Adaptive Variable Structure Maneuvering Control and Vibration Reduction of Three-axis Stabilized Flexible Spacecraft

Qinglei Hu* and Guangfu Ma
Department of Control Science and Engineering, Harbin Institute of Technology, Harbin, 150001, China

This paper proposes a robust control algorithm for stabilization of a three-axis flexible spacecraft in the presence of model uncertainty, external disturbances and control input nonlinearities. This control algorithm is based on variable structure output feedback control design technique that explicitly accounts for the control input nonlinearities in the stability analysis. Asymptotically and exponentially stable design methods are investigated for constructing the controller to stabilized uncertain system with input nonlinearities. Two kinds of the controller are presented that both ensures the global reaching condition of the sliding mode of the spacecraft dynamics system. Moreover, in the sliding mode, the dynamics system under study still bears the insensitivity to the uncertainties and disturbances as the system with linear input. An additional attractive feature of the control method is that the structure of controller is independent of the elastic mode dynamics of the spacecraft, since in practice the measurement of flexible modes is not easy or feasible. It is also shown that an adaptive version of the proposed controller is achieved through removing requirements of knowing the bounds of the uncertainties and perturbations in advance. Furthermore, a modified adaptation control law is given to improve the adaptive performances such that a new controller is designed which can guarantee the boundedness of the closed-loop system. Numerical simulations show that the precise attitude and vibration suppression can be accomplished using the derived controller for both cases with and without adaptive control.

Keywords: Flexible spacecraft; variable structure output feedback control; attitude control; vibration suppression; input nonlinearities

1. Introduction

Fine attitude control and stabilization are the key technologies of some modern flexible spacecraft, whose missions, such as stereoscopic mapping, requires high pointing accuracy and stabilization. However, some orbiting operations, such as attitude slewing or maneuvering, will introduce certain levels of vibration to flexible appendages, which will deteriorate its pointing performance. In addition, in the realistic environment, the knowledge about system parameters such as inertia matrix and modal frequencies is usually unknown, and various disturbances, including gravitational torque, aerodynamic torque, radiation torque, and other environmental and non-environmental torques, are also presented. Therefore, disturbance rejection control strategies that are also robust to parametric uncertainty and effectively suppress the induced vibration are of great interest in spacecraft applications.

Variable structure control (VSC) as discussed by Utkin [19] and as discussed elsewhere [12] is an effective approach to deal with uncertainties and disturbances for nonlinear systems due to its simplicity and effectiveness as well as its robustness. Over the last few years, VSC has been applied to the design of

*Correspondence to: Q. Hu; E-mail: huqinglei@hit.edu.cn

Received 12 November 2005; Accepted 24 November 2006
Recommended by S.M. Veres, D. Normand-Cyrot
attitude control system for spacecraft as discussed in Refs. [3,6,13,20]. However, the conventional VSC is always limited to the systems with full-state feedback. In practical application, full measurement of state might be neither possible nor feasible, such as the measure of the variables describing the flexible motion, the modal position, and velocity of the flexible spacecrafts. Even though some asymptotic observers and dynamic compensators have been used in variable structure system to deal with the unavailability of states as discussed in Ref. [5,8], they possibly increase the complexity of the system. Therefore, the direct output feedback design in VSC is worth investigating. Several authors have considered VSC in the static output feedback format as discussed in Refs. [7,9,21,23,25]. Linear systems using the variable structure output feedback control (VSOFC) have considered by Heck and Ferri [9]. Zak et al. [10] studied the use of the output feedback in variable structure control with uncertainties for a class of controllers. Yallapragada et al. [23] have considered the reaching condition design for the variable structure control with static output feedback. For variable structure control system, an important assumption is that the uncertainties are bounded and that their bounds are available to the designer. These bounds are an important clue to the possibility of guaranteed stability of a closed-loop. Occasionally, due to the complexity of the structure of uncertainties, their bounds may not be easily obtained. Adaptive control is also known as an effective and robust strategy with uncertain system as discussed in Refs. [10,18,22,24]. Using online identification, one can assure global stability for a class of systems with known structure but unknown parameters. The advantages of these controllers are that the information of the upper bound of perturbation is not required. However, the adaptive switching gains of controller will slowly increase boundlessly due to the fact that the state variable will not be exactly equal to zero in practice. Moreover, this study concentrated on the uncertain systems with full-state feedback.

Another important problem encountered in practice is that of control input nonlinearities. As it is well known, due to physical structure and energy consumption of the actuators, there do exist nonlinear characteristics such as saturation in the control inputs. This can lead to substantial performance deterioration and even to instability of the system. Hence, globally stable algorithms that take control input nonlinearities explicitly into account are of interest in practice. Although the problem of stabilization on system with nonlinearities input has been worked out, e.g. [10,11,16], the studies about variable structure system control mostly concentrated on the uncertain systems with full-state feedback. Such control problems are of particular interest in spacecraft control where the control objectives are to be achieved with limited control authority. In Ref. [15], the authors considered this problem using the equivalent control approach. The control design was found to be effective in simulation studies. However, the stability analysis when the input was saturated was lacked. In other related studies [1,2], globally stable control algorithms for stabilization of rigid spacecraft attitude dynamics were reported. However, to the authors’ best knowledge, for flexible spacecraft attitude control system, the problem of VSOFC for systems with control input nonlinearities has not been fully addressed yet.

The objective of this paper is to design a non-linear controller to achieve the attitude maneuver for a three-axis stabilized flexible spacecraft while actively suppressing the vibrations of the flexible appendages under the model uncertainty, external disturbances and control input nonlinearities. The control design method proposed is based on the continuous version of variable structure output feedback control design technique and only the attitude and angular rate information is used for feedback. The designed controller effectively suppresses the vibration of the flexible structures, and achieves disturbance rejection with control input nonlinearities, which is also robust to model uncertainty in the spacecraft model. Moreover, the developed variable structure output feedback controller is also extended through adaptive variable structure control without the limitation of knowing the bounds of the uncertainties and the perturbations in advance, that is, an adaptive variable structure output feedback controller is proposed. In addition, by adding a negative feedback term, an improved adaptation law is also developed which solves the problem that the adaptive switching gains of controller will slowly increase boundlessly due to the fact that the restriction to the sliding surface cannot always be achieved, and guarantees the boundedness of estimated control gains.

