + 2\alpha_i \mathcal{E}_i(A)\right) \cdot (\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1} \\
&= ((\alpha_i I_n - C_{0i}) \otimes (\alpha_i I_n - C_{0i}) + 2\alpha_i \mathcal{E}_i(A))(\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1} \\
&= I_{n^2} + 2\alpha_i \mathcal{E}_i(A)(\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1}.
\end{aligned} \tag{27}$$

With this, we clearly see that

$$\lim_{\alpha_i \to 0^+} \sum_{s=0}^{r} \mathcal{A}_{si} \otimes \mathcal{A}_{si} = I_{n^2}, \quad \forall i \in \mathcal{N}. \tag{28}$$

On the other hand, we know that there exists a set of scalars $\beta_i \in (0, \infty),\ \forall i \in \mathcal{N}$, such that $\left\|\frac{1}{\alpha_i} C_{0i}\right\| < \frac{1}{2}$ is satisfied for all $\alpha_i > \beta_i,\ \forall i \in \mathcal{N}$. Since (see, for example, [31])

$$\left\|\left(I_n - \frac{C_{0i}}{\alpha_i}\right)^{-1}\right\| \leqslant \frac{1}{1 - \|\frac{1}{\alpha_i} C_{0i}\|}, \quad \forall i \in \mathcal{N},$$

we can compute

$$\begin{aligned} \|2\alpha_i \mathcal{E}_i(A)(\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1}\| &= \left\|\frac{2}{\alpha_i}\mathcal{E}_i(A)\left(I_n - \frac{C_{0i}}{\alpha_i}\right)^{-1} \otimes \left(I_n - \frac{C_{0i}}{\alpha_i}\right)^{-1}\right\| \leqslant \frac{2}{\alpha_i}\|\mathcal{E}_i(A)\|\left\|\left(I_n - \frac{C_{0i}}{\alpha_i}\right)^{-1}\right\|^2 \\ &\leqslant \frac{2}{\alpha_i}\|\mathcal{E}_i(A)\|\left(\frac{1}{1 - \|\frac{1}{\alpha_i} C_{0i}\|}\right)^2 \leqslant \frac{8}{\alpha_i}\|\mathcal{E}_i(A)\|, \quad \forall i \in \mathcal{N}. \end{aligned}$$

Therefore, we get

$$\lim_{\alpha_i \to \infty} 2\alpha_i \mathcal{E}_i(A)(\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1} = 0, \quad \forall i \in \mathcal{N}.$$

Hence, in view of (27), we conclude that

$$\lim_{\alpha_i \to \infty} \sum_{s=0}^{r} \mathcal{A}_{si} \otimes \mathcal{A}_{si} = I_{n^2}, \quad \forall i \in \mathcal{N}. \tag{29}$$

On the other hand, we can obtain

$$\mathcal{B}_i \otimes \mathcal{B}_i = \sqrt{2\alpha_i}(\alpha_i I_n - C_{0i})^{-1} \otimes \sqrt{2\alpha_i}(\alpha_i I_n - C_{0i})^{-1} = 2\alpha_i(\alpha_i I_n - C_{0i})^{-1} \otimes (\alpha_i I_n - C_{0i})^{-1}, \quad \forall i \in \mathcal{N}.$$

Then similar to (28) and (29), we are able to show

$$\lim_{\alpha_i \to 0^+} \mathcal{B}_i \otimes \mathcal{B}_i = \lim_{\alpha_i \to \infty} \mathcal{B}_i \otimes \mathcal{B}_i = 0, \quad \forall i \in \mathcal{N}. \tag{30}$$

Combining (28)–(30) gives

$$\lim_{\alpha_i \to 0^+} H = \lim_{\alpha_i \to \infty} H = I_{n^2 N}, \quad \forall i \in \mathcal{N}.$$

Since $\rho(H) < 1$ is true for all $\alpha_i > 0,\ \forall i \in \mathcal{N}$, there exists at least one $\alpha_i^* > 0,\ \forall i \in \mathcal{N}$, such that $\rho(H)$ is minimized, namely, the asymptotic exponential convergence rates of iteration (25) is maximized. The proof is finished. □

However, how to compute the optimal scalars $\alpha_i^*,\ i \in \mathcal{N}$, is not clear at present and requires a further study.

