On semi-global stabilization of linear periodic systems with control magnitude and energy saturations

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Abstract

This paper establishes a systematic approach to solve the $L_\infty(l_\infty)$ and $L_2(l_2)$ semi-global stabilization problem of linear periodic systems with controls having bounded magnitude and energy, respectively. The developed approach will be referred to as $L_\infty(l_\infty)$ and $L_2(l_2)$ low gain feedback. Definitions, properties, and characterizations of this new concept are also provided, and particularly, the characterizations are based upon differential (difference) Lyapunov inequalities. Design of $L_\infty(l_\infty)$ and $L_2(l_2)$ low gain feedback by solving differential (difference) Riccati equations is proposed. Both continuous-time and discrete-time linear periodic systems are studied and both state feedback and observer based output feedback are considered. In the discrete-time setting, a linear matrix inequalities (LMIs) based solution to the $l_\infty$ and $l_2$ semi-global stabilization problem is also established and the LMIs conditions are shown to be always solvable. Applications of the proposed approach to the elliptical spacecraft rendezvous system show the effectiveness of the established theory.

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1. Introduction

Control of time-varying systems has received much attention in the past several decades and many results have been published in the literature (see [31] and the references therein). Periodic systems as special cases of time-varying systems have also been extensively studied in the literature (see [2,3,7,10,16,17,32] and the references given there). It has been clear that linear
periodic systems possess many important properties belonging to linear time-invariant systems because of the well-known Floquet–Lyapunov theory [1]. Also, it is clear that continuous-time linear periodic systems are quite different from discrete-time linear periodic systems since the latter ones can be reformulated as time-invariant systems by using some lifting techniques (see, for example, [1] and [15]). Hence, it is believed that control problems, if not all, for continuous-time linear periodic systems are harder to solve than discrete-time linear periodic systems [28]. On the other hand, saturation nonlinearity exists in every practical control systems and the ignorance of such a nonlinearity will lead to performance degradation and even instability of the practical control system [8]. Hence control systems design by taking the actuator saturation nonlinearity into consideration has received much attention during the past several decades and a great number of results have been published in the literature (see [8,11,18,19,27,29,30] and the references therein).

The design of periodic control systems with controls having either bounded magnitude or bounded energy is important since it has some potential applications in engineering. For example, the spacecraft rendezvous problem belongs to this type since the linearized Tschauner–Hempel equations characterizing the relative motion is a continuous-time linear periodic system and the control accelerations being the control of the system are always bounded in both the $L_\infty$ and $L_2$ sense [26]. For the constrained control problems of linear periodic systems, some results are already available in the literature. For example, a discrete-time linear plant that is asymptotically null controllable by bounded controls is semi-globally stabilized by bounded periodic feedback via a lifted representation of the time-invariant plant in [6]; a polynomial approach is developed in [5] to solve the local stabilization problem of an SISO discrete-time linear periodic system with bounded controls; local stability analysis and stabilization of discrete-time linear periodic systems with bounded controls are considered in [24] by using the concept of periodic invariant sets, semi-global stabilization of discrete-time and continuous-time linear periodic systems are respectively studied in [25,33] and [28] by using periodic Lyapunov differential and difference equations, and the global stabilization of neutrally stable linear periodic systems by bounded feedback was solved very recently in [34] by linear feedback.

By considering that the solutions to the semi-global stabilization problems of linear periodic systems established in our recent work [25,33,28] are constructive, this paper will establish a systematic approach for solving $L_\infty(l_\infty)$ and $L_2(l_2)$ semi-global stabilization problems for linear periodic systems with controls having bounded magnitude and bounded energy, respectively. The established approaches will be referred to as periodic low gain feedback. Characterizations of periodic low gain feedback in terms of Lyapunov differential and difference inequalities are established. Analytical methods based on the periodic differential Riccati equation (DREs) and periodic difference Riccati equations (DcREs) will be developed. These results generalize our recent work on linear time-invariant systems in [27,29] to the periodic setting. For the $l_\infty$ and $l_2$ semi-global stabilization of discrete-time linear periodic systems, a computational approach based on linear matrix inequalities (LMIs) is also established. An application of the proposed approach to the elliptical spacecraft rendezvous system will be carried out to illustrate the effectiveness of the proposed approach.

The remainder of this paper is organized as follows. Semi-global stabilization of continuous-time and discrete-time linear periodic systems are respectively studied in Sections 2 and 3 and applications of the developed theory to the elliptical spacecraft rendezvous system will be investigated in Section 4. Finally, Section 5 concludes this paper.

**Notation:** Throughout this paper, we will use standard notation. For a matrix $A$, we use $|A|$ and $|A|_\infty$ to denote its 2 and $\infty$ norms and we use $\lambda(A), A^T$, and $\text{tr}(A)$ to denote its eigenvalue set, its
transpose, and its trace, respectively. For an integer \( k \), we use \( \mathbb{I}[k, \infty) \) to denote the integer set \( \{k, k + 1, \ldots\} \). For the positive definite matrix \( P \in \mathbb{R}^{n \times n} \) we denote \( \mathcal{E}(P) = \{x \in \mathbb{R}^n : x^TPx \leq 1\} \).

For a periodic time-varying square matrix \( A(t) \), we use \( \mathcal{M}(A(t)) \) to denote its characteristic multiplier set. Throughout this paper, we say that a matrix function \( P(t) : [t_0, \infty) \rightarrow \mathbb{R}^{n \times n} \) is positive definite if there exist two positive constants \( p_1 \) and \( p_2 \) such that

\[
p_1I_n \leq P(t) \leq p_2I_n, \quad \forall t \geq t_0.
\]

In this case, we denote \( P(t) > 0 \) for short. Finally, for an expression \( f(z) \), we denote \( \mathcal{C}^{f(z)} = \{z \in \mathbb{C} : f(z) \text{ is satisfied} \} \). For example, \( \mathcal{C}^{\leq 1} = \{z \in \mathbb{C} : |z| \leq 1\} \).

### 2. Semi-global stabilization of continuous-time periodic systems

#### 2.1. Problems formulation

Consider a linear \( \omega \)-periodic system in the form of

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,
\]

where \( A(t) : [t_0, \infty) \rightarrow \mathbb{R}^{n \times n} \) and \( B(t) : [t_0, \infty) \rightarrow \mathbb{R}^{n \times m} \) are \( \omega \)-periodic matrices, namely,

\[
A(t + \omega) = A(t), \quad B(t + \omega) = B(t), \quad \forall t \in [t_0, \infty).
\]

The problems to be solved in this section for system (2) are stated below.

**Problem 1.** Consider the linear \( \omega \)-periodic system (2). For any given set \( \Omega \subset \mathbb{R}^n \) that is arbitrarily large and bounded, find a linear state feedback control such that the closed-loop system is asymptotically stable with \( \Omega \) contained in the domain of attraction, and, for any \( x_0 \in \Omega \),

1. **(\( L_\infty \) Semi-global stabilization) the control satisfies**
   \[
   \|u\|_{L_\infty} \triangleq \sup_{t \in [t_0, \infty)} |u(t)|_\infty \leq 1.
   \]

2. **(\( L_2 \) Semi-global stabilization) the control satisfies**
   \[
   \|u\|_{L_2} \triangleq \left( \int_{t_0}^{\infty} |u(t)|^2 \, dt \right)^{1/2} \leq 1.
   \]

We have shown in [28] recently that Problem 1 is solvable if and only if the following assumption is satisfied.

**Assumption 1.** The \( \omega \)-periodic matrix pair \((A(t), B(t))\) is stabilizable and \( \mathcal{M}(A(t)) \subset \mathcal{C}^{\leq 1} \).

#### 2.2. Semi-global stabilization by periodic low gain feedback

We consider a parameterized linear state feedback

\[
u(t) = F(\gamma, t)x(t), \quad F(\gamma, t) : (0, D] \times [t_0, \infty) \rightarrow \mathbb{R}^{m \times n},
\]

where \( D > 0 \) is a constant and

\[
F(\gamma, t + \omega) = F(\gamma, t), \quad \forall t \in [t_0, \infty),
\]

is a parameterized \( \omega \)-periodic feedback gain, which is bounded for all \( \gamma \in (0, D) \) and \( t \in [t_0, \infty) \).
Assume that $A(t) + B(t)F(\gamma, t)$ is asymptotically stable for all $\gamma \in (0, D]$. The closed-loop system becomes

$$\dot{x}(t) = (A(t) + B(t)F(\gamma, t))x(t), \quad x(t_0) = x_0.$$  \hfill (8)

It follows that

$$u(t) = F(\gamma, t)\Phi_{A+BF}(t, t_0)x_0,$$  \hfill (9)

in which $\Phi_{A+BF}(t, t_0)$ is the state transition matrix for system (8). With this, we have

$$\|u\|_{L_\infty} = \sup_{t \in [t_0, \infty)} |F(\gamma, t)\Phi_{A+BF}(t, t_0)x_0|_{\infty},$$  \hfill (10)

$$\|u\|_{L_2} = x_0^T \left( \int_{t_0}^{\infty} \Phi_{A+BF}^T(t, t_0)F^T(\gamma, t)F(\gamma, t)\Phi_{A+BF}(t, t_0) \, dt \right)x_0.$$  \hfill (11)

With this observation, we can give the following definition:

**Definition 1.** Assume that $(A(t), B(t))$ satisfies Assumption 1. Let $F(\gamma, t) : (0, D] \times [t_0, \infty) \rightarrow \mathbb{R}^{n \times n}$ be a bounded, $\omega$-periodic in $t$, and such that $A(t) + B(t)F(\gamma, t)$ is asymptotically stable for all $\gamma \in (0, D]$. Then

1. $F(\gamma, t)$ is called a periodic $L_\infty$ low gain for $(A(t), B(t))$ if
   $$\lim_{\gamma \downarrow 0} \sup_{t \in [t_0, \infty)} |F(\gamma, t)\Phi_{A+BF}(t, t_0)| = 0,$$  \hfill (12)

2. $F(\gamma, t)$ is called a periodic $L_2$ low gain for $(A(t), B(t))$ if
   $$\lim_{\gamma \downarrow 0} \int_{t_0}^{\infty} |F(\gamma, t)\Phi_{A+BF}(t, t_0)|^2 \, dt = 0.$$  \hfill (13)

By this definition, we obtain immediately the following result.