The paper is organized as follows: In Section 2, mathematical mode of a flexible spacecraft and some properties of the uncertain system with input nonlinearities are described. The robust variable structure output feedback controller is designed in Section 3, while variable structure output feedback controller is extended in Section 4 to the case without knowing the bounds of the uncertainty and the disturbances. Simulation examples are presented in Section 5 to demonstrate the effectiveness and the potential of the developed techniques. The paper is concluded in Section 6.
2. System Statement

A general description of uncertain dynamical system with nonlinear input is given in the form of
\[
\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)\Phi(u) + f(t) \\
y = Cx 
\]
(1)
where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u \in \mathbb{R}^m\) the control input of the system, and \(y \in \mathbb{R}^p\) the output vector. \(A \in \mathbb{R}^{n \times n}\) is the state matrix, \(B \in \mathbb{R}^{n \times m}\) the input matrix and \(C \in \mathbb{R}^{p \times n}\) the output matrix, and \(\Phi(u)\) a continuous function and \(\Phi(0) = 0\), where \(\Phi : \mathbb{R}^m \to \mathbb{R}^m\) with the law \(u \to \Phi(u)\) and \(\Phi(u) : \mathbb{R}^m \to \mathbb{R}^m\). \(\Delta A\) and \(\Delta B\) are the uncertainty matrix of \(A\) and \(B\), respectively. \(f(t) \in \mathbb{R}^n\) stands for the disturbance. Note that for the real flexible spacecraft, the motion equation is really more nonlinear than that shown in Eq. (1). While in view of control system design, the flexible spacecraft equation given later can be transformed into this case through appropriate assumption and simplification.

Throughout the remainder of this paper, the following assumptions are taken:

**Assumption 1.** For the nominal part of the uncertain dynamic system with nonlinearities indicated in Eq. (1), the triplet \((A,B,C)\) of the nominal system is controllable and observable.

**Assumption 2.** There exist matrix functions \(H, E\) and \(d\) such that the following matching conditions hold:
\[
\Delta A = BH \quad \|H\| \leq \beta_1 \\
\Delta B = BE \quad \|E\| \leq \beta_2 \\
f = Bd \quad \|d\| \leq \beta_3
\]
(2)
where \(\|E\| < 1\). Note that \(\|W\|\) represents the Euclidean norm when \(W\) is a vector or the induced norm when \(W\) is a matrix.

**Assumption 3.** The nonlinear input \(\Phi(u)\) applied to the system is inside sector \([h_1, h_2]\) and satisfies
\[
h_1u_i^2 \leq u_i\Phi_i \leq h_2u_i^2 \quad i = 1, 2, \ldots, m
\]
(3)
where \((u_i, \Phi_i)\) is the \(i\)th scalar element of \((u, \Phi)\).

When the elements of the nonlinear input function inside sector satisfy the property shown in Eq. (3), there exists
\[
h_1u^T u \leq u^T \Phi(u) \leq h_2u^T u
\]
(4)
where \(h_2 = \max(h_{2,i})\) is referred to as the gain margin, and \(h_1 = \min(h_{1,i})\) as the gain reduction tolerance.

It also notes that \(h_1\|u\| \leq \|\Phi(u)\| \leq h_2\|u\|\) is implied by Eq. (4).

**Lemma 1.** If the nonlinear input satisfies the property shown in Eq. (4), there exists a continuous function \(\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+\), \(\phi(0) = 0\), and \(\phi(q) > 0\) for \(q > 0\). Therefore, if \(\|u(t)\| = \phi(q)\), then
\[
hu^Tu \geq q\phi(q) \quad \forall q \geq 0
\]
(5)
**Proof:** Omitted for brevity; see, e.g., [11] and references therein. This lemma will be employed later to derive the variable structure output feedback control law.

3. Variable Structure Output Feedback Control Design

A linear switching surface is defined as
\[
S(t) = Gy = GCx(t)
\]
(6)
where \(G \in \mathbb{R}^{m \times p}\) is a constant matrix such that \(GCB\) is invertible.

In consequence, asymptotically and exponentially stable design methods are analyzed for development of the variable structure output feedback control law.

3.1. Asymptotically Stable Design

With appropriately selected \(G\), the following condition is desired
\[
S(t) = 0
\]
(7)

Once the above sliding mode \(S(t) = 0\) is reached, it is always accompanied with \(\dot{S}(t) = 0\). Therefore, the equivalent control \(\Phi_{eq}\) in the sliding mode \(S(t) = 0\) can be derived from
\[
\dot{S}(t) = GC\dot{x}(t) = 0
\]
(8)
or
\[
GC\dot{x} = GC(A + BH)x + GC(B + BE)\Phi_{eq} + GCBd = 0
\]
(9)
which yields
\[
\Phi_{eq} = -(I+H)^{-1}(GCB)^{-1}GCAX + HX + d
\]
(10)

**Remark 1.** It is noted that the equivalent control \(\Phi_{eq}\) is only a mathematically derived tool for the analysis of a sliding motion rather than a real control law being generated in practical systems. In fact, \(\Phi_{eq}\) is not realizable through a nonlinear controller even if the system is nominal, or the system is in the absence of uncertainties. Therefore, the equivalent
control generates an ideal sliding motion on the switching hyperplanes while the real variable structure controller generates a trajectory close to the ideal sliding motion around the switching hyperplanes.