4.2. Implicit iteration

In the above subsection, an iterative algorithm is proposed by using an auxiliary transformation, which may require additional computation. In this subsection, similar to the discrete-time case, we propose an alterative iterative algorithm without transformation.

We begin our development by noting that the stochastic Lyapunov equation (21) can be written as

$$\mathcal{A}_i^{\mathrm{T}} P_i + P_i \mathcal{A}_i = -\sum_{s=1}^{r} A_{si}^{\mathrm{T}} P_i A_{si} - \sum_{j=1, j\neq i}^{N} \pi_{ij} P_j - \beta_i P_i - Q_i, \quad i \in \mathcal{N},$$

where

$$\mathcal{A}_i = A_{0i} + \frac{\pi_{ii}}{2} I_n - \frac{\beta_i}{2} I_n.$$

Based on this presentation, we can construct the implicit iteration:

$$\mathcal{A}_i^{\mathrm{T}} P_i(k+1) + P_i(k+1)\mathcal{A}_i = -\sum_{s=1}^{r} A_{si}^{\mathrm{T}} P_i(k) A_{si} - \sum_{j=1, j\neq i}^{N} \pi_{ij} P_j(k) - \beta_i P_i(k) - Q_i, \quad i \in \mathcal{N}. \tag{31}$$

Notice that at every step of the iteration, N standard continuous-time Lyapunov equations in the form of

$$A^{\mathrm{T}} X + XA = -Z, \quad Z > 0 \tag{32}$$

should be solved. Fortunately, there are many numerically stable algorithms for solving such equations. One of the most effective methods is the well-known Hessenberg-Schur form based approach ([9]) which has been imbedded in the Matlab function `lyap`.

Regarding the convergence of the algorithm (31), we have the following result.

Theorem 5. *If the continuous-time Markovian jump Itô stochastic system (20) is mean square stable, then for any $\beta_i \geqslant 0,\ i \in \mathcal{N}$, the iteration in (31) converges to the unique solution $P^* = (P_1^*, P_2^*, \ldots, P_N^*) \in \mathcal{R}_+^{n\times n}$ of the continuous-time stochastic Lyapunov matrix equation (21) for any initial condition $P(0) = (P_1(0), P_2(0), \ldots, P_N(0)) \in \mathcal{C}^{n\times n}$.*

Proof. Define the operator $\mathcal{R}$ as

$$\mathcal{R}(P) = (\mathcal{R}_1(P), \mathcal{R}_2(P), \ldots, \mathcal{R}_N(P)),$$
$$\mathcal{R}_i(P) = \mathcal{A}_i^{\mathrm{T}} P_i + P_i \mathcal{A}_i, \quad i \in \mathcal{N},$$

and the operator $\mathcal{S}$ as

$$\mathcal{S}(P) = (\mathcal{S}_1(P), \mathcal{S}_2(P), \ldots, \mathcal{S}_N(P)),$$
$$\mathcal{S}_i(P) = \sum_{s=1}^{r} A_{si}^{\mathrm{T}} P_i A_{si} + \sum_{j=1, j\neq i}^{N} \pi_{ij} P_j + \beta_i P_i, \quad i \in \mathcal{N},$$

where $P = (P, P, \ldots, P_N)$. We notice that $\mathcal{S}(P)$ is a positive operator. Then the rest of the proof is similar to the proof of Theorem 2 and is hence omitted for brevity. $\square$

The following two remarks for iteration (31) parallel those in Remarks 3 and 4.

Remark 5. Under the condition of mean square stability of the continuous-time Markovian jump Itô stochastic system (20), numerical experiences show that the smaller the values of $\beta_i,\ i \in \mathcal{N}$, the faster the iteration in (31) converges. Hence, similar to the discrete-time case, we can choose $\beta_i = 0,\ i \in \mathcal{N}$.