**Theorem 1.** Assume that $(A(t), B(t))$ satisfies Assumption 1 and $F(\gamma, t)$ is stated in Definition 1.

1. If $F(\gamma)$ is a periodic $L_\infty$ low gain for $(A(t), B(t))$, then there exists a $\gamma^*_\infty \in (0, D]$ such that the $L_\infty$ semi-global stabilization problem (namely, Item 1 of Problem 1) is solved by the periodic $L_\infty$ low gain feedback (6) for all $\gamma \in (0, \gamma^*_\infty]$.
2. If $F(\gamma)$ is a periodic $L_2$ low gain for $(A(t), B(t))$, then there exists a $\gamma^*_2 \in (0, D]$ such that the $L_2$ semi-global stabilization problem (namely, Item 2 of Problem 1) is solved by the periodic $L_2$ low gain feedback (6) for all $\gamma \in (0, \gamma^*_2]$.

The characterization and design of periodic low gain feedback will be discussed in the next two subsections.

### 2.3. Characterization of periodic $L_\infty$ and $L_2$ low gain feedback

The following proposition is a consequence of Definition 1.

**Proposition 1.** Assume that $(A(t), B(t))$ satisfies Assumption 1 and $F(\gamma, t)$ is stated in Definition 1. Then $F(\gamma, t)$ is an $L_\infty$ (or $L_2$) low gain for $(A(t), B(t))$ only if

$$\lim_{\gamma \downarrow 0} F(\gamma, t) = 0, \quad \forall t \in [t_0, \infty).$$  \hfill (14)
Now we present the following characterization of periodic low gain feedback.

**Theorem 2.** Assume that \((A(t), B(t))\) satisfies Assumption 1 and \(F(\gamma, t)\) is stated in Definition 1. Denote \(A_c(\gamma, t) = A(t) + B(t)F(\gamma, t)\). Then

1. \(F(\gamma, t)\) is a periodic \(L_{\infty}\) low gain for \((A(t), B(t))\) if there exist a scalar \(\gamma^* \in (0, D]\) and a matrix \(P_\infty(t) = P_\infty(\gamma, t) : (0, \gamma^*_\infty) \times [t_0, \infty) \rightarrow R^{n \times n}\), which is continuous in \(\gamma\), positive definite for all \(\gamma \in (0, \gamma^*_\infty]\), \(\omega\)-periodic in \(t\), namely,

\[
P_\infty(\gamma, t) = P_\infty(\gamma, t + \omega), \quad \forall t \in [t_0, \infty),
\]

and such that, for all \(\gamma \in (0, \gamma^*_\infty]\) and for all \(t \geq t_0\),

\[
\frac{\partial}{\partial t} P_\infty(t) + A_c^T(\gamma, t)P_\infty(t) + P_\infty(t)A_c(\gamma, t) < 0,
\]

(16)

\[
F(\gamma, t) \leq P_\infty(t),
\]

(17)

\[
\lim_{\gamma \downarrow \infty} P_\infty(\gamma, t) = 0.
\]

(18)

2. \(F(\gamma, t)\) is a periodic \(L_2\) low gain for \((A(t), B(t))\) if there exist a scalar \(\gamma^*_2 \in (0, D]\) and a matrix \(P_2(t) = P_2(\gamma, t) : (0, \gamma^*_2) \times [t_0, \infty) \rightarrow R^{n \times n}\), which is continuous in \(\gamma\), positive definite for all \(\gamma \in (0, \gamma^*_2]\), \(\omega\)-periodic in \(t\), namely,

\[
P_2(\gamma, t) = P_2(\gamma, t + \omega), \quad \forall t \in [t_0, \infty),
\]

(19)

and such that, for all \(\gamma \in (0, \gamma^*_2]\) and for all \(t \geq t_0\),

\[
\frac{\partial}{\partial t} P_2(t) + A_c^T(\gamma, t)P_2(t) + P_2(t)A_c(\gamma, t) + F(\gamma, t)F(\gamma, t) < 0,
\]

(20)

\[
\lim_{\gamma \downarrow \infty} P_2(\gamma, t) = 0.
\]

(21)

**Proof.** Proof of Item 1: Let \(V_\infty(x) = x^TP_\infty(\gamma, t)x\) whose time-derivative along the trajectories of the closed-loop system \((8)\) satisfies

\[
\dot{V}_\infty(x(t)) = x^T(t)\left(\frac{\partial}{\partial t} P(t) + A_c^T(\gamma, t)P_\infty(t) + P_\infty(t)A_c(\gamma, t)\right)x(t)
\]

\[
< 0, \quad \forall t \geq t_0, \quad \forall x(t) \neq 0,
\]

(22)

where we have used Eq. (16). With this we notice from Eq. (17) that

\[
|u(t)|^2 = x^TF(\gamma, t)F(\gamma, t)x(t)
\]

\[
\leq x^TP_\infty(\gamma, t)x(t)
\]

\[
= V_\infty(x(t))
\]

\[
\leq V_\infty(x(t_0)), \quad \forall t \geq t_0, \quad \forall \gamma \in (0, \gamma^*_\infty]\.
\]

(23)

Hence, for any \(|x_0| < \infty\), it follows from the above relation that

\[
\lim_{\gamma \downarrow \infty} \sup_{t \geq t_0} |u(t)| \leq \lim_{\gamma \downarrow \infty} \sqrt{V_\infty(x(t_0))} = 0,
\]

(24)

where we have noticed Eq. (18). Then by Eq. (9) and the arbitrariness of \(x(t_0)\), we conclude that \(F(\gamma, t)\) satisfies Eq. (12) and is thus a periodic \(L_{\infty}\) low gain for \((A(t), B(t))\).
Without loss of generality, we assume that

\[ x(t) = \frac{\partial}{\partial t}P_2(\gamma, t)x(t) + A_0^T(\gamma, t)P_2(\gamma, t) + P_2(\gamma, t)A_0(\gamma, t) \]

Then by a direct computation, the periodic DRE (28) is equivalent to

\[ -\dot{V}(x(t)) = -|u(t)|^2, \quad \forall t \geq t_0, \quad \forall \gamma \in (0, \gamma_s^*) \].

Taking integration on both sides of Eq. (25) gives

\[
\int_{t_0}^{\infty} |u(t)|^2 \, dt \leq - \int_{t_0}^{\infty} \dot{V}_2(x(t)) \, dt
\]

\[
= V_2(x_0) - \lim_{t \to \infty} V_2(x(t))
\]

\[
= V_2(x_0),
\]

where we have noticed that \( A(t) + B(t)F(\gamma, t) \) is asymptotically stable for all \( \gamma \in (0, \gamma_s^*) \). It then follows from Eq. (21) that

\[
\lim_{\gamma \to 0} \int_{t_0}^{\infty} |u(t)|^2 \, dt \leq \lim_{\gamma \to 0} V_2(x_0) = 0, \quad \forall |x_0| < \infty.
\]

By Eq. (9) and the arbitrariness of \( x_0 \), we conclude that \( F(\gamma, t) \) satisfies Eq. (13) and is thus a periodic \( L_2 \) low gain. The proof is thus completed. \( \square \)

### 2.4. Design of periodic low gain feedback by periodic DRE

In this subsection, we provide a periodic DRE based approach to the design of periodic \( L_\infty \) and \( L_2 \) low gain feedback. To this end, we need the following technical lemma.

**Lemma 1.** Assume that the \( \omega \)-periodic matrix pair \( (A(t), B(t)) \in (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}) \) is stabilizable and \( \mathcal{M}(A(t)) \subset \mathcal{C}^{\omega \leq 1} \). Then the unique \( \omega \)-periodic positive semi-definite solution to the following periodic differential Riccati equation (DRE) is \( P(t) = 0 \):

\[
\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) = 0.
\]

**Proof.** Without loss of generality, we assume that \( P(t) \geq 0 \) takes the following structure:

\[
P(t) = \begin{bmatrix}
0 & 0 \\
0 & P_2(t)
\end{bmatrix},
\]

where \( P_2(t) = P_2(t + \omega) \in \mathbb{R}^{n_2 \times n_2} \) is positive definite for at least a \( t_0 \geq t_0 \). Let \( (A(t), B(t)) \) be partitioned accordingly as

\[
A(t) = \begin{bmatrix}
A_{11}(t) & A_{12}(t) \\
A_{21}(t) & A_{22}(t)
\end{bmatrix}, \quad B(t) = \begin{bmatrix}
B_1(t) \\
B_2(t)
\end{bmatrix}.
\]

Then by a direct computation, the periodic DRE (28) is equivalent to \( P_2(t)A_{21}(t) = 0 \), and

\[
-\dot{P}_2(t) = A_{22}^T(t)P_2(t) + P_2(t)A_{22}(t) - P_2(t)B_2(t)B_2^T(t)P_2(t).
\]