Substituting Eq. (10) into system (1) yields the equivalent dynamical system with nonlinear input in the sliding mode as

$$
\dot{x}(t) = [I - B(GCB)^{-1}GC]Ax(t)
$$

(11)

From the above analysis, it is shown that how to enforce the system trajectory onto the sliding mode is the key task for system stabilization. The reaching condition of sliding mode is given below.

**Lemma 2.** The motion of the sliding mode (7) is asymptotically stable, if the following condition is held:

$$
S^T(t)\dot{S}(t) < 0 \quad \forall t \geq 0
$$

(12)

**Proof.** Omitted for brevity. To fulfill the condition state in (12), the desired switching control is suggested by

$$
u(t) = -\frac{B^TC^TGTs(t)}{\|B^TC^TGTs(t)\|}\phi(\vartheta)
$$

(13)

where $\phi(\vartheta) = \mu\vartheta(y,t), \mu > 1/h, \vartheta(y,t) = [-STGBNv + \alpha\|S\|]/\|B^TC^TGTs(t)\|$.

Define the singular value decomposition of GC as $GC = U\Sigma V^T$, where $U \in \mathbb{R}^{n,m}$, $\Sigma \in \mathbb{R}^{m \times m}$, and $V \in \mathbb{R}^{m \times m}$. Theorem 1. Consider the uncertain system (1) with nonlinearities satisfying Assumptions 1–3. For the control law in (13), choose

$$
N = -(\gamma I + GCA\Sigma^{-1}U^TG)
$$

(14)

where $\gamma > \beta_1\|\Sigma^{-1}\|$ is a scalar. If $\alpha$ is chosen to satisfy $\alpha > (\|GA\| + \beta_2h_2\rho + \beta_3)$ and $\vartheta(\cdot) \geq 0$ where $\Omega > 0$, and $\rho = \max\|u\|$, then the reaching condition $S^T\dot{S} < 0$ is satisfied for $\{x : \|x_k\| \leq \Omega\}$.

**Proof.** Let $V(t) = \frac{1}{2}\|S(t)\|^2$ be the Lyapunov function of system (1). Then, from (8), one can obtain

$$
\dot{V}(t) = S^T(t)\dot{S}(t) = S^T(t)GCx
$$

$$
= S^TGC[(A + \Delta A)x + (B + \Delta B)\Phi(u) + f]
$$

$$
= S^TGC[Ax + B\Phi(u) + B(Hx + E\Phi(u) + d)]
$$

(15)

From (5) and (12), we get

$$
u^T\Phi(u) = -\frac{S(t)GCB}{\|B^TC^TGTs(t)\|}\phi(\vartheta)\Phi(u)
$$

$$
\geq \delta u^T u \geq \vartheta \phi(\vartheta)
$$

(16)

Therefore, it follows from (16)

$$
S^TGB\Phi(u) \leq -\vartheta(y,t)\|B^TC^TGTs(t)\|
$$

(17)

Substituting (17) into (15) gives

$$
\dot{V}(t) = S^T\dot{S} \leq S^TGCx + S^TGBNv
$$

$$
- \alpha\|S\| + S^TGB(Hx + E\Phi(u) + d)
$$

(18)

Note that a nonsingular transformation of switching surface will not change the sliding mode dynamics. If a particular switching surface $S_1 = G_1\gamma$ is chosen, it can be transformed to $S = G_1\gamma$, where $G = (G_1CB)^{-1}G_1$. Therefore, without loss of generality, we can assume $GCB = I$.

By substituting $N$ into the reaching condition (12), and use $GCB = I$, we have

$$
S^T\dot{S} \leq S^TGA(C - BGC\Sigma^{-1}U^TG)x
$$

$$
- \gamma S^T - \alpha\|S\| + \beta_2h_2\|u\|\|S\| + S^T(Hx + d)
$$

(19)

If $x$ is decomposed as $x = x_k + x_p$, where $x_k \in N(GC)$ and $x_p \in N^\perp(GC)$, then $GCx = GCx_p$. Simplify the first term on the right-hand side in (19), and then decompose it as follows

$$
S^TGA(I - V_1\Sigma^{-1}U^TG)x
$$

$$
= S^TGCx_k + S^TGC(I - V_1\Sigma^{-1}U^TG)x_p
$$

(20)

If $GC = U\Sigma^TV^T$, then $x_p = V_1\bar{\beta}$ for some vector $\bar{\beta}$. By using $V_1^TV_1 = I$ and $U^TU = I$, the second term on the right-hand side of (20) is zero, and then the expression in (19) is reduced to

$$
S^T\dot{S} \leq S^TGCx_k - \gamma S^T - \alpha\|S\|
$$

$$
+ S^THV_1\Sigma^{-1}U^TS + \beta_2h_2\|u\|\|S\| + S^Td
$$

(21)

If $\Omega \geq \|x_k\|$, the upper bound for the expression in (21) is

$$
S^T\dot{S} < \|S\|\|GCA\|\Omega - \gamma S^TS
$$

$$
+ \beta_1\|\Sigma^{-1}\|S^TS - \alpha\|S\| + \beta_2h_2\|u\|\|S\| + \beta_3\|S\|
$$

(22)
Choose $\alpha$ and $\gamma$ such that $\alpha > (\|GCA\|\Omega + \beta_2h_2\rho + \beta_3)$, and $\gamma > \beta_1\|\Sigma^{-1}\|$, respectively. It can be shown now that

$$S^T(t)\dot{S}(t) < 0 \quad \text{or} \quad \dot{V}(t) < 0$$

According to the Lyapunov stability theorem, condition (23) ensures that $S(t)$ is toward the switching surface and the sliding mode is asymptotically stable. The proof is completed.