Remark 6. Similar to the discrete-time case, the mean square stability of system (20) is only a sufficient condition for the convergence of the iteration in (31). An illustrative example can be constructed as follows: Let $N = 1,\ n = 1$ and $r = 1$. Then Eq. (21) becomes

$$A_0^{\mathrm{T}} P + P A_0 = -A_1^{\mathrm{T}} P A_1 - Q,$$

where $A_1 \neq 0$. The iteration in (31) can then be written as

$$P(k+1) = \frac{-A_1^2 - \beta}{2A_0 - \beta} P(k) - \frac{1}{2A_0 - \beta} Q, \quad \beta \geqslant 0,$$

which converges if and only if

$$\rho\left(\frac{-A_1^2 - \beta}{2A_0 - \beta}\right) = \left|\frac{A_1^2 + \beta}{2A_0 - \beta}\right| < 1. \tag{33}$$

On the other hand, the associated continuous-time Markovian jump Itô stochastic system (20) is mean square stable if and only if

$$2A_0 + A_1^2 < 0. \tag{34}$$

Inequality (34) implies (33) but the converse is not true. For example, if we choose $\beta = 0$ and $A_0 = A_1^2$, then (33) is satisfied but (34) is not satisfied.

Based on Remark 6, we can also generalize Theorem 3 to iteration (31) by providing necessary and sufficient condition for its convergence. Similarly, we let

$$E = -\mathrm{diag}\{E_1, E_2, \ldots, E_N\},$$

with $E_i = I_n \otimes A_{0i} + A_{0i} \otimes I_n + (\pi_{ii} - \beta_i) I_{n^2}$, and denote

$$G = E^{-1} V, \tag{35}$$

where

$$V = \begin{bmatrix} V_{11} & \pi_{12} I_{n^2} & \cdots & \pi_{1N} I_{n^2} \\ \pi_{21} I_{n^2} & V_{22} & \cdots & \pi_{2N} I_{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1} I_{n^2} & \cdots & \pi_{N,N-1} I_{n^2} & V_{NN} \end{bmatrix},$$

with $V_{ii} = \sum_{k=1}^{r} A_{ki} \otimes A_{ki} + \beta_i I_{n^2}, \; i \in \mathcal{N}$. Similar to the discrete-time case, we should assume that $I_{n^2} \otimes \mathcal{A}_i + \mathcal{A}_i \otimes I_{n^2} = E$ is non-singular. The above assumption is again not restrictive since we can always find $\beta_i, \; i \in \mathcal{N}$, such that it is satisfied. We have the following theorem, its proof is similar to that of Theorem 3 and hence omitted.

Theorem 6. *Assume that the continuous-time Markovian jump Itô stochastic system (21) has a unique solution* $P^* = (P_1^*, P_2^*, \ldots, P_N^*) \in \mathcal{R}_+^{n\times n}$. *Then for any initial condition* $P(0) = (P_1(0), P_2(0), \ldots, P_N(0)) \in \mathcal{C}^{n\times n}$, *the iteration in (31) converges to* P^* *if and only if* $\rho(G) < 1$. *Moreover, the asymptotic exponential convergence rate of iteration (31) is* $-\ln \rho(G)$.

5. Numerical Examples

In this section, we work out two numerical examples to validate the effectiveness of the proposed algorithms.

Example 1. Consider the discrete-time Lyapunov equation (3) with $N = 2, \; r = 1, \; Q_1 = Q_2 = I_4$, and

$$\Pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{bmatrix}.$$

The coefficient matrices $A_{ij}, \; i = 0, 1, \; j = 1, 2$ are randomly generated by Matlab:

$$A_{01} = \begin{bmatrix} -0.2854 & -0.0409 & 0.3464 & 0.00320 \\ -0.4621 & 0.3521 & 0.09170 & -0.0220 \\ -0.2135 & -0.3569 & 0.4254 & -0.4799 \\ -0.0975 & 0.4726 & -0.4324 & 0.0353 \end{bmatrix},$$

$$A_{02} = \begin{bmatrix} -0.0240 & -0.0354 & 0.2295 & -0.0041 \\ 0.1039 & 0.2905 & 0.1248 & -0.3693 \\ -0.3534 & 0.2842 & -0.4000 & -0.2181 \\ 0.0337 & -0.0222 & -0.2950 & -0.1377 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 0.2663 & 0.2283 & -0.1163 & 0.2558 \\ -0.1242 & -0.2103 & -0.0799 & -0.0284 \\ -0.1655 & 0.0055 & 0.2577 & 0.1995 \\ -0.0447 & -0.0318 & 0.1742 & -0.2439 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -0.4347 & 0.4695 & 0.0237 & 0.3432 \\ -0.1249 & -0.1579 & -0.3366 & 0.3062 \\ -0.1265 & -0.2473 & -0.0136 & 0.3578 \\ -0.0160 & 0.0849 & -0.0039 & 0.1098 \end{bmatrix}.$$