Consider a positive semi-definite matrix function

\[
Q_2(t) = Q_2(t, t_0) = \Phi_{A_{22}}^T(t, t_0)P_2(t)\Phi_{A_{22}}(t, t_0), \quad \forall t \geq t_0.
\]
Then it follows from Eq. (31) that
\[
\dot{Q}_{2}(t) = \left( \frac{\partial}{\partial t} \Phi_{A_{22}}(t, s) \right)^T P_{2}(t) \Phi_{A_{22}}(t, s) + \Phi_{A_{22}}^T(t, s) P_{2}(t) \left( \frac{\partial}{\partial t} \Phi_{A_{22}}(t, s) \right) \\
+ \Phi_{A_{22}}^T(t, s) \dot{P}_{2}(t) \Phi_{A_{22}}(t, s) \\
= \Phi_{A_{22}}^T(t, s) \left( A_{22}^T(t) P_{2}(t) + P_{2}(t) A_{22}(t) \right) \Phi_{A_{22}}(t, s) \\
+ \Phi_{A_{22}}^T(t, s) \dot{P}_{2}(t) \Phi_{A_{22}}(t, s) \\
= \Phi_{A_{22}}^T(t, s) \left( \dot{P}_{2}(t) + A_{22}^T(t) P_{2}(t) + P_{2}(t) A_{22}(t) \right) \Phi_{A_{22}}(t, s) \\
\geq 0. \tag{33}
\]
Since \(Q(t)\) is positive definite with \(t = t_p\), we conclude from Eq. (33) that it is also positive definite for all \(t \geq t_p\). The same is \(P_{2}(t)\) since \(\Phi_{A_{22}}(t, t_0)\) is nonsingular for all \(t\) and \(t_0\). As \(P_{2}(t)\) is \(\omega\)-periodic, we conclude that \(P_{2}(t)\) is positive definite for all \(t \geq t_0\).

Hence it follows from \(P_{2}(t)A_{21}(t) = 0\) that \(A_{21}(t) = 0\) and, as a result, \(A(t)\) can be rewritten as
\[
A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix}, \quad \forall t \geq t_0. \tag{34}
\]
Rearrange the reduced-order periodic DRE (31) as
\[
\dot{W}_{2} = W_{2}(t) A_{22}^T(t) + A_{22}(t) W_{2}(t) - B_{2}(t) B_{2}^T(t), \tag{35}
\]
where \(W_{2}(t) = P_{2}^{-1}(t)\). Consider the linear periodic system \(\dot{x}_{2}(t) = -A_{22}^T(t)x(t), \forall t \geq t_0,\) and the Lyapunov function \(V(x_2) = x_{2}^T W_{2}(t)x_{2}.\) Then it follows from Eq. (35) that
\[
\dot{V}(x_{2}(t)) = -\left| B_{2}^T(t)x_{2}(t) \right|^2 \leq 0, \quad \forall t \geq t_0. \tag{36}
\]
By the Lyapunov stability theory, such a system is marginally stable. Hence \(\mathcal{M}(-A_{22}^T) \subset \mathbb{C}^{2|}\leq 1\}, or equivalently, \(\mathcal{M}(A_{22}) \subset \mathbb{C}^{2|\geq 1\}.\) As \((A_{22}(t), B_{2}(t))\) is stabilizable, we conclude that \((A_{22}(t), B_{2}(t))\) must be further controllable. Therefore, by standard Lyapunov equation theory, we further get from Eq. (35) that \(\mathcal{M}(-A_{22}^T) \subset \mathbb{C}^{2|<1\},\) or equivalently, \(\mathcal{M}(A_{22}) \subset \mathbb{C}^{2|>1\}.\) However, this contradicts with
\[
\mathcal{M}(A_{22}) \subset \mathcal{M}(A) \subset \mathbb{C}^{2|\leq 1}. \tag{37}
\]
Hence we have \(P(t) = 0\) and the proof is finished. \(\Box\)

Then we can present the following lemma.

**Lemma 2** (Periodic DRE based periodic low gain feedback). Assume that \((A(t), B(t))\) satisfies Assumption 1. Let \(Q(\gamma, t) : (0, D) \times [t_0, \infty) \to \mathbb{R}^{n \times n}\) be continuously differentiable on \(\gamma\), positive definite, and such that
\[
Q(\gamma, t + \omega) = Q(\gamma, t) \geq 0, \quad \frac{\partial}{\partial \gamma} Q(\gamma, t) \geq 0 \quad \lim_{\gamma \to 0} Q(\gamma, t) = 0, \tag{38}
\]
and \((A(t), Q(\gamma, t))\) is observable for all \(\gamma \in (0, D].\) Then the following periodic DRE
\[
-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(\gamma, t), \tag{39}
\]
has a unique \(\omega\)-periodic positive definite solution \(P(t) = P(\gamma, t)\) for all \(\gamma \in (0, D]\). Let

\[
F(\gamma, t) = -B^T(t)P(t).
\]

(40)

Then the closed-loop system matrix \(A_c(t) = A(t) + B(t)F(\gamma, t)\) is asymptotically stable. Moreover,

\[
\frac{\partial}{\partial \gamma} P(\gamma, t) \geq 0, \quad \lim_{t \to 0} P(\gamma, t) = 0.
\]

(41)

Particularly, \((\partial / \partial \gamma) P(\gamma, t) > 0\) if \((\partial / \partial \gamma) Q(\gamma, t) > 0\).

Proof. The existence of a periodic positive definite solution \(P(\gamma, t)\) to the periodic DRE (39) for any \(\gamma \in (0, D]\) is a standard result and can be found, for example, in [17]. Moreover, it is also known that such a periodic positive definite solution \(P(\gamma, t)\) is a stabilizing solution, namely, \(A_c(t)\) is asymptotically stable.

Taking partial derivative on both sides of Eq. (39) with respect to \(\gamma\) gives

\[
\frac{d}{dt} \frac{\partial P(\gamma, t)}{\partial \gamma} + A_c^T(t) \frac{\partial P(\gamma, t)}{\partial \gamma} + \frac{\partial P(\gamma, t)}{\partial \gamma} A_c(t) = - \frac{\partial Q(\gamma, t)}{\partial \gamma}.
\]

(42)

Since \(Q(\gamma, t)\) is \(\omega\)-periodic in \(\gamma\), we know that \((\partial / \partial \gamma) Q(\gamma, t)\) is also \(\omega\)-periodic in \(t\). Hence \((\partial / \partial \gamma) Q(\gamma, t)\) is an \(\omega\)-periodic positive semi-definite matrix. As a result, it follows from the asymptotic stability of \(A_c(t)\) and the standard Lyapunov equation theory that \((\partial / \partial \gamma) P(\gamma, t)\) is an \(\omega\)-periodic positive semi-definite matrix. Moreover, \((\partial / \partial \gamma) P(\gamma, t) > 0\) if \((\partial / \partial \gamma) Q(\gamma, t) > 0\).

Consequently, \(\lim_{t \to 0} P(\gamma, t)\) exists and \(P_0(t) = \lim_{t \to 0} P(\gamma, t) \geq 0, \forall t \in \mathbb{R}\). Taking limits on both sides of Eq. (39) gives

\[
-\dot{P}_0(t) = A^T(t)P_0(t) + P_0(t)A(t) - P_0(t)B(t)B^T(t)P_0(t).
\]

(43)

Since \((A(t), B(t))\) is stabilizable and \(\mathcal{M}(A(t)) \subset C^{[d]} \leq 1\), by Lemma 1, we have \(P_0(t) = 0\). The proof is finished. \(\square\)

By this lemma we can show immediately the following result.

Proposition 2. Let the gain \(F(\gamma, t)\) be defined in Eq. (40). Then \(F(\gamma, t)\) is both a periodic \(L_\infty\) and \(L_2\) low gain in the sense of Definition 1.

Proof. Since \(\lim_{t \to 0} P(\gamma, t) = 0\), it follows from

\[
F^T(\gamma, t)F(\gamma, t) = P(t)B(t)B^T(t)P(t) \\
\leq \text{tr}(B^T(t)P(\gamma, t)B(t))P(\gamma, t),
\]

(44)

that there exists a \(\gamma^*_\infty > 0\) such that

\[
F^T(\gamma, t)F(\gamma, t) \leq P(\gamma, t), \quad \forall \gamma \in (0, \gamma^*_\infty].
\]

(45)

On the other hand, the periodic DRE (39) can be written as

\[
\dot{P}(t) + A_c^T(t)P(t) + P(t)A_c(t) = -Q(\gamma, t) - F^T(\gamma, t)F(\gamma, t).
\]

(46)

It follows that the gain \(F(\gamma, t)\) satisfies Eqs. (16)–(18) with \(P_\infty(\gamma, t) = P(\gamma, t)\) and is thus a periodic \(L_\infty\) low gain for \((A(t), B(t))\) by Item 1 of Theorem 2.

The inequality (46) is also in the form of Eq. (20), and hence, by Item 2 of Theorem 2, \(F(\gamma, t)\) is also a periodic \(L_2\) low gain for \((A(t), B(t))\). The proof is finished. \(\square\)
It is clear that if \((A(t), B(t))\) and \(Q(\gamma, t)\) are independent of \(t\), the developed approach reduces to the existing ARE based low gain design approach for linear time-invariant systems [12] (we point out that our result is more general since \(Q(\gamma)\) is only required to be positive semi-definite here).

2.5. Semi-global stabilization by dynamic output feedback

In this subsection, we will solve the observer based output feedback \(L_\infty\) semi-global stabilization problem by assuming that the linear periodic system (2) has an output

\[
y(t) = C(t)x(t), \quad C(t) \in \mathbb{R}^{p \times n},
\]

in which the \(\omega\)-periodic matrix pair \((A(t), C(t))\) is assumed to be detectable.