### 3.2. Exponential Stability

If the variable structure control law (13) is modified as

$$\ddot{u}(t) = \frac{B^TC^GT^TS(t)}{\|B^TC^GT^TS(t)\|} \phi_a(\dot{\vartheta})$$

where $\phi_a(\dot{\vartheta}) = \dot{\vartheta} + \frac{q}{2\alpha}(\|S(t)\|/\|B^TC^GT^TS(t)\|)$, $q>0$, then we have the following stability result.

**Theorem 2.** Consider the uncertain nonlinear dynamic system (1) under Assumptions 1–3. The sliding mode control law is chosen as (24). Then the system trajectories globally exponentially converge to the sliding mode (6).

**Proof.** Let the Lyapunov function candidate also be $\dot{V}(t) = \frac{1}{2}\|S(t)\|^2$. Using (24) and the same manipulations as those in (15)–(22), the time derivative of $V(t)$ becomes

$$\dot{V} \leq S^T \dot{S} \leq \|S\|\|GCA\|\Omega - \gamma S^T S + \beta_1\|\Sigma^{-1}\| S^T S$$

$$- \alpha\|S\| + \beta_2 h_2 \|u\| S + \beta_3\|S\| - \frac{1}{2} q S^T S$$

$$\leq - \frac{1}{2} q\|S\|^2 = - q V \leq 0$$

It yields

$$V(t) \leq V(t_0) e^{-q(t-t_0)}$$

where $t_0$ is the initial time. Therefore, the system trajectories globally and exponentially converge to the sliding mode (6).

### 4. Adaptive Variable Structure Output Feedback Control Design

In Section 3, we have shown how to design a stable system by dynamic variable structure control for the systems with mismatched uncertainty. However, the bounds of the uncertainties/disturbances must be known in advance. In general, this bound is difficult to measure in practical applications; therefore, the bound of the uncertainty is usually chosen large enough to ensure robust stability. However, a large parameter will result in substantial chattering of the control effort. On the other hand, if the bound is chosen too small, the robust stability cannot be guaranteed. To relax the requirement for the bound of uncertainty, an adaptive design method for the robust gain is proposed. Here, we recall the Barbalat lemma as discussed [14] in the follows. Barbalat lemma will be a tool used in the proof of the subsequent main result of Theorem 3.

**Lemma 3.** If $g : R \rightarrow R$ is uniformly continuous for $t \geq 0$, and if the limit of the integral

$$\lim_{t \rightarrow \infty} \int_0^t |g(\tau)|d\tau$$

exists and is finite, then

$$\lim_{t \rightarrow \infty} g(t) = 0$$

Now, the following adaptive variable structure output feedback control is proposed as

$$\ddot{u}(t) = \frac{B^TC^GT^TS(t)}{\|B^TC^GT^TS(t)\|} \dot{\vartheta}(\vartheta)$$

where $\dot{\vartheta}(\vartheta) = \mu \ddot{\vartheta}(y, t), \mu > 1/\hat{h}$, and $\ddot{\vartheta}(y, t) = [\frac{-S^TGCB\hat{N}+a\|S\|}{\|B^TC^GT^TS\|} + \hat{N}]$.

Note that $k$ is any positive constant, $\hat{N}$ and $\hat{\alpha}$ are chosen to be $\hat{\alpha} > (\|GCA\|\Omega + \beta_2h_2\rho + \beta_3)$ and $\hat{\beta} = (\gamma I + GCA\hat{V}_1\Sigma^{-1}U^T)\hat{G}$, respectively, where $\gamma > \beta_1\|\Sigma^{-1}\|$, $\epsilon_i$ is any positive constant, and $\hat{\beta}$ is the estimate of $\beta_i$. $\hat{\beta}_i$ and $\beta_i$ can be obtained from the following dynamics

$$\hat{\beta}_1 = \frac{1}{p_1}\|S^TGCB\|\|\Sigma^{-1}\|\|S\|$$

$$\hat{\beta}_2 = \frac{1}{p_2}\|h_2\rho\|S^TGCB\|$$

$$\hat{\beta}_3 = \frac{1}{p_3}\|S^TGCB\|$$

where $p_i$ is a positive constant. Let $\hat{\beta}_i = \beta_i - \beta_i$ denote the adaptation error. Because $\beta_i$ is assumed to be constant, then we have the following

$$\hat{\beta}_i = -\dot{\beta}_i$$
**Theorem 3.** Consider system (1) under Assumptions 1–3, if this system is controlled by \(u(t) = \tilde{u}(t)\) in (29) with adaptation law (30), then the system trajectory asymptotically converges to the sliding surface \(S(t) = 0\).

**Proof:** Consider the following Lyapunov functional candidate

\[
\dot{V} = \frac{1}{2} \left[ \|S\|^2 + \dot{\beta}^T \Gamma \dot{\beta} \right] \tag{32}
\]

where \(\dot{\beta} = [\dot{\beta}_1 \; \dot{\beta}_2 \; \dot{\beta}_3]^T\), \(\Gamma = \text{diag} \{p_1, p_2, p_3\}\).

Taking the derivative of \(\dot{V}\) with respect to time \(t\) and combing with (30) and (31) yield

\[
\dot{V} \leq \dot{S}^T GCA x_k - \hat{\gamma} S^T S - \alpha \|S\| - k \|S\| \\
+ \dot{S}^T d + \dot{S}^T H V \Sigma^{-1} U^T S + \dot{\beta}_2 h_2 \|u\| \|S\| \\
+ \dot{\beta}^T \Gamma \dot{\beta} \leq \|S\| \|GCA\| \Omega - \hat{\gamma} S^T S \\
+ \dot{\beta}_1 \|\Sigma^{-1} S^T S - \alpha \|S\| - k \|S\| \\
+ \dot{\beta}_2 h_2 \|u\| \|S\| + \dot{\beta}_3 \|S\| + \dot{\beta}^T \Gamma \dot{\beta} \\
< -k \|S\| + (\beta_1 - \dot{\beta}_1) \|\Sigma^{-1} S^T S \\
+ (\beta_2 - \dot{\beta}_2) h_2 \rho \|S\| + (\beta_3 - \dot{\beta}_3) \|S\| + \dot{\beta}^T \Gamma \dot{\beta} \\
= -k \|S\| + \dot{\beta}_1 \|\Sigma^{-1} S^T S + \dot{\beta}_2 h_2 \rho \|S\| \\
+ \dot{\beta}_3 \|S\| + \dot{\beta}^T \Gamma \dot{\beta} \tag{33}
\]