We first use the implicit iteration (7) to produce approximate solutions of Eq. (3) with different values of $\gamma_i, \; i = 1, 2$. The results are recorded in Fig. 1 where the y-axis corresponds to the iteration error $\log(\Delta(k))$ with

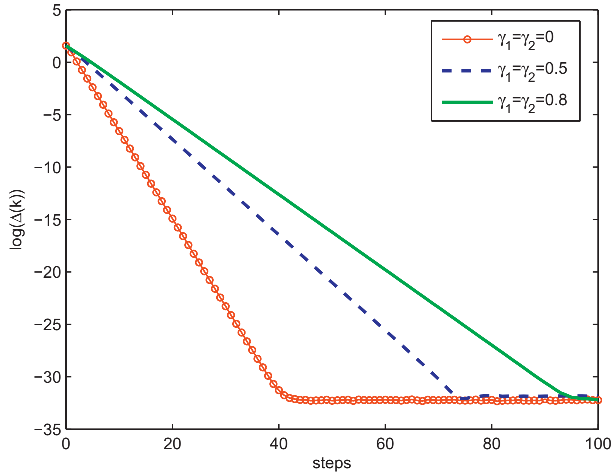

Fig. 1. Convergence performance of implicit iteration (7) for different values of $\gamma_i, \; i \in \mathcal{N}$.

$$\Delta(k)=\sqrt{\sum_{i=0}^{N}\left\|\sum_{s=0}^{r}A_{si}^{T}\left(\sum_{j=1}^{N}\pi_{ij}P_j(k)\right)A_{si}-P_i(k)+Q_i\right\|_F^2}.$$

From the figure we see that different values of $\gamma_i,\ i=1,2$, lead to different convergence rates of the iteration. Especially, we see that the convergence rate of the iteration is maximized if $\gamma_1=\gamma_2=0$. Our observation has been confirmed by a couple of numerical examples that are not recorded in this paper. For this reason, we can simply choose $\gamma_i=0$ in iteration (7) as Remark 2 states. For $\gamma_1=\gamma_2=0$, the iteration error is $\|\Delta(40)\|=4.2620\times10^{-15}$ and the resulting approximate solution is given by

$$P_1(40)=\begin{bmatrix}1.7520 & -0.0374 & -0.3924 & 0.2664\\ -0.0374 & 2.2980 & -0.8302 & 0.6242\\ -0.3924 & -0.8302 & 2.2450 & -0.4910\\ 0.2664 & 0.6242 & -0.4910 & 1.8180\end{bmatrix},$$
$$P_2(40)=\begin{bmatrix}1.6970 & -0.4055 & 0.3677 & -0.3606\\ -0.4055 & 1.8790 & -0.0550 & -0.1827\\ 0.3677 & -0.0550 & 1.8370 & -0.0578\\ -0.3606 & -0.1827 & -0.0578 & 1.8440\end{bmatrix}.$$

Next we compare our proposed algorithms (4) and (7) to the gradient based iterative algorithm presented in [28]. Here we choose $\gamma_i=0,\ i=1,2$ in iteration (7) and the step size in the gradient based iteration of [28] is chosen as the optimal one. The results are recorded in Fig. 2. We can observe the following two facts from the figure: 1. The implicit iteration in (7) converges faster than the direct iteration in (4) and the gradient based iteration in [28]; 2. The accuracy that the implicit iteration in (7) can achieve is better than that of the direct iteration in (4). This is because the accuracy of the implicit iteration relies on the accuracy of solutions of the normal discrete-time Lyapunov equation in the form of (8). Based on these two facts, we suggest to use the implicit iteration in (7) if the requirement on accuracy is not so demanding. Otherwise, we should use the direct iteration in (4).