**Theorem 3.** Let Assumption 1 be satisfied. Then there exists a \(\gamma^* > 0\) such that Problem 1 (notice that in this case \(\Omega \subset \mathbb{R}^{2n}\)) is solved by the observer based periodic low gain feedback

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + L(t)(C(t)x(t) - y(t)), \\
u(t) &= F(\gamma, t)x(t), \quad \forall \gamma \in (0, \gamma^*),
\end{align*}
\]

where \(F(\gamma, t)\) is defined in Eq. (40) and \(L(t)\) is any \(\omega\)-periodic matrix such that \(A(t) + L(t)C(t)\) is asymptotically stable.

**Proof.** In the absence of magnitude saturation, the closed-loop system can be expressed as

\[
\dot{\eta}(t) = (A_\eta(t) + B_\eta(t)F_\eta(\gamma, t))\eta(t), \quad u(t) = F_\eta(\gamma, t)\eta(t),
\]

in which \(\eta = [x^T, x^T - \hat{x}^T]^T\) and

\[
A_\eta(t) = \begin{bmatrix} A(t) & 0 \\ 0 & A(t) + L(t)C(t) \end{bmatrix}, \quad B_\eta = \begin{bmatrix} B(t) \\ 0 \end{bmatrix},
\]

\[
F_\eta(\gamma, t) = [F(\gamma, t) - F(\gamma, t)].
\]

Hence in the following, we need only to verify that \(F_\eta(t) = F_\eta(\gamma, t)\) is an \(L_\infty\) low gain for \((A_\eta(t), B_\eta(t))\).

Let \(\hat{R}(t)\) be an \(\omega\)-periodic positive definite matrix function that solves the following Lyapunov differential equation:

\[
\dot{R}(t) + (A(t) + L(t)C(t))^T R(t) + R(t)(A(t) + L(t)C(t)) = -I_n.
\]

We also denote

\[
P_\eta(t) = P_\eta(\gamma, t) = \begin{bmatrix} P(\gamma, t) & 0 \\ 0 & \phi(\gamma)R(t) \end{bmatrix} > 0, \quad \forall t > t_0,
\]

where \(P(t) = P(\gamma, t)\) solves the periodic DRE (39) and the scalar function \(\phi(\gamma)\) is defined as

\[
\phi(\gamma) = \sup_{t \geq t_0} |F(\gamma, t)| = \max_{t \in [t_0, t_0 + \omega]} |F(\gamma, t)|.
\]

By the property of \(P(t)\), we clearly have

\[
\lim_{\gamma \downarrow 0} P_\eta(\gamma, t) = 0.
\]
Therefore, there exists a $\gamma^* > 0$ such that
\[
F_\eta(t)P_\eta^{-1}(t)F_\eta^T(t) = F(\gamma, t)P^{-1}(t)F^T(\gamma, t) + \frac{1}{\phi(\gamma)} F(\gamma, t)R^{-1}(t)F^T(\gamma, t)
\]
\[
\leq \begin{bmatrix} B^T(t)P(t)B(t) + |F(\gamma, t)R^{-1}(t)| I_m \end{bmatrix} = I_m, \quad \forall \gamma \in (0, \gamma^*],
\]
from which it follows that, by a Schur complement,
\[
F_\eta^T(\gamma, t)F_\eta(\gamma, t) \leq P_\eta(\gamma, t), \quad \forall \gamma \in (0, \gamma^*].
\] (57)

On the other hand, by the periodic DRE (39), we can compute
\[
\frac{\partial}{\partial t} P_\eta(\gamma, t) + A_c^T(\gamma, t)P_\eta(\gamma, t) + P_\eta(\gamma, t)A_c(\gamma, t)
\]
\[
= \begin{bmatrix} -Q(\gamma, t) - F^T(\gamma, t)F(\gamma, t) & -P(\gamma, t)B(t)F(\gamma, t) \\
-F^T(\gamma, t)B^T(t)P(\gamma, t) & -\phi(\gamma)I_n \end{bmatrix},
\] (58)
where $Q(\gamma)$ is defined in Lemma 2 and $A_c(\gamma, t) = A_\eta(t) + B_\eta(t)F_\eta(\gamma, t)$. Consider the Schur complement of the above matrix:
\[
- Q(\gamma, t) - F^T(\gamma, t)F(\gamma, t) + \frac{1}{\phi(\gamma)} P(\gamma, t)B(t)F(\gamma, t)F^T(\gamma, t)B^T(t)P(\gamma, t)
\]
\[
\leq - Q(\gamma, t) - (1 - |F(\gamma, t)|^2)I_m F^T(\gamma, t)F(\gamma, t).
\] (59)

It follows from $\lim_{\gamma \to 0} F(\gamma, t) = 0$ that there exists a $\gamma^*_1 \in (0, \gamma^*]$ such that $1 - |F(\gamma, t)|^2 \geq 0, \forall \gamma \in (0, \gamma^*_1], \forall t \geq t_0$. As a result,
\[
\frac{\partial}{\partial t} P_\eta(\gamma, t) + A_c^T(\gamma, t)P_\eta(\gamma, t) + P_\eta(\gamma, t)A_c(\gamma, t) < 0, \quad \forall \gamma \in (0, \gamma^*_1].
\] (60)

Notice that Eqs. (60), (57), and (55) are respectively in the form of Eqs. (16), (17) and (18); hence it follows from Item 1 of Theorem 2 that $F_\eta(\gamma, t)$ is a periodic $L_\infty$ low gain for $(A_\eta(t), B_\eta(t))$.

By a similar approach we can show that $F_\eta(\gamma, t)$ is also a periodic $L_2$ low gain for $(A_\eta(t), B_\eta(t))$. The proof is finished. $\square$

3. Semi-global stabilization of discrete-time periodic systems

3.1. Problems formulation

Consider the following discrete-time linear periodic system:
\[
x(k + 1) = A(k)x(k) + B(k)u(k), \quad \forall k \geq k_0
\] (61)
where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are, respectively, the state vector and the input vector, and the matrices $A(k) \in \mathbb{R}^{n \times n}$ and $B(k) \in \mathbb{R}^{n \times m}$ are $\omega$-periodic, where $\omega \geq 1$ is an integer, namely,
\[
A(k + \omega) = A(k), \quad B(k + \omega) = B(k), \quad \forall k \geq k_0
\] (62)

For this system, the problem to be solved in this section is stated as follows.
Problem 2. Consider the discrete-time $\omega$-periodic linear system (61). For any given set $\Omega / C26 \subset \mathbb{R}^n$ that is arbitrarily large yet bounded, find a linear state feedback control such that the closed-loop system is asymptotically stable with $\Omega$ contained in the domain of attraction, and, for any $x_0 \in \Omega$,

1. (l_\infty Semi-global stabilization) the control satisfies
   $$\|u\|_{l_\infty} \triangleq \sup_{k \in \mathbb{I}[k_0, \infty)} |u(k)|_\infty \leq 1.$$  
   (63)

2. (l_2 Semi-global stabilization) the control satisfies
   $$\|u\|_{l_2} \triangleq \left( \sum_{k=k_0}^{\infty} |u(k)|^2 \right)^{1/2} \leq 1.$$  
   (64)

We have shown recently in [25,33] that Problem 2 is solvable if and only if the following assumption for system (61) is satisfied.

Assumption 2. The discrete-time $\omega$-periodic linear system (61) is stabilizable and $\mathcal{M}(A(k)) \subset C[|z| \leq 1].$

3.2. Semi-global stabilization by periodic low gain feedback

For the discrete-time linear periodic system (61) we consider the following parametrized periodic linear state feedback:

$$u(k) = F(\gamma, k)x(k), \quad F(\gamma, k) : (0, D] \times \mathbb{I}[k_0, \infty) \to \mathbb{R}^{m \times n},$$  
(65)

where $D>0$ is some constant and

$$F(\gamma, k + \omega) = F(\gamma, k), \quad \forall k \in \mathbb{I}[k_0, \infty),$$  
(66)

is a parameterized periodic feedback gain, which is bounded for all $\gamma \in (0, D]$ and $k \in \mathbb{I}[k_0, \infty).$

Assume that $A(k) + B(k)F(\gamma, k)$ is asymptotically stable for all $\gamma \in (0, D].$ Write the closed-loop system as

$$x(k+1) = (A(k) + B(k)F(\gamma, k))x(k), \quad x(k_0) = x_0.$$  
(67)

Notice that the closed-loop system is also an $\omega$-periodic linear system. Hence, it follows from

$$u(k) = F(\gamma, k)\Phi_{A+BF}(k, k_0)x_0,$$  
(68)

that we can write

$$\|u\|_{l_\infty} = \sup_{k \in \mathbb{I}[k_0, \infty)} |F(\gamma, k)\Phi_{A+BF}(k, k_0)|_\infty,$$  
(69)

$$\|u\|_{l_2} = x_0^T \left( \sum_{k=k_0}^{\infty} \Phi_{A+BF}^T(k, k_0)F^T(\gamma, k)F(\gamma, k)\Phi_{A+BF}(k, k_0) \right) x_0,$$  
(70)

where $\Phi_{A+BF}(k, k_0)$ is the state transition matrix for system (67).