where \(\dot{\beta}_i = \dot{\beta}_i\) is given in (31). Therefore, by substituting (30) into (33), we obtain the following inequality

\[
\dot{V} \leq -k \|S\| = -g(t) \leq 0 \tag{34}
\]

where \(g(t) = k \|S\|\). Integrating (34) from zero to \(t\), it yields

\[
\dot{V}(0) \geq \dot{V}(t) + \int_0^t g(\tau) d\tau \geq \int_0^t g(\tau) d\tau \geq 0 \tag{35}
\]

As \(t\) goes to infinity, the above integral is always less than or equal to \(\dot{V}(0)\). However, \(\dot{V}(0)\) is finite and positive, then we have \(g(t) \to 0\) as \(t \to \infty\) from Lemma 3, that is

\[
\lim_{t \to \infty} g(t) = \lim_{t \to \infty} k \|S\| = 0 \tag{36}
\]

Since \(GCB = I\) is nonsingular, (36) guarantees that \(S(t) \to 0\) as \(t\) goes to infinity. Then the proof is completed.

However, the adaptive switching gains of controller will slowly increase boundlessly due to the fact that the restriction to the sliding surface cannot always be achieved. In order to address this problem, we propose a modified adaptive control law by introducing a negative feedback term \(-\mu_i(t)\dot{\beta}_i\), i.e.,

\[
\dot{\beta}_1 = -\mu_1(t)\dot{\beta}_1 + \frac{1}{p_1} \left| \dot{S}^T GCB \right| \|\Sigma^{-1}\| \|S\| \tag{37a}
\]

\[
\dot{\beta}_2 = -\mu_2(t)\dot{\beta}_2 + \frac{1}{p_2} \left| \dot{h}_2 \rho \right| \|S\| \tag{37b}
\]

\[
\dot{\beta}_3 = -\mu_3(t)\dot{\beta}_3 + \frac{1}{p_3} \left| \dot{S}^T GCB \right| \tag{37c}
\]

\[
\dot{\mu}_i(t) = -r_i \mu_i(t) \tag{37d}
\]

where \(\mu_i(0) > 0\) and \(r_i > 0\) \((i = 1, 2, 3)\).

**Theorem 4.** Consider system (5) under Assumptions 1–3, if this system is controlled by \(u(t) = \tilde{u}(t)\) in (29) with the adaptive law (37), then the system trajectory asymptotically converges to the sliding surface \(S(t) = 0\).

**Proof.** Consider the following Lyapunov functional candidate

\[
\dot{V} = \frac{1}{2} \left[ \|S\|^2 + \dot{\beta}^T \Gamma \dot{\beta} \right] + \dot{\beta}^T \Gamma R^{-1} \mu \tag{38}
\]

where \(\dot{\beta} = [\dot{\beta}_1 \; \dot{\beta}_2 \; \dot{\beta}_3]^T, \dot{\beta} = [\dot{\beta}_1 \; \dot{\beta}_2 \; \dot{\beta}_3]^T, \Gamma = \text{diag} \{p_1, p_2, p_3\}, R = \text{diag} \{r_1, r_2, r_3\}, \mu = \text{diag} \{\mu_1, \mu_2, \mu_3\}\) and \((\mu) = [\mu_1 \beta_1 \mu_2 \beta_2 \mu_3 \beta_3]^T\).

Differentiating \(\dot{V}\) with respect to time \(t\), and combing with (7) yields

\[
\dot{V} = \dot{S}^T \dot{S} + \dot{\beta}^T \Gamma \dot{\beta} + \dot{\beta}^T \Gamma R^{-1} \mu \beta \\
= \left( S^T GCBx + S^T GCB \dot{\beta} \right) y \\
- \hat{\alpha} \|S\| - k \left( \dot{B}^T C^T G^T S \right) + S^T GCBHx \\
+ S^T GCB \dot{\beta} + \dot{\beta}^T \Gamma R^{-1} \mu \beta \tag{39}
\]

Using the fact \(\dot{\mu} = -R(\mu, \beta)\), then

\[
\dot{V} \leq -k \|S\| + \dot{\beta}^T \Gamma (\mu \beta) - \beta^T \Gamma (\mu \beta) = -k \|S\| + \Xi \tag{40}
\]

where

\[
\Xi = \beta^T \Gamma (\mu \beta) - \beta^T \Gamma (\mu \beta) = \sum_{i=1}^{3} (\beta_{pi} \mu_i \beta_i - \beta_i^T \mu_i) 
\]

Here, definite
where \( \beta_i \beta_j \beta_k \) is obtained from the well-known Lagrangian approach.

The dynamics of the spacecraft with flexible appendages (solar arrays, antennas, etc.), actuated by reaction wheels, by considering small displacements, can be written as

\[ \dot{\mathbf{S}} \rightarrow 0 \] as \( t \) goes to infinity. Then the proof is completed.

**Remark 2.** If the proposed adaptive sliding mode control laws \( \hat{u} \) in (29) is modified by the following form \( u \) in (43), then the system trajectories will exponentially converge:

\[ \hat{u}(t) = - \frac{B^T \mathbf{C}^T \mathbf{G}^S(t)}{\|B^T \mathbf{C}^T \mathbf{G}^S(t)\|} \hat{\phi}_u(\bar{\vartheta}) \]

where \( \hat{\phi}_u(\bar{\vartheta}) = \hat{\phi}(x, t) + (q/2\alpha) \left( (S^T S) / \|B^T \mathbf{C}^T \mathbf{G}^S(t)\| \right) \).