Example 2. Consider a continuous-time Lyapunov equation in the form of (21) with $N=2,\ r=2,\ Q_1=Q_2=I_4$, and

$$\Pi=\begin{bmatrix}-0.6 & 0.6\\ 1 & -1\end{bmatrix}.$$

The matrices A_{01} and A_{02} are chosen as Hurwitz stable matrices, namely,

$$A_{01}=\begin{bmatrix}-1.0000 & -1.0000 & -2.0000 & 1.0000\\ -0.6667 & -3.5000 & 1.0000 & 1.1670\\ 1.0000 & 0.5000 & -3.0000 & 0.5000\\ -2.0000 & -2.5000 & 1.0000 & -2.5000\end{bmatrix},$$

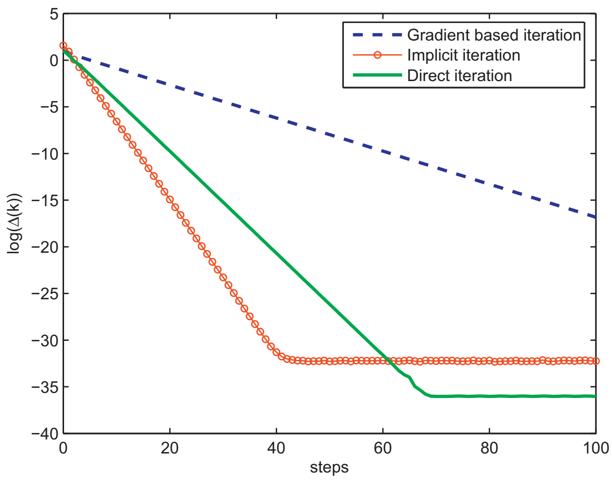

Fig. 2. Comparison of convergence rates for different iterative algorithms for discrete-time stochastic Lyapunov equation (3).

$$A_{02} = \begin{bmatrix} -1.3333 & 1.0000 & -2.0000 & 0.3333 \\ 0.0000 & -3.5000 & 1.0000 & 1.5000 \\ 1.0000 & 1.5000 & -4.0000 & -0.5000 \\ -1.3333 & -3.5000 & 1.0000 & -1.1667 \end{bmatrix},$$

while the matrices A_{11} and A_{12} are generated randomly by Matlab:

$$A_{11} = \begin{bmatrix} 0.9003 & 0.7826 & 0.6428 & 0.8436 \\ -0.5377 & 0.5242 & -0.1106 & 0.4764 \\ 0.2137 & -0.0871 & 0.2309 & -0.6475 \\ -0.0280 & -0.9630 & 0.5839 & -0.1886 \end{bmatrix},$$
$$A_{12} = \begin{bmatrix} 0.8709 & -0.8842 & -0.7222 & -0.4556 \\ 0.8338 & -0.2943 & -0.5945 & -0.6024 \\ -0.1795 & 0.6263 & -0.6026 & -0.9695 \\ 0.7873 & -0.9803 & 0.2076 & 0.4936 \end{bmatrix}.$$

We first study the convergence of the transformation based direct iteration in (25) for different values of $\alpha_i,\ i \in \mathcal{N}$. The spectral radius of the matrix H defined in (26) as a function of α_1 and α_2 is plotted in Fig. 3. Clearly, we see that there exist α_1 and α_2 such that $\rho(H)$ is minimized, namely, the convergence rate of iteration (25) is maximized. The optimal values are approximately equal to $\alpha_1 = 2.7$ and $\alpha_2 = 3.0$. This statement is further approved by Fig. 4 in which the convergence performances of the iteration for different values of $\alpha_i,\ i = 1, 2$ are recorded. Here the y-axis is $\log(\Delta(k))$ with

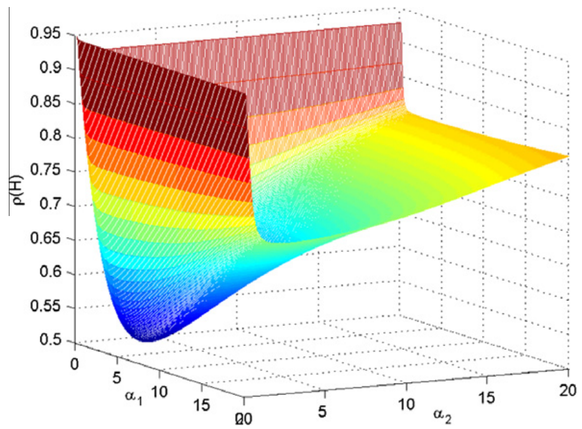

Fig. 3. Spectral radius of matrix H defined in (26) for different values of $\alpha_i,\ i \in \mathcal{N}$.