With this observation, similar to the continuous-time setting, we can give the following definition:
Definition 2. Assume that $(A(k), B(k))$ satisfies Assumption 2. Let $F(\gamma, k) : (0, D] \times I[k_0, \infty) \to \mathbb{R}^{m \times n}$ be a bounded, satisfy Eq. (66), and such that $A(k) + B(k)F(\gamma, k)$ is asymptotically stable for all $\gamma \in (0, D]$. Then

1. $F(\gamma, k)$ is called a periodic $l_\infty$ low gain for the $\omega$-periodic linear system $(A(k), B(k))$ if
   \[
   \lim_{\gamma \to 0} \sup_{k \in I[k_0, \infty)} \left| F(\gamma, k)\Phi_{A+BF}(k, k_0) \right|_\infty = 0, \tag{71}
   \]
2. $F(\gamma, k)$ is called a periodic $l_2$ low gain for the $\omega$-periodic linear system $(A(k), B(k))$ if
   \[
   \lim_{\gamma \to 0} \sum_{k=k_0}^\infty \left| F(\gamma, k)\Phi_{A+BF}(k, k_0) \right|^2 = 0. \tag{72}
   \]

The following theorem is then a consequence of the above definition.

Theorem 4. Assume that $(A(k), B(k))$ satisfies Assumption 2 and $F(\gamma, k)$ is stated in Definition 2.

1. If $F(\gamma, k)$ is a periodic $l_\infty$ low gain for $(A(k), B(k))$, then there exists a $\gamma^*_\infty \in (0, D]$ such that the $l_\infty$ semi-global stabilization problem (namely, Item 1 of Problem 2) is solved by the periodic $l_\infty$ low gain feedback (65) for all $\gamma \in (0, \gamma^*_\infty]$.
2. If $F(\gamma, k)$ is a periodic $l_2$ low gain for $(A(k), B(k))$, then there exists a $\gamma^*_2 \in (0, D]$ such that the $l_2$ semi-global stabilization problem (namely, Item 2 of Problem 2) is solved by the $l_2$ low gain feedback (65) for all $\gamma \in (0, \gamma^*_2]$.

The characterization and design of periodic low gain feedback will be discussed in the next two subsections.

3.3. Characterization of periodic $l_\infty$ and $l_2$ low gain feedback

Parallel to the continuous-time setting, we have the following characterization of periodic low gain feedback.

Theorem 5. Assume that $(A(k), B(k))$ satisfies Assumption 2 and $F(\gamma, k)$ is stated in Definition 2. Denote $A_c(\gamma, k) = A(k) + B(k)F(\gamma, k)$. Then

1. $F(\gamma, k)$ is a periodic $l_\infty$ low gain for $(A(k), B(k))$ if there exist a scalar $\gamma^*_\infty \in (0, D]$ and a matrix $P_\infty(k) = P_\infty(\gamma, k) : (0, \gamma^*_\infty] \times I[k_0, \infty) \to \mathbb{R}^{n \times n}$, which is continuous in $\gamma$, positive definite for all $\gamma \in (0, \gamma^*_\infty]$, $\omega$-periodic in $k$, namely,
   \[
   P_\infty(\gamma, k) = P_\infty(\gamma, k + \omega), \quad \forall k \in I[k_0, \infty), \tag{73}
   \]
   and such that, for all $\gamma \in (0, \gamma^*_\infty]$ and for all $k \geq k_0$,
   \[
   A_c^T(\gamma, k)P_\infty(k + 1)A_c(\gamma, k) - P_\infty(k) < 0, \tag{74}
   \]
   \[
   F^T(\gamma, k)F(\gamma, k) \leq P_\infty(k), \tag{75}
   \]
   \[
   \lim_{\gamma \to 0} P_\infty(\gamma, k) = 0. \tag{76}
   \]
2. $F(\gamma, k)$ is a periodic $l_2$ low gain for $(A(k), B(k))$ if there exist a scalar $\gamma^*_2 \in (0, D]$ and a matrix $P_2(k) = P_2(\gamma, k) : (0, \gamma^*_2] \times I[k_0, \infty) \to \mathbb{R}^{n \times n}$, which is continuous in $\gamma$, positive definite for all
\( \gamma \in (0, \gamma_2^b] \), \( \omega \)-periodic in \( k \), namely,
\[
P_2(\gamma, k) = P_2(\gamma, k + \omega), \quad \forall k \in \mathbb{I}[k_0, \infty),
\] (77)
and such that, for all \( \gamma \in (0, \gamma_2^b] \) and for all \( k \geq k_0 \),
\[
A_c^T(\gamma, k)P_2(k + 1)A_c(\gamma, k) - P_2(k) + F^T(\gamma, k)F(\gamma, k) < 0,
\] (78)
\[
\lim_{\gamma \downarrow 0} P_2(\gamma, k) = 0.
\] (79)

**Proof.** Proof of Item 1: Let \( V_\infty(x) = x^TP_\infty(\gamma, k)x \) whose time-shift along the trajectories of the closed-loop system (67) satisfies
\[
V_\infty(x(k + 1)) - V_\infty(x(k)) = x^T(k)(A_c^T(\gamma, k)P_\infty(k + 1)A_c(\gamma, k) - P_\infty(k))x(k)
\]
\[
< 0, \quad \forall k \geq k_0, \ \forall x(k) \neq 0,
\] (80)
where we have used Eq. (74). Now we notice from Eq. (75) that
\[
|u(k)|^2 = x^T(k)F^T(\gamma, k)F(\gamma, k)x(k)
\]
\[
\leq x^T(k)P_\infty(\gamma, k)x(k)
\]
\[
= V_\infty(x(k))
\]
\[
\leq V_\infty(x_0), \quad \forall k \in \mathbb{I}[k_0, \infty).
\] (81)
Hence, by definition, we get from Eq. (76) that
\[
\lim_{\gamma \downarrow 0} \|u(k)\|_{\ell_\infty} = \lim_{\gamma \downarrow 0} \sup_{k \geq k_0} |u(k)|_\infty \leq \lim_{\gamma \downarrow 0} \sqrt{V_\infty(x(k_0))} = 0.
\] (82)
Then by Eq. (68) and the arbitrariness of \( x_0 \), we conclude from the above relation that \( F(\gamma, k) \) satisfies Eq. (71) and is thus a periodic \( l_\infty \) low gain for \( (A(k), B(k)) \). The proof is completed.

**Proof of Item 2:** For the closed-loop system (67), the time-shift of the positive definite function \( V_2(x) = x^TP_2(k)x \) satisfies
\[
V_2(x(k + 1)) - V_2(x(k)) = x^T(k)(A_c^T(\gamma, k)P_2(k + 1)A_c(\gamma, k) - P_2(k))x(k)
\]
\[
< -x^T(k)F^T(\gamma, k)F(\gamma, k)x(k)
\]
\[
= -|u(k)|^2, \quad \forall k \geq k_0, \ \forall \gamma \in (0, \gamma_2^b].
\] (83)
Taking sum on both sides of Eq. (83) gives
\[
\sum_{k = k_0}^{\infty} |u(k)|^2 \leq - \sum_{k = k_0}^{\infty} (V_2(x(k + 1)) - V_2(x(k)))
\]
\[
= V_2(x_0) - \lim_{k \to \infty} V_2(x(k))
\]
\[
= V_2(x_0), \quad \forall \gamma \in (0, \gamma_2^b],
\] (84)
where we have noticed that \( A(k) + B(k)F(\gamma, k) \) is asymptotically stable. It then follows from Eq. (79) that
\[
\lim_{\gamma \downarrow 0} \sum_{k = k_0}^{\infty} |u(k)|^2 \leq \lim_{\gamma \downarrow 0} V_2(x_0) = 0, \quad \forall |x_0| < \infty.
\] (85)
By Eq. (68) and the arbitrariness of \( x_0 \), we conclude that \( F(\gamma, k) \) satisfies Eq. (72) and is a periodic \( l_2 \) low gain for \( (A(k), B(k)) \). The proof is thus completed. \( \square \)
Remark 1. It is not difficult to see from the proof of Theorem 5 that $\mathcal{E}(P_{\infty}(k))$ is a contractively invariant set for the closed-loop system in the $l_{\infty}$ case, and $\mathcal{E}(P_{2}(k))$ is a contractively invariant set for the closed-loop system in the $l_{2}$ case.

3.4. Design of periodic low gain feedback by periodic DcRE

In this subsection, we provide a periodic DcRE approach to the design of periodic $l_{\infty}$ and $l_{2}$ low gain feedback. We first need the following lemma.

Lemma 3. Assume that the $\omega$-periodic matrix pair $(A(k), B(k)) \in (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}), k \in \mathbb{N}[0, \infty)$, is stabilizable and $\mathcal{M}(A) \subset \mathbb{C}[\omega] \leq 1$. Then the unique positive semi-definite solution to the following DcRE

$$0 = A^T(k)P(k + 1)A(k) - P(k) - A^T(k)P(k + 1)B(k)S^{-1}(k)B^T(k)P(k + 1)A(k),$$

is $P = 0$, where $S(k) = I_m + B^T(k)P(k + 1)B(k)$.