This can be worked out by the same techniques as those used in the proofs of Theorems 2 and 3.

5. Application to a Flexible Spacecraft

5.1. Mathematical Model

The design of control system for spacecraft is a complex task as a result of coupled nonlinear dynamics and attached flexible structures. While for the flexible spacecraft with a small Euler angle rotations, the dynamic model in Eq. (44) can be approximated as

\[ \begin{bmatrix} (J_T - J_R) & \delta^T \\ \delta \\ I \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\Psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} I \end{bmatrix} (\Pi(\cdot) - u_e) \]

where \( \varphi, \theta \) and \( \Psi \) are roll, pitch and yaw attitude angles, and \( \Pi(\cdot) \) are function of the moments of inertia of the flexible spacecraft, the coupling matrix been attitude and vibrations modes, the orbital angular velocity, the modal frequencies, the flywheels relative angular velocity, and the external disturbances.

**Remark 3.** Note that under the assumption that a small Euler angle rotation is implemented and the orbit angular rate is small and slow, a simple model given by Eq. (45) is derived by neglecting some nonlinear coupling terms. This simplified equation is easy to manipulate and more suitable for control design. In fact, the exact model described by Eq. (39) is more nonlinear and more difficult to handle with. In order to verify the effectiveness of the control law, the simplified mode is used to derive the attitude control law to accomplish the rational maneuver and simultaneously suppress vibration. However, in the simulation section, the exact model of Eq. (44) is adopted and analyzed.

Introducing a new variable

\[ z = [\varphi \ \theta \ \Psi \ \eta]^T \]
and setting $\mathbf{M} = \begin{bmatrix} (J_T - J_R) & \delta^T \\ \delta & I \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$, $\mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$, $\mathbf{B} = [I \ O]^T$.

one can obtain

$$M\ddot{z} + D\dot{z} + Kz = B[\Pi(\cdot) - u_s] \quad (47)$$

Define the state variable $x_s = [\varphi \ \theta \ \eta \ \dot{\varphi} \ \dot{\theta} \ \dot{\eta}]^T \in \mathbb{R}^{2(N+3)}$, the state-space spacecraft model describing the dynamics behavior of the flexible spacecraft can be described by

$$\dot{x}_s(t) = (A_s + \Delta A_s)x_s(t) + (B_s + \Delta B_s)\Phi_s(u_s) + f_s(x_s, u_s)$$

$$y_s = C_s x_s \quad (48)$$

where $A_s = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}$, $B_s = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}$, $f(x_s, u_s) = B_s \Pi(\cdot)$, $C_s = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times N} & 0_{3 \times 3} & 0_{3 \times N} \\ 0_{3 \times 3} & I_{3 \times N} & 0_{3 \times 3} & 0_{3 \times N} \end{bmatrix}$, $\Delta A_s$ and $\Delta B_s$ are the uncertainty matrix of $A_s$ and $B_s$, respectively. Here the nonlinearities in the input $\Phi_s(u_s)$ is taken as saturation whose saturated value is 1 Nm.

### 5.2. Simulation Results

In this section, numerical application of the proposed control schemes to the attitude control of a flexible spacecraft is presented using MATLAB/SIMULINK software. The nominal inertia and coupling matrices are from Ref. [4]

$$J_T = \begin{bmatrix} 1001 & 19.6 & 8.3 \\ 19.6 & 623 & 13.6 \\ 8.3 & 13.6 & 703 \end{bmatrix} \quad \text{kgm}^2, \quad J_R = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \quad \text{kgm}^2,$$

$$\delta = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \quad \text{kg}^{1/2}\text{m}, \quad \text{and the first four elastic modes have been taken into account in the model used for simulating the spacecraft at } \omega_{n1} = 1.9, \quad \omega_{n2} = 4.1, \quad \omega_{n3} = 5.8, \quad \omega_{n4} = 6 \text{ rad/s with damping } \xi_1 = 0.08, \xi_2 = 0.30, \xi_3 = 0.60, \xi_4 = 0.75.\text{ Suppose the initial values of the three attitude angles are } \varphi(0) = 6^\circ, \theta(0) = 4^\circ, \text{ and } \Psi(0) = -4^\circ. \text{ It is required to control each of the attitude angles approaching zero from their initial value within } t_f = 150s, \text{ and each of the vibration mode coordinates satisfies } \eta_i = \eta_i(t_f) = \dot{\eta}_i(t_f) = 0, i = 1, 2, 3, 4. \text{ Here, } \Delta A_s, \Delta B_s \text{ and } f(x_s, u_s) \text{ are usually imprecise and satisfies the Assumption 2, and suppose } H = 0.012A_s, E = 0.5B_s \text{ and } d \text{ is a random disturbance torque, given by}

$$d = d_{\text{max}} \begin{bmatrix} N(\cdot) \\ N(\cdot) \end{bmatrix} \quad (49)$$

whose maximum absolute $d_{\text{max}}$ has been fixed to 0.25 Nm; $N(\cdot)$ denotes the normal distribution with mean zero and standard deviation one.

In the numerical simulation, for comparison, three cases are conducted: (1) attitude maneuver control using the proposed variable structure output feedback control technique; (2) attitude maneuver control using a linear static output feedback control; (3) attitude maneuver control using the proposed adaptive variable structure output feedback control technique. The attitude response and vibration suppression are analyzed to study the performances of the controllers. In addition, the vibration energy level described by $E = \dot{\eta}^T \eta + \eta^T K \eta$ is also used to express the effectiveness of vibration control. All computations and plots shown in the paper were performed using MATLAB/SIMULINK software package.