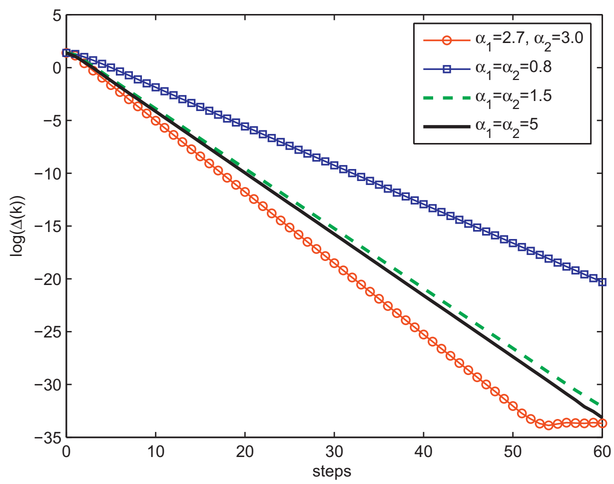

Fig. 4. Convergence performances of direct iteration (25) for different values of $\alpha_i,\ i \in \mathcal{N}$.

$$\Delta(k) = \sqrt{\sum_{i=1}^{N} \left\| A_{0i}^{T} P_i(k) + P_i(k) A_{0i} + \sum_{k=1}^{r} A_{ki}^{T} P_i(k) A_{ki} + \sum_{j=1}^{N} \pi_{ij} P_j(k) + Q_i \right\|_F^2}.$$

Moreover, when $\alpha_i, \ i = 1, 2$ are taken as the optimal values and the initial conditions are chosen $P_1(0) = P_2(0) = 0$, we can obtain the approximate solution as

$$P_1(50) = \begin{bmatrix} 0.6062 & -0.1174 & -0.1992 & 0.0301 \\ -0.1174 & 0.3156 & 0.0937 & -0.0175 \\ -0.1992 & 0.0937 & 0.4302 & 0.1149 \\ 0.0301 & -0.0175 & 0.1149 & 0.3843 \end{bmatrix},$$

$$P_2(50) = \begin{bmatrix} 0.5396 & -0.0148 & -0.2386 & -0.0689 \\ -0.0148 & 0.4183 & 0.0473 & -0.1514 \\ -0.2386 & 0.0473 & 0.3474 & 0.1163 \\ -0.0689 & -0.1514 & 0.1163 & 0.3898 \end{bmatrix}.$$

The iteration error is $\|\Delta(50)\| = 4.3034 \times 10^{-15}$.

We next study the implicit iteration in (31). For different values of β_1 and β_2, the convergence rates of the iteration are recorded in Fig. 5. Similar to the discrete-time case, from the figure we can conclude that the smaller the values of β_i, the faster the iteration will converge.

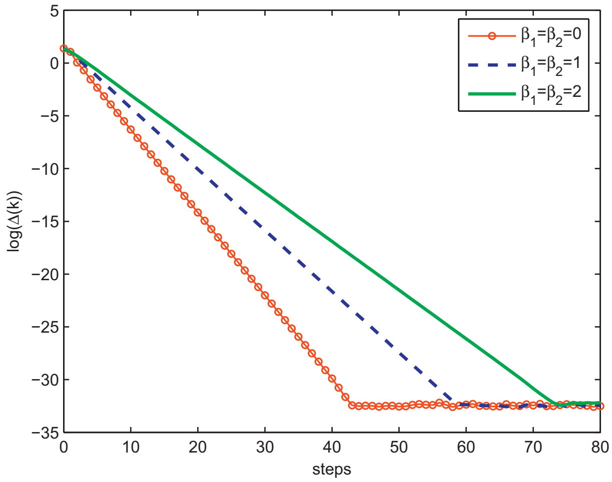

Fig. 5. Convergence rates of implicit iteration (31) for different values of $\beta_i, \ i \in \mathcal{N}$.