Proof. Assume that $P(k)$ takes the form

$$P(k) = \begin{bmatrix} 0 & 0 \\ 0 & P_2(k) \end{bmatrix},$$

where $P_2(k) = P_2(k + \omega), k \geq k_0$ is positive semi-definite and is positive definite for at least one $k = k_p \geq k_0$. The matrices $A(k)$ and $B(k)$ are partitioned accordingly as

$$A = \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{bmatrix}, \quad B = \begin{bmatrix} B_1(k) \\ B_2(k) \end{bmatrix}.$$  

Then the periodic DcRE (86) reduces to

$$\begin{cases} 0 = A_{21}^T(k)P_2(k + 1)A_{22}(k) - A_{21}^T(k)P_2(k + 1)B_1(k)S^{-1}(k)B_1^T(k)P_2(k + 1)A_{22}(k), \\ 0 = A_{21}^T(k)P_2(k + 1)A_{21}(k) - A_{21}^T(k)P_2(k + 1)B_1(k)S^{-1}(k)B_1^T(k)P_2(k + 1)A_{21}(k), \\ P_2(k) = A_{22}^T(k)P_2(k + 1)A_{22}(k) - A_{22}^T(k)P_2(k + 1)B_2(k)S^{-1}(k)B_2^T(k)P_2(k + 1)A_{22}(k), \end{cases}$$

where $S_2(k) = I_m + B_2^T(k)P_2(k + 1)B_2(k) > 0$. It follows from the third equation in the above that

$$A_{22}^T(k)P_2(k + 1)A_{22}(k) \geq P_2(k).$$

This implies that $A_{22}(k)$ is nonsingular and $P_2(k + 1) > 0$ with $k = k_p$. Applying the above procedure recursively we conclude that $A_{22}(k)$ is nonsingular and $P_2(k) > 0$ for all $k \geq k_0$ (we have also noticed that they are periodic). Hence from the first equation of Eq. (89) we can derive

$$A_{21}^T(k) = A_{21}^T(k)P_2(k + 1)B_1(k)S^{-1}(k)B_2^T(k),$$

substitution of which into the second equation of Eq. (89) gives

$$0 = -A_{21}^T(k)P_2(k + 1)B_1(k)S^{-1}(k)S^{-1}(k)B_1^T(k)P_2(k + 1)A_{21}(k).$$

Since $S^{-1}(k)S^{-1}(k) > 0$, the above equation implies $B_1^T(k)P_2(k + 1)A_{21}(k) = 0$. Hence we know...
from Eq. (91) that $A_{21}(k) = 0$. Thus $A(k)$ takes the form

$$A(k) = \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{bmatrix}. \quad (93)$$

Now rewrite the third equation in Eq. (89) as

$$P_2(k) = A_{22}^T(k)(P_2^{-1}(k + 1) + B_{22}(k)B_{22}^T(k))^{-1}A_{22}(k),$$

which, by using the Sherman–Morrison–Woodbury formula (Lemma C.6 in [26]), is equivalent to

$$A_{22}^{-1}(k)P_2^{-1}(k + 1)A_{22}^{-T}(k) - P_2^{-1}(k) = -A_{22}^{-1}(k)B_2(k)B_2^T(k)A_{22}^{-T}(k). \quad (95)$$

Consider the discrete-time linear periodic system $x_2(k + 1) = A_{22}^{-T}(k)x(k)$ and a Lyapunov function $V(x_2(k)) = x_2^T(k)P_2^{-1}(k)x_2(k)$. Then

$$V(x_2(k + 1)) - V(x_2(k)) = -B_2^T(k)A_{22}^{-T}(k)x_2(k)^2,$$

which, by the Lyapunov stability theory, implies that the $x_2$-system is marginally stable. Therefore, $\mathcal{M}(A_{22}^T) \subset C^{[\gamma]} \leq 1$, or equivalently, $\mathcal{M}(A_{22}) \subset C^{[\gamma]} \geq 1$. As $(A_{22}(k), B_2(k))$ is stabilizable, we conclude that $(A_{22}(k), B_2(k))$ is moreover controllable. By some intricate yet straightforward computation, we can verify that $(A_{22}^{-1}(k), A_{22}^{-1}(k)B_2(k))$ is also controllable. Then, by the standard Lyapunov equation theory, we know from $P_2(k) > 0$ that $\mathcal{M}(A_{22}^T) \subset C^{[\gamma]} \leq 1$, or equivalently, $\mathcal{M}(A_{22}) \subset C^{[\gamma]} \geq 1$, which contradicts with $\lambda(A_{22}) \subset \lambda(A) \subset C^{[\gamma]} \leq 1$. Hence we have $P = 0$ and the proof is finished. \(\square\)

Then we can present the following lemma, which improves our recent results in [33].

**Lemma 4** (Periodic DcRE based periodic low gain feedback). Assume that $(A(k), B(k))$ satisfies Assumption 2. Let $Q(\gamma, k) : (0, D) \times [k_0, \infty) \rightarrow \mathbb{R}^{n \times n}$ be continuously differentiable on $\gamma$, positive definite, and such that

$$Q(\gamma, k + \omega) = Q(\gamma, k) \geq 0, \quad \frac{\partial}{\partial \gamma} Q(\gamma, k) \geq 0, \quad \lim_{\gamma \downarrow 0} Q(\gamma, k) = 0,$$

and $(A(k), Q(\gamma, k))$ is observable for all $\gamma \in (0, D]$. Then the following periodic DcRE

$$0 = A^T(k)P(k + 1)A(k) - P(k) + Q(\gamma, k)$$

$$-A^T(k)P(k + 1)B(k)S^{-1}(k)B^T(k)P(k + 1)A(k), \quad (98)$$

where $S(k) = I_n + B^T(k)P(k + 1)B(k)$, has a unique $\omega$-periodic positive definite solution $P(k) = P(\gamma, k) = P(\gamma, k + \omega)$. Let

$$F(k) = -S^{-1}(k)B^T(k)P(k + 1)A(k). \quad (99)$$

Then the closed-loop system matrix $A_c(k) = A(k) + B(k)F(\gamma, k)$ is asymptotically stable. Moreover,

$$\frac{\partial}{\partial \gamma} P(\gamma, k) \geq 0, \quad \lim_{\gamma \downarrow 0} P(\gamma, k) = 0.$$

(100)

Particularly, $(\partial / \partial \gamma) P(\gamma, k) > 0$ if $(\partial / \partial \gamma) Q(\gamma, k) > 0$. 

Proof. It is a standard result that the periodic DcRE (98) has a periodic positive definite solution $P(k) = P(\gamma, k)$ for any $\gamma \in (0,D]$ if $(A(k), B(k))$ is stabilizable and $(A(k), Q(\gamma, k))$ is observable (see [1]). Moreover, such a periodic positive definite solution $P(k)$ is also a stabilizing solution, namely, $A_c(k)$ is asymptotically stable.

Differential on both sides of Eq. (98) with respect to $\gamma$ gives

$$A_c^T(k) \frac{\partial P(\gamma, k + 1)}{\partial \gamma} A_c(k) - \frac{\partial P(\gamma, k)}{\partial \gamma} = - \frac{\partial Q(\gamma, k)}{\partial \gamma}, \quad \forall k \geq k_0.$$  \hfill (101)

Since $A_c(k)$ is asymptotically stable and $(\partial/\partial \gamma)Q(\gamma, k) \geq 0$, it follows from the standard Lyapunov equation theory that the above periodic Lyapunov difference equation has a unique $\omega$-periodic semi-positive definite solution $(\partial/\partial \gamma)P(\gamma, k) \geq 0, \forall k \geq k_0, \forall \gamma \in (0,D]$. Particularly, $(\partial/\partial \gamma)P(\gamma, k) > 0$ if $(\partial/\partial \gamma)Q(\gamma, k) > 0$.

It follows that $\lim_{\gamma \downarrow 0} P(\gamma, k)$ exists and we denote it by $\lim_{\gamma \downarrow 0} P(\gamma, k) = P_0(k) \geq 0$. Then $P_0(k)$ satisfies

$$0 = A_c^T(k)P_0(k + 1)A_c(k) - P_0(k) - A_c^T(k)P_0(k + 1)B(k)S_0^{-1}(k)B^T(k)P_0(k + 1)A_c(k),$$  \hfill (102)

where $S_0(k) = I_m + B^T(k)P_0(k + 1)B(k)$. Since $(A(k), B(k))$ is stabilizable and $\mathcal{M}(A(k)) \subset C_{\|\|} \leq 1$, it follows from Lemma 3 that $P_0(k) = 0$. The proof is finished. \hfill $\square$

Rearrange the periodic DcRE (98) as

$$A_c^T(k)P(k + 1)A_c(k) - P(k) = -Q(k) - F^T(\gamma, k)F(\gamma, k),$$  \hfill (103)

which implies

$$F^T(\gamma, k)F(\gamma, k) \leq P(k), \quad \forall k \geq k_0.$$  \hfill (104)

With this, we get immediately the following proposition.

**Proposition 3.** Let the gain $F(\gamma, k)$ be defined in Eq. (99). Then $F(\gamma, k)$ is both a periodic $l_\infty$ low gain and a periodic $l_2$ low gain in the sense of Definition 2. Particularly, all the conditions of Theorem 5 are satisfied with $P_\infty(\gamma, k) = P_2(\gamma, k) = P(\gamma, k)$ where $P(\gamma, k)$ is defined in Lemma 4.

3.5. Semi-global stabilization by dynamic output feedback

We assume that the discrete-time linear periodic system (61) possesses an output equation

$$y(k) = C(k)x(k), \quad k \geq k_0,$$  \hfill (105)

where $C(k) \in \mathbb{R}^{p \times n}$ is also an $\omega$-periodic matrix function and such that $(A(k), C(k))$ is detectable.