#### 5.2.1. Variable Structure Output Feedback Control Case

In order to reduce chattering in an actual implementation, the discontinuous control given as (13) often is replaced by a smoothed control of the form:

$$u(t) = -\frac{B^T C^T G^T S(t)}{\|B^T C^T G^T S(t)\| + \varepsilon} \phi(\theta) \quad (50)$$

where $\varepsilon > 0$ is small. This creates a small boundary layer about the switching surface in which the system trajectory will remain, as opposed to ideal sliding where the trajectory remains on the switching surface.

The lemma 1 and Theorem 1 given in Section III suggest the following algorithm, by which the controller in the Eq. (13) can be designed.

**Algorithm.**

(Step 0) Input $A_s, B_s, C_s, G_1$ and $\Omega$, where $G_1$ defines the switching surface.

(Step 1) Transform $G_1$ by $G = (G_1 C_s B_s)^{-1} G_1$.

(Step 2) Decompose $G C_s$ as $G C_s = U \Sigma V^T$.

(Step 3) Let $2 \gamma_2 = -\lambda_{\text{min}}(G C_s A_s V_1 \Sigma^{-1} U^T + U \Sigma^{-1} V_1^T A_s^T C_s G^T)$

(Step 4) Choose $\gamma \geq \gamma_{\text{min}}$, where $\gamma_{\text{min}} = \max(\xi_1 + \gamma_2, \beta_1 \|\Sigma^{-1}\|)$ and $\varepsilon > 0$ is small.

(Step 5) Let $N = -\gamma G - G C_s A_s V_1 \Sigma^{-1} U^T G$.

(Step 6) For control law (50), choose $\alpha \geq \alpha_{\text{min}}$, where $\alpha_{\text{min}} = (\|G C_s A_s\| \Omega + \beta_2 h_2 \rho + \beta_3)$. 


In this controller parameter design procedure, four design parameters $\Omega$, $\gamma$, $\alpha$ and $\varepsilon$ are involved. Considerable simulations have been done for determining the parameters $\Omega$, $\gamma$, $\alpha$ and $\varepsilon$, and here only one case of this numerical studies is given for the space limitation: $\Omega = 10$, $\gamma = 10$, $\alpha = 5$ and $\varepsilon = 0.01$.

The disturbances considered $d$ in the simulation are shown in Fig. 1. With the designed controller in Eq. (50), time responses of angles, velocity, control torque, vibration mode and vibration energy are shown in Fig. 2. From Fig. 2(c), we can see that the inner-torque of each flywheel approach to zero at the time of 100 s. It is also clear that each of the three-attitude angles response approaches zero at the time of 150 s, and the steady error of each angle is close to zero. It is noted that an acceptable angle rate response was achieved as show in Fig. 2(b). In addition, responses of the first three vibrations modes and the

![Fig. 1. Time responses of disturbance torques.](image1)

![Fig. 2. Time responses of variable structure output feedback control (asymptotically stable) case. (a) attitude angle; (b) angular velocity; (c) inner-torque of flywheels; (d) first three modes and vibration energy.](image2)
vibration energy are also illustrated in Fig. 2(d). The maximum induced vibration coordinate of the 1st mode is effectively suppressed below 0.005 kg$^{1/2}$m and the maximum amplitude of vibration energy is less than 0.006 Nm. It is noted that for the current design system, fast and precise attitude control and vibration suppression for the flexible appendages are achieved even in the presence of control input nonlinearities and external perturbations.

For the exponentially stable design case, the controller (29) can also be modified into the form similar to the Eq. (50) by introducing the boundary layer. Here, parameter $q$ is selected 15 through considerable numerical simulations. The parameters $N$, $\Omega$, $\beta$, $\varepsilon$, $\gamma$ and $\alpha$ remain the same for a fair comparison. The same initial condition and disturbances are also used as above case. The simulations are shown in Fig. 3. The three-axis attitude responses almost approaches zero at the time of 100 s, and the steady error of each angle is also less than 0.002$^\circ$ as shown in Fig. 3(a). Fig. 3(b) shows the responses of angular velocity of the flexible spacecraft with the steady error no more than 0.002$^\circ$/s. It is clear that from comparison of Figs 2(a–b) and 3(a–b) the attitude and rate responses under the exponential design case are faster than the asymptotical design case, but the some chattering is observable due to the additional term $(q/2\alpha)((S^TS)/\|B^TC^TG^TS(t)\|)$. Note that when the parameter $q$ is changed by extensively numerical simulations analysis, the chattering can be reduced. In addition, the maximum amplitude of the 1st mode and the vibration energy is also effectively suppressed below 0.1 kg$^{1/2}$m and 0.004 Nm as shown in Fig. 3(d), respectively.

5.2.2. Linear Static Output Feedback Control Case

For comparison, consider a linear static control of $u = K_1y$, where $K_1$ is chosen to give closed-loop
eigenvalues as approximate as possible to the eigenvalues of the sliding mode equations, such that the nominal behavior of the linear control and the proposed variable structure output feedback control are comparable. The responses of the linear control to the same initial condition, uncertainties and disturbances are given in Fig. 4. The excessive changes of the control signal are observed in Fig. 4(c). As a result, a significant amount of the oscillations of the attitude angle and rate occurred during the maneuvering as demonstrated in Figs 4(a) and 4(b). Moreover, the oscillations do not settle within 40 s. The disturbance rejection is markedly worse that that achieved by the sliding mode control. In addition, from comparison of Figs 2(d) or 3(d), the linear control introduces much more vibrations to the vibration modes as shown in Fig. 4(d). From the responses we can see that the maximum induced vibration coordinate of the 1st mode is nearly close to 0.1 kg\(^{1/2}\)m and the vibration energy 0.02 Nm, respectively. Furthermore, the oscillatory induced by attitude maneuver converges very slowly.