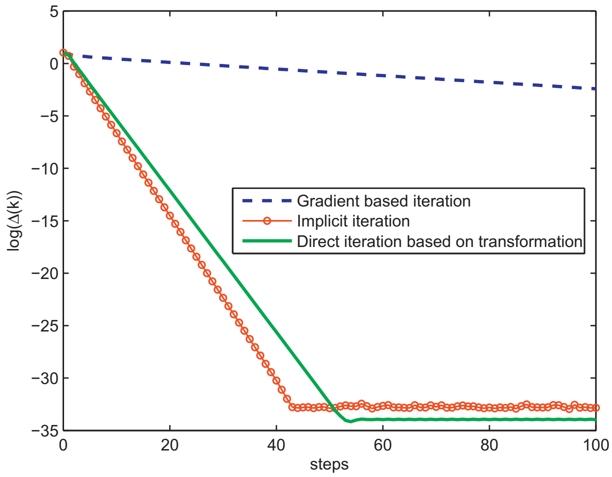

Fig. 6. Comparison of convergence rates for different algorithms for stochastic Lyapunov equation (21).

Finally, we compare the proposed direct iteration in (25) and the implicit iteration in (31) to the gradient based iteration given in [28]. Again, we see from Fig. 6 that the implicit iteration can lead to the fastest converge rate while the transformation based direct iteration can lead to more accurate approximate solutions.

6. Conclusion

This paper has considered the numerical solution of Lyapunov equations for Itô stochastic systems with Markovian jump which includes the Markovian jump system as a special case by using positive operator theory. Two iterative algorithms are established for Lyapunov equations in both discrete-time and continuous-time cases. If the associated stochastic system is mean square stable, it is shown by using positive operator theory that these algorithms converge to the unique solution of the Lyapunov equation. The proposed algorithms are very efficient in obtaining numerical solutions and may take important functions in related control problems. Some numerical examples are given to validate the effectiveness of the proposed methods.