**Theorem 6.** Let the $\omega$-periodic matrix pair $(A(k), B(k))$ satisfies Assumption 2 and the $\omega$-periodic matrix pair $(A(k), C(k))$ is detectable. Then there exists a $\gamma^* > 0$ such that the following family of observer based output feedback laws

$$\begin{align*}
\dot{x}(k + 1) &= A(k)x(k) + B(k)u(k) + L(k)(C(k)x(k) - y(k)), \\
u(k) &= F(\gamma, k)\dot{x}(k), \quad \forall \gamma \in (0, \gamma^*),
\end{align*}$$

solves Problem 2, where $F(\gamma, k)$ is related with Eq. (99) and $L(k) \in \mathbb{R}^{n \times p}, \forall k \geq k_0$ is any $\omega$-periodic matrix such that $A(k) + L(k)C(k)$ is asymptotically stable.
Proof. Express the closed-loop system as
\[ \eta(k + 1) = (A_\eta(k) + B_\eta(k)F_\eta(\gamma, k))\eta(k), \quad u(k) = F_\eta(\gamma, k)\eta(k) \]  \hspace{1cm} (107)
where \( A_\eta(k), B_\eta(k) \) and \( F_\eta(\gamma, k) \) are in the form of Eqs. (50)–(51). In the following, we will show that \( F_\eta(\gamma, k) \) is a periodic \( l_\infty \) and \( l_2 \) low gain for \( (A_\eta(k), B_\eta(k)) \). Let \( R(k) \) be an \( \omega \)-periodic matrix function that solves
\[ (A(k) + L(k)C(k))^T R(k + 1) (A(k) + L(k)C(k)) - R(k) = -I_n. \]  \hspace{1cm} (108)
Then define
\[ P_\eta(k) = (n + 1) \begin{bmatrix} P(k) & 0 \\ 0 & \delta(\gamma) R(k) \end{bmatrix}, \]  \hspace{1cm} (109)
where \( P(k) \) is the unique \( \omega \)-periodic positive definite solution to the periodic DcRE (98) and
\[ \delta(\gamma) = \max_{k \in [k_0, k_0 + \omega]} \left[ P(\gamma, k) \right] = \max_{k \geq k_0} \left[ P(\gamma, k) \right]. \]  \hspace{1cm} (110)
Then, by direct computation, there exists a \( \gamma^*_1 > 0 \) such that
\[ F_\eta(\gamma, k)P_\eta^{-1}(k)F_\eta^T(\gamma, k) \leq \frac{1}{(n + 1)} \text{tr} \left( P(\gamma, k) \right) F_\eta(\gamma, k) - R(k) \]
\[ \leq \frac{1}{(n + 1)} \left( \text{tr} \left( P^{-1/2}(k) F_\eta^T(\gamma, k) F_\eta(\gamma, k) P^{-1/2}(k) \right) + \left| F(\gamma, k) R^{-1}(k) \right| \right) I_m \]
\[ \leq \frac{1}{(n + 1)} \left( n + \left| F(\gamma, k) R^{-1}(k) \right| \right) I_m \]
\[ < I_m, \quad \forall \gamma \in (0, \gamma^*_1], \]  \hspace{1cm} (111)
where we have noticed Eq. (104). By a Schur complement, the above relation is equivalent to
\[ F_\eta^T(\gamma, k) F_\eta(\gamma, k) < P_\eta(k), \quad \forall \gamma \in (0, \gamma^*_1]. \]  \hspace{1cm} (112)
Now let \( A_c(k) = A_\eta(k) + B_\eta(k)F_\eta(\gamma, k) \) and compute
\[ A_c^T(k)P_\eta(k + 1)A_c(k) - P_\eta(k) \]
\[ = (n + 1) \begin{bmatrix} -Q(k) - F^T(\gamma, k) F(\gamma, k) & F^T(\gamma, k) F(\gamma, k) \\ F^T(\gamma, k) F(\gamma, k) & -\delta(\gamma) I_n \end{bmatrix}, \]  \hspace{1cm} (113)
where we have used Eq. (103) and noticed that
\[ F(\gamma, k) = -B^T(k) P(k + 1) (A(k) + B(k) F(\gamma, k)), \]  \hspace{1cm} (114)
(see Eq. (A.432) in [26]). The Schur complement of Eq. (113) is
\[ -Q(k) - F^T(\gamma, k) F(\gamma, k) + \delta^{-1}(\gamma) F^T(\gamma, k) F(\gamma, k) F^T(\gamma, k) F(\gamma, k) \]
\[ \leq -Q(k) - (1 - |F(\gamma, k)|) F^T(\gamma, k) F(\gamma, k), \]  \hspace{1cm} (115)
from which it follows that there exists a \( \gamma^* \in (0, \gamma^*_1] \) such that
\[ A_c^T(k)P_\eta(k + 1)A_c(k) - P_\eta(k) < 0, \quad \forall \gamma \in (0, \gamma^*]. \]  \hspace{1cm} (116)
This shows that \( F_\eta(\gamma, k) \) is a periodic \( l_\infty \) low gain for \( (A_\eta(k), B_\eta(k)) \) in view of Theorem 5. The \( l_2 \) case can be shown similarly and the proof is finished. \( \square \)
It follows that the separation principle holds true for the observer based output feedback semi-global stabilization.

3.6. Semi-global stabilization design by LMIs

Since \( \Omega \) is bounded, there exists an \( S>0 \) such that \( \Omega \subset \mathcal{E}(S) \). Then we have the following result.

**Theorem 7.** Let Assumption 2 be satisfied. Then Problem 2 is solved by the periodic linear state feedback

\[
u(k) = F(k)x(k) = G(k)Q^{-1}(k)x(k),
\]

where \( G(k) \) and \( Q(k) > 0 \) are \( \omega \)-periodic matrices and such that

1. **in the \( l_\infty \) case,** \( G(k_0 + i) = G_\infty(i), Q(k_0 + i) = Q_\infty(i), i \in [0, \omega - 1] \) with \( Q_\infty(\omega) = Q_\infty(0) \) satisfying the following set of LMIs:

\[
\begin{bmatrix}
  -Q_\infty(i) & Z_\infty^T(i) \\
  Z_\infty(i) & -Q_\infty(i + 1)
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
  -Q_\infty(i) & G_\infty^T(i) \\
  G_\infty(i) & -I_m
\end{bmatrix} < 0,
\]

\[
S^{-1} - Q(0) < 0,
\]

where \( Z_\infty(i) = A(k_0 + i)Q_\infty(i) + B(k_0 + i)G_\infty(i), i \in [0, \omega - 1] \).

2. **in the \( l_2 \) case,** \( G(k_0 + i) = G_2(i), Q(k_0 + i) = Q_2(i), i \in [0, \omega - 1] \) with \( Q_2(\omega) = Q_2(0) \) satisfying the following set of LMIs:

\[
\begin{bmatrix}
  -Q_2(i) & Z_2^T(i) & G_2^T(i) \\
  Z_2(i) & -Q_2(i + 1) & 0 \\
  G_2(i) & 0 & -I_m
\end{bmatrix} < 0,
\]

\[
S^{-1} - Q_2(0) < 0,
\]

where \( Z_2(i) = A(k_0 + i)Q_2(i) + B(k_0 + i)G_2(i), i \in [0, \omega - 1] \).

**Moreover, in both cases, the set of LMIs (118)–(120) and the set of LMIs (121)–(122) are always solvable.**

**Proof.** According to the proof of Item 1 of Theorem 5 and Remark 1, the \( l_\infty \) semi-global stabilization problem is solved by the periodic state feedback \( u = F(k)x(k) \) if there exists an \( \omega \)-periodic positive definite matrix function \( P_\infty(k) \) such that

\[
(A(k) + B(k)F(k))^TP_\infty(k + 1)(A(k) + B(k)F(k)) - P_\infty(k) < 0,
\]

\[
F^T(k)F(k) < P_\infty(k),
\]

\[\mathcal{E}(S) \subseteq \mathcal{E}(P_\infty(k_0)).\]
By Schur complements, the above three inequalities are respectively equivalent to Eqs. (118)–(120) by setting
\[ Q_\infty(i) = P_\infty^{-1}(k_0 + i), \quad G_\infty(i) = F(k_0 + i)P_\infty^{-1}(k_0 + i), \]
for all \( i \in \{0, \omega - 1\} \).

Similarly, the \( l_2 \) semi-global stabilization problem is solved by the periodic state feedback \( u(k) = F(k)x(k) \) if there exists an \( \omega \)-periodic positive definite function \( P_2(k) \) such that, by setting \( A_c(k) = A(k) + B(k)F(k) \),
\[ A_c^T(k)P_\infty(k + 1)A_c(k) - P_\infty(k) + F^T(k)F(k) < 0, \]
which, by Schur complements, are equivalent to Eqs. (121)–(122) in view of Eq. (126) where \( P_\infty(i) \) and \( G_\infty(i) \) are respectively replaced by \( P_2(i) \) and \( G_2(i) \).

Finally, the solvability of Eqs. (118)–(120) (or equivalently, Eqs. (123)–(125)) and the solvability of Eqs. (121)–(122) (or equivalently, Eqs. (127)–(128)) follow from Proposition 3, namely, we choose \( F(k) = F(\gamma, k) \) as Eq. (99) and \( P_\infty(k) = P_2(k) = P(\gamma, k) \) where \( P(\gamma, k) \) is the unique positive definite solution to the periodic DcRE (98) and satisfies \( \lim_{\gamma \downarrow 0} P(\gamma, k) = 0 \). The proof is finished. \( \square \)

4. Applications to the elliptical spacecraft rendezvous

In this section, we use the obtained theory to design the elliptical spacecraft rendezvous control systems.

4.1. The model and problem formulation

Firstly, we introduce the elliptical spacecraft rendezvous system model. Consider a target spacecraft in an eccentric orbit with the eccentricity \( e \in [0, 1) \). Let
\[ \rho = 1 + e \cos \theta, \quad k = \frac{\mu^{1/4}}{(a(1 - e^2))^{3/4}}, \]
where \( \theta = \theta(t) \) is the true anomaly, \( a \) is the semi-major axis and \( \mu = GM \) is the geocentric gravitational constant with \( G \) being the universal gravitational constant and \( M \) being the mass of the Earth.