5.2.3. Adaptive Variable Structure Output Feedback Control Case

As in the preceding case, in the following simulations we used the soothed function described in Eq. (51) in the control law to avoid chattering of the control input.

$$u(t) = -\frac{B^T C^T G^T S(t)}{\|B^T C^T G^T S(t)\| + \varepsilon} \phi(\theta)$$ (51)

The controller parameter design process is as same as the algorithm aforementioned and parameters \(\Omega, \gamma, \alpha\) and \(\varepsilon\) for remain the same for a fair composition. The attitude angles and rates response of each axis are shown in Figs 5(a) and 5(b), respectively. As expected, the responses of both the angle displacement and the

![Fig. 4. Time responses of linear static output feedback control case. (a) attitude angle; (b) angular velocity; (c) inner-torque of flywheels; (d) first three modes and vibration energy.](image-url)
angular velocity also converge to zero in 150 s. The time response of flexible modes vibrations and energy are shown in Fig. 5(d). The maximum induced vibration coordinate of the 1st mode is effectively suppressed below 0.008 $kg^{1/2}m$ and energy less than 0.0004 Nm, respectively. It is seen that the adaptive control law, as expected, is essentially capable of suppressing the vibrations while maintaining the attitude maneuvering capability of flexible spacecraft.

For the adaptively and exponentially stable design case, the controller (29) can also be modified into the form similar to the Eq. (51) by introducing the boundary layer. Here, parameter $q$ is also selected 15 for a fair comparison. The parameters $N, \Omega, \beta, \varepsilon, \gamma$ and $\alpha$ also remain the same. The simulations are shown in Fig. 6. The responses of the attitude angles almost approaches zero at the time of 100 s, and the steady error of each angle is also less than 0.001° as shown in Fig. 6(a). Fig. 6(b) shows the responses of angular velocity of the flexible spacecraft with the steady error much less than 0.002°/s. It is clear that, from comparison of Figs 5(a–b) and 6(a–b), the attitude and rate responses under the exponential design case are also faster than the asymptotical design case even if there is some control chattering due to the additional term $(q/2\alpha)(S^T S)/\|B^T C T G^T S(t)\|$). In addition, the maximum induced vibration coordinate of the 1st mode is also effectively suppressed much less than 0.008 $kg^{1/2}m$ and energy 0.0001 Nm as shown in Fig. 6(d).

In addition, the case of asymptotically stable design with modified adaptive control law is also studied by simulation. In this case, $\mu_1(0) = 0.01$ and $r_i = 0.5$ ($i = 1, 2, 3$) are used in the simulation and the other parameters $q, N, \Omega, \beta, \varepsilon, \gamma$ and $\alpha$ also remain the same with above case. The responses of the attitude angles almost approaches zero at the time of 100 s, and the steady error of each angle is also less than 0.001° as shown in Fig. 7(a). Fig. 7(b) shows the responses of angular velocity of the flexible spacecraft with the
steady error much less than 0.002/s. Moreover, the maximum induced vibration coordinate of the first mode is also effectively suppressed much less than 0.0005 kg\(^{1/2}\)m and the vibration energy less than 0.0001 Nm as shown in Fig. 7(d). At the same time, the chattering is also significantly reduced by involved the negative terms in the adaptive law. For the case of exponentially stable design with modified adaptive control law, same trends with Fig. 7 were also found (figures not shown because of space limitation).

From the comparison of above several cases, it is shown that the proposed approach can not only accomplish the high precision attitude control during maneuvers, but also simultaneously suppress the undesired vibrations of the flexible appendages even though the input nonlinearity is explicitly considered. Furthermore, the information of upper bound of the perturbations and uncertain is not required beforehand when the adaptive version of the proposed variable structure output feedback control is adopted.

6. Conclusions

In this paper, an approach to vibration suppression of three-axis stabilized flexible spacecraft is investigated during attitude maneuver in the presence of bounded model uncertainty, external disturbances and control input nonlinearities. The approach of the current study is the construction of variable structure output feedback controller to stabilize uncertain dynamics system with explicitly taking into account control input nonlinearities. We also propose an adaptive version of this algorithm, which removes the disadvantages of the sliding mode controller, and can results in precise response to attitude control.

Fig. 6. Time responses of adaptive variable structure output feedback control (exponentially stable) case. (a) attitude angle; (b) angular velocity; (c) inner-torque of flywheels; (d) first three modes and vibration energy.
command and effective suppression to the vibration of its flexible appendages in the presence of disturbances, uncertainty and input nonlinear characters. In both cases we included rigorous proof of stability of the resulting closed-loop system, as well as computer simulations to evaluate the overall performance. The significant advantages of this control include the following: (i) presented variable structure output feedback controller can derive the trajectories of the uncertain system with nonlinear input onto the sliding mode; (ii) the uncertain system with input nonlinearities still keeps insensitive to the parameter variations and/or disturbance as that of a system with linear input in the sliding mode; (iii) the variable structure output feedback control can also be achieved through adaptive control without knowing the bounds of the uncertainty and perturbation terms in advance; (iv) the modified adaptation law can guarantee the boundedness of the control gains even if the restriction to the sliding surface cannot be achieved.

**Acknowledgments**

This work was supported by the Research Fund for the Doctoral Program of Higher Education of China (Project Number: 20050213010). The authors wish to thank Prof. Gangbing Song from the University of Houston, Prof. Brij N. Agrawal from the Naval Postgraduate School, and Seung-Bok Choi from the Inha University, for their useful design and insightful comments. The authors also thank the editor and anonymous referees for the help suggestions that lead to the improvements in the presentation of the results.

**References**

1. Bošković JD, Li SM, Mehra RK. Robust adaptive variable structure control of spacecraft under control input saturation. J Guid Control Dynam, 2001; 24: 14–22
2. Bošković JD, Li SM, Mehra RK. Robust tracking control design for spacecraft under control input saturation. J Guid Control Dynanm, 2004; 27: 627–633
11. Hsu KC. Variable structure control design for uncertain dynamic system with sector nonlinearities. Automatica, 1998; 34: 505–508