References

[1] I. Borno, Z. Gajic, Parallel algorithm for solving coupled algebraic Lyapunov equations of discrete-time jump linear systems, Computers and Mathematics with Applications 30 (7) (1995) 1–4.
[2] M. Dehghan, M. Hajarian, An iterative algorithm for the reflexive solutions of the generalized coupled Sylvester matrix equations and its optimal approximation, Applied Mathematics and Computation 202 (2, 15) (2008) 571–588.
[3] M. Dehghan, M. Hajarian, The general coupled matrix equations over generalized bisymmetric matrices, Linear Algebra and its Applications 432 (6, 1) (2010) 1531–1552.
[4] F. Ding, P.X. Liu, J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, Applied Mathematics and Computation 197 (1) (2008) 41–50.
[5] F. Ding, P.X. Liu, Y. Shi, Convergence analysis of estimation algorithms for dual-rate stochastic systems, Applied Mathematics and Computation 176 (1) (2006) 245–261.
[6] F. Ding, T. Chen, Iterative least-squares solutions of coupled Sylvester matrix equations, Systems & Control Letters 54 (2) (2005) 95–107.
[7] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Transactions on Automatic Control 50 (8) (2005) 1216–1221.
[8] H. Gao, J. Lam, S. Xu, C. Wang, Stabilization and $\mathcal{H}_\infty$ control of two-dimensional Markovian jump systems, IMA Journal of Mathematical Control and Information 21 (4) (2004) 377–392.
[9] G.H. Golub, S. Nash, C.F. Van Loan, A Hessenberg–Schur method for the matrix problem $AX + XB = C$, IEEE Transactions on Automatic Control 24 (6) (1979) 909–913.
[10] S.J. Hammarling, Numerical solution of the stable, non-negative definite Lyapunov equation, IMA Journal of Numerical Analysis 2 (1982) 303–323.
[11] Y. Ji, H.J. Chizeck, Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, IEEE Transactions on Automatic Control 35 (7) (1990) 777–788.
[12] Z. Jia, M. Wei, Solvability and sensitivity analysis of polynomial matrix equation $X^s + A^TX^tA = Q$, Applied Mathematics and Computation 209 (2, 15) (2009) 230–237.
[13] Z. Li, Y. Wang, B. Zhou, G. Duan, On unified concepts of detectability and observability for continuous-time stochastic systems, Applied Mathematics and Computation 217 (2, 15) (2010) 521–536.
[14] Z. Li, Y. Wang, B. Zhou, G. Duan, Least squares solution with the minimum-norm to general matrix equations via iteration, Applied Mathematics and Computation 215 (10, 15) (2010) 3547–3562.
[15] J. Li, X. Hu, L. Zhang, The submatrix constraint problem of matrix equation $AXB + CYD = E$, Applied Mathematics and Computation 215 (7, 1) (2009) 2578–2590.
[16] A. Liao, G. Yao, X. Duan, Thompson metric method for solving a class of nonlinear matrix equation, Applied Mathematics and Computation 216 (6, 15) (2010) 1831–1836.
[17] H. Schneider, Positive operators and an inertia theorem, Numerische Mathematik 7 (1965) 11–17.
[18] R. Schmid, T. Ying, Performance analysis of iterative algorithms for Sylvester equations, in: 2010 Eigth IEEE International Conference on Control and Automation 9–11, June, 2010, pp. 1848–1852.
[19] P. Shi, E.K. Boukas, Y. Shi, On stochastic stabilization of discrete-time Markovian jump systems with delay in state, Stochastic Analysis and Applications 21 (4) (2003) 935–951.
[20] C.E. de Souza, M.D. Fragoso, $\mathcal{H}_\infty$ filtering for discrete-time linear systems with Markovian jumping parameters, International Journal of Robust and Nonlinear Control 13 (14) (2003) 1299–1316.
[21] Z. Wang, J. Lam, X. Liu, Robust filtering for discrete-time Markovian jump delay systems, IEEE Signal Processing Letters 11 (8) (2004) 659–662.
[22] Q. Wang, J. Lam, Y. Wei, T. Chen, Iterative solutions of coupled discrete Markovian jump Lyapunov equations, Computers and Mathematics with Applications 55 (4) (2008) 843–850.
[23] J. Xiong, J. Lam, Stabilization of discrete-time Markovian jump linear systems via time-delayed controllers, Automatica 42 (5) (2006) 747–753.
[24] S. Xu, T. Chen, J. Lam, Robust $\mathcal{H}_\infty$ filtering for uncertain Markovian jump systems with mode-dependent time delays, IEEE Transactions on Automatic Control 48 (5) (2003) 900–907.
[25] H. Yan, M.Q.H. Meng, H. Zhang, H. Shi, Robust $\mathcal{H}_\infty$ exponential filtering for uncertain stochastic time-delay systems with Markovian switching and nonlinearities, Applied Mathematics and Computation 215 (12, 15) (2010) 4358–4369.
[26] W. Zhang, Y. Huang, H. Zhang, Stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control for discrete-time systems with state and disturbance dependent noise, Automatica 43 (3) (2007) 513–521.
[27] W. Zhang, Y. Huang, L. Xie, Infinite horizon stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control for discrete-time systems with state and disturbance dependent noise, Automatica 44 (9) (2008) 2306–2316.
[28] B. Zhou, G.R. Duan, Z. Li, Gradient based iterative algorithm for solving coupled matrix equations, Systems & Control Letters 58 (2009) 327–333.
[29] B. Zhou, J. Lam, G.R. Duan, Convergence of gradient-based iterative solution of coupled Markovian jump Lyapunov equations, Computers and Mathematics with Applications 56 (2008) 3070–3078.
[30] B. Zhou, Z. Li, G. Duan, Y. Wang, Solutions to a family of matrix equations by using the Kronecker matrix polynomials, Applied Mathematics and Computation 212 (2 15) (2009) 327–336.
[31] R.A. Horn, Matrix Analysis, Cambridge University Press, 1985.
[32] L. Xie, Y.J. Liu, H.Z. Yang, Gradient based and least squares based iterative algorithms for matrix equations $AXB + CX^TD = F$, Applied Mathematics and Computation 217 (5) (2010) 2191–2199.

[33] Y.J. Liu, Y.S. Xiao, X.L. Zhao, Multi-innovation stochastic gradient algorithm for multiple-input single-output systems using the auxiliary model, Applied Mathematics and Computation 215 (4) (2009) 1477–1483.
[34] L.Y. Wang, L. Xie, X.F. Wang, The residual based interactive stochastic gradient algorithms for controlled moving average models, Applied Mathematics and Computation 211 (2) (2009) 442–449.
[35] J. Ding, Y.J. Liu, F. Ding, Iterative solutions to matrix equations of form $A_iXB_i = F_i$, Computers & Mathematics with Applications 59 (11) (2010) 3500–3507.
[36] L. Xie, J. Ding, F. Ding, Gradient based iterative solutions for general linear matrix equations, Computers & Mathematics with Applications 58 (7) (2009) 1441–1448.
[37] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM Journal on Control and Optimization 44 (6) (2006) 2269–2284.