Let \( (x, y, z) \) be a rotating right-hand reference frame, where the origin is in the target spacecraft, \( x \) is in the radial direction, \( y \) is in the flying direction, and \( z \) completes the right-hand frame [22]. Consider a chaser spacecraft near the target spacecraft. Denote the position of the chaser spacecraft in the \( (x, y, z) \) coordinate by \( (x, y, z) \). Let
\[
\begin{align*}
\xi(\theta) &= \begin{bmatrix} \rho_x & \rho_y & \rho_z \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \end{bmatrix}^T, \\
u(\theta) &= \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T,
\end{align*}
\]
where \( a_x, a_y \) and \( a_z \) are the accelerations on the \( x-, y- \) and \( z- \) axes due to thrust forces on the chaser spacecraft. Then, by Newton’s rule, the linearized equations of relative motion can be expressed
as [22,26]
\[
\begin{align*}
\dot{x}(\theta) &= A(\theta)x(\theta) + B(\theta)u(\theta), \\
y(\theta) &= C(\theta)x(\theta),
\end{align*}
\] (131)
where \(A(\theta)\) is given by
\[
A(\theta) = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{2}{\rho} & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
\end{bmatrix},
\] (132)
and \(B(\theta) = (1/k^4\rho^3)B, C(\theta) = C\) with \((B, C)\) given by
\[
B = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}^T, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}.
\] (133)
We notice that system (131) is 2\(\pi\)-periodic since \(\rho\) is 2\(\pi\)-periodic. Eq. (131) is known as T-H equation [4]. A further property of the T-H equation is recalled as follows.

**Proposition 4 ([26]).** In the T-H (131), \((A(\theta), B(\theta))\) is controllable, \((A(\theta), C(\theta))\) is observable and \(\mathcal{M}(A) = \{1, 1, 1, 1, 1, 1\}\).

The whole rendezvous process can be described by the transformation of state vector \(x(\theta)\) from nonzero initial condition \(x(\theta_0)\) to the terminal one \(x(\theta_T) = 0\), where \(\theta_T = \theta(t_T)\) with \(t_T\) being the rendezvous time. In order to meet the requirements of actual conditions, especially the limited energy (power) of the actuator and the maximal control signals that the actuator can generate, controllers should be designed to solve the above problem by imposing the bounds
\[
\|u\|_{L_\infty} \leq \mu_\infty, \quad \|u\|_{L_2} \leq \mu_2,
\] (134)
where \(\mu_\infty\) and \(\mu_2\) are some given positive scalars.

### 4.2. Solutions to the elliptical spacecraft rendezvous problem

By Proposition 4, the matrix pair \((A(\theta), B(\theta))\) in the T-H equation (131) satisfies Assumption 1, then it follows from Theorem 2 and Proposition 2 that, for any initial condition \(x(\theta_0) \in \omega \subset \mathbb{R}^{6\times6}\) where \(\omega\) is bounded yet can be arbitrarily large, there exists a parameterized periodic \(L_\infty\) and \(L_2\) low gain feedback
\[
u(\theta) = F(\gamma, \theta)x(\theta), \quad \forall \gamma \in (0, \gamma^*],
\] (135)
such that the constraints (134) can be satisfied. We use the periodic DRE based approach in Lemma 2 to design the periodic low gain \(F(\gamma, \theta)\). Let \(P(\theta) = P(\gamma, \theta)\) be the unique periodic positive definite
solution to the following DRE:

\[- \frac{dP(\theta)}{d\theta} = A^T(\theta)P(\theta) + P(\theta)A(\theta) - P(\theta)B(\theta)R^{-1}(\theta)B^T(\theta)P(\theta) + \gamma I_6,\]

(136)

where \(R(\theta) = 1/k^8\rho^6\) is such that \(B(\theta)R^{-1}(\theta)B^T(\theta)\) is a constant matrix (notice that in Lemma 2 we have chosen \(R(t) = I_m\). However, non-identity periodic positive definite matrix \(R(t)\) can be absorbed

Table 1
The \(L_\infty\) and \(L_2\) norms of \(u(\theta)\) and the rendezvous time \(\theta_f\) for different \(\gamma\).

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>(|u|<em>{L</em>\infty})</th>
<th>(|u|_{L_2})</th>
<th>(\theta_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>0.03884</td>
<td>0.02369</td>
<td>5.8</td>
</tr>
<tr>
<td>1.00</td>
<td>0.03407</td>
<td>0.02224</td>
<td>6.1</td>
</tr>
<tr>
<td>0.80</td>
<td>0.03293</td>
<td>0.02183</td>
<td>6.8</td>
</tr>
<tr>
<td>0.50</td>
<td>0.03094</td>
<td>0.02103</td>
<td>7.2</td>
</tr>
<tr>
<td>0.30</td>
<td>0.02923</td>
<td>0.02025</td>
<td>8.1</td>
</tr>
<tr>
<td>0.10</td>
<td>0.02638</td>
<td>0.01884</td>
<td>14.5</td>
</tr>
<tr>
<td>0.05</td>
<td>0.02497</td>
<td>0.01817</td>
<td>18.5</td>
</tr>
</tbody>
</table>

Fig. 1. The periodic positive definite matrix \(P(\theta)\) satisfying Eq. (136) with \(\gamma=0.5\).
by the matrix $B(t)$ and will not change the results.) Then the periodic low gain is given by

$$F(\gamma, \theta) = -R^{-1}(\theta)B^T(\theta)P(\gamma, \theta).$$

By computation we find that $P(\theta)$ possesses the following structure:

$$P(\theta) = \begin{bmatrix}
p_{11}(\theta) & p_{12}(\theta) & 0 & p_{14}(\theta) & p_{15}(\theta) & 0 \\
p_{12}(\theta) & p_{22}(\theta) & 0 & p_{24}(\theta) & p_{25}(\theta) & 0 \\
0 & 0 & p_{33} & 0 & 0 & p_{36} \\
p_{14}(\theta) & p_{24}(\theta) & 0 & p_{44}(\theta) & p_{45}(\theta) & 0 \\
p_{15}(\theta) & p_{25}(\theta) & 0 & p_{54}(\theta) & p_{55}(\theta) & 0 \\
0 & 0 & p_{63} & 0 & 0 & p_{66}
\end{bmatrix},$$

where $p_{33}, p_{36}, p_{63}$ and $p_{66}$ are constants. To explain this, we notice that $p_{33}, p_{36}, p_{63}$ and $p_{66}$ correspond to the $(\xi_3, \xi_6)$-system. Since the $(\xi_3, \xi_6)$-system (which denotes the out-of-plane

Fig. 2. State response of the closed-loop system consisting of Eqs. (131), (135) and (137) with $\gamma = 0.5$.

Fig. 3. Control signals of the closed-loop system consisting of Eqs. (131), (135) and (137) with $\gamma = 0.5$. 
dynamics [22]) and the \((\xi_1, \xi_2, \xi_4, \xi_5)\)-system (which denotes the in-plane dynamics [22]) are uncoupled, the \((\xi_3, \xi_6)\)-system is time-invariant, and \(B(\theta)R^{-1}(\theta)B^T(\theta)\) is a constant matrix, the scalars \(p_{33}, p_{36}, p_{63}\) and \(p_{66}\) actually obey an ARE, namely, they are constant.

For the simulation purpose, we suppose that the target spacecraft is in the geostationary transfer orbit, which is a temporary orbit to inject a satellite into the geostationary Earth orbit (see, for example, [22,26]). The orbital parameters are as follows: the semimajor axis \(a = 24,616\) km, the eccentricity \(e = 0.73074\). It follows that \(k = 0.02267\). Then for different values of \(\gamma\), and a fixed initial condition

\[
\xi(\theta_0) = [10\ 000\ 10\ 000\ 6000\ 10 - 10\ 10]^T,
\]

where \(\theta_0 = 0\), the \(L_\infty\) and \(L_2\) norms of \(u(\theta)\) are recorded in Table 1. It follows that they are decreasing if \(\gamma\) is decreasing, indicating that the constraints (134) can be satisfied if \(\gamma < \gamma^*\) for some \(\gamma^*\) dependent on \(\mu_\infty\) and \(\mu_2\). However, from this table we can also see that a smaller \(\gamma\) means a longer rendezvous time \(\theta_f\).

For the illustration purpose, if we choose \(\gamma = 0.5\), the periodic positive definite matrix \(P(\theta)\), the state response \(x(\theta)\), and the control signals \(u(\theta)\) are respectively recorded in Figs. 1, 2 and 3. It follows that the rendezvous mission is accomplished at about \(\theta_f = 7.2\) rad with \(\|u\|_{L_\infty} = 0.03094\) and \(\|u\|_{L_2} = 0.02103\).

5. Conclusion

This paper has proposed a systematic approach to solve the \(L_\infty(l_\infty)\) and \(L_2(l_2)\) semi-global stabilization problems of linear periodic systems with both magnitude saturation and energy saturation. The developed approach generalizes for linear time-invariant systems and is referred to as periodic low gain feedback. The definitions, characterizations, properties, and design methods for periodic low gain feedback are carefully studied in this paper. Both continuous-time and discrete-time linear periodic systems are investigated, and both state feedback and observer based output feedback controllers are established. For \(l_\infty\) and \(l_2\) semi-global stabilization of discrete-time linear periodic systems, an LMIs based approach is also established. The developed theory is applied on the elliptical spacecraft rendezvous control systems to illustrate its effectiveness. A possible further study is to consider nonlinear periodic systems with both actuator saturation and time delays by combining the approach in this paper and those advanced approaches for time-delay systems (see, for instance, [20,21,26]) and nonlinear systems (see, for example, [9,13,14,23]).

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