Brief paper

Stability and stabilization of discrete-time periodic linear systems with actuator saturation

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A B S T R A C T
This paper is concerned with the problems of stability and stabilization for discrete-time periodic linear systems subject to input saturation. Both local results and global results are obtained. For local stability and stabilization, the so-called periodic invariant set is used to estimate the domain of attraction. The conditions for periodic invariance of an ellipsoid can be expressed as linear matrix inequalities (LMIs) which can be used for both enlarging the domain of attraction with a given controller and synthesizing controllers. The periodic enhancement technique is introduced to reduce the conservatism in the methods. As a by-product, less conservative results for controller analysis and design for discrete-time time-invariant systems with input saturation are obtained. For global stability, by utilizing the special properties of the saturation function, a saturation dependent periodic Lyapunov function is constructed to derive sufficient conditions for guaranteeing the global stability of the system. The corresponding conditions are expressed in the form of LMIs and can be efficiently solved. Several numerical and practical examples are given to illustrate the theoretical results proposed in the paper.

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1. Introduction
During the past several decades, control systems with actuator saturation have received much attention. This is because, on the one hand, saturation nonlinearity exists in every practical control system as a practical physical actuator can only generate bounded signals, and on the other hand, the control of plants with actuator saturation is challenging theoretically. Various control problems for systems with actuator saturation have been extensively studied in the literature. For example, semi-global stabilization is solved in Lin and Saberi (1993) and Teel (1995) by linear feedback, global stabilization is studied in Sussmann, Sontag, and Yang (1994) and Teel (1992) by nonlinear feedback, and robust global stabilization is addressed in Lin (1998). However, in these papers, it is required that the open-loop plant is asymptotically null controllable by bounded controls (ANCBC), namely, it is stabilizable in the ordinary sense and does not contain any strictly unstable poles (Sussmann et al., 1994). For a plant that is not ANCBC, only local results can be expected. In the recent two decades, with the help of the powerful tool Linear Matrix Inequalities (LMIs), many researchers have been able to devote themselves to studying plenty of control problems for actuator saturated linear plants that are not ANCBC (see Grimm, Teel, and Zaccarian (2008), Hindi and Boyd (1998), Hu and Lin (2001), Zheng and Wu (2008) and the references therein). These problems include local stability analysis and stabilization (Gomes da Silva & Tarbouriech, 2001; Hindi & Boyd, 1998), semi-global stabilization and output regulation in the null controllable region (Hu & Lin, 2001; Hu, Lin, & Shamash, 2001), and L2 and/or L∞ disturbance rejection (Fang, Lin, & Hu, 2004). For more related work on this topic, see the references cited in the above mentioned papers.

Periodic linear systems as the simplest time-varying system can model many practical control systems such as multivariate data systems (Chen & Francis, 1995) and systems that operate periodically (for example, satellites). Another important source of periodic systems is the closed-loop system consisting of a time-invariant plant and a periodic controller, because a periodic controller is more efficient in upgrading system performances than a time-invariant one (Khargonekar & Ozguler, 1994). Periodic
Consider the following discrete-time periodic linear system
\[ x(k + 1) = A_0 x(k) + B_0 \text{sat}(u(k)), \quad \forall k \in \mathbb{Z}, \]
where the period \( \omega \geq 1 \) is an integer, and \( \text{sat} : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a vector valued standard saturation function, i.e.,
\[ \text{sat}(u) = [\text{sat}(u_1) \, \text{sat}(u_2) \, \cdots \, \text{sat}(u_m)]^T. \]

**Lemma 1** (Bittanti & Colaneri, 1996). The discrete-time periodic linear system (1) with \( u(k) \equiv 0 \) is asymptotically stable if and only if the so-called monodromy matrix \( A_{\omega-1} A_{\omega-2} \cdots A_1 A_0 \) is Schur stable.

Assume that an \( \omega \)-periodic linear state feedback
\[ u(k) = F_k x(k), \quad F_k = F_k (\omega \nu), \quad \forall k \in \mathbb{Z}, \]
has been designed such that the closed-loop system in the absence of input saturation is asymptotically stable. Now in the absence of input saturation, the closed-loop system consisting of (1) and (2) is expressible as
\[ x(k + 1) = A_0 x(k) + B_0 \text{sat}(F_k x(k)), \quad \forall k \in \mathbb{Z}. \]

Because of the presence of input saturation, global stability of the closed-loop system (3) is generally not preserved and only local stability can be guaranteed. In particular, the domain of attraction is important in studies of local stability of dynamic systems (Hu & Lin, 2001). To introduce this notion, we consider the following general discrete-time nonlinear dynamic system
\[ x(k + 1) = A_0 x(k) + B_0 \text{sat}(f_k(x(k))), \quad \forall k \in \mathbb{Z}. \]

where \( f_k(\cdot) \) are some given nonlinear functions. For an initial condition \( x(0) \in \mathbb{R}^n \), denote the solutions to system (4) by \( x(k, x(0)) \).

**Definition 1.** Consider the discrete-time nonlinear dynamic system (4). The domain of attraction of the origin is the set \( \mathcal{D} = \{ x(0) \in \mathbb{R}^n : \lim_{k \to \infty} \| x(k, x(0)) \| = 0 \} \).

Since it is generally not possible to obtain the domain of attraction of a nonlinear system analytically (Hu & Lin, 2001), it is more realistic to consider an estimate of it. Then the problem considered in this paper can be stated as follows.

**Problem 1.** For the discrete-time periodic system (3).

1. Find an estimate of the domain of attraction which is as large as possible with respect to some given bounded and convex reference set \( \mathcal{X}_R \subset \mathbb{R}^n \).

2. Design the feedback gains \( F_k, k \in \mathbb{Z} \) in (2) such that an estimate of the domain of attraction for the resulting closed-loop system (3) is as large as possible with respect to some given bounded and convex reference set \( \mathcal{X}_R \subset \mathbb{R}^n \).

3. Find conditions under which the domain of attraction of the discrete-time periodic nonlinear system (3) is \( \mathbb{R}^n \), namely, the discrete-time periodic nonlinear system (3) is globally asymptotically stable.

Since saturation nonlinearity belongs to the sector bound \([-1, 1]\], global stability of systems containing saturation nonlinearity is also known as absolute stability. Then Item 3 of Problem 1 is solved if absolute stability of the discrete-time periodic system (3) is guaranteed.

Recall that a set is said to be (contractively) invariant if all the trajectories starting from it will remain in it forever (and converge to zero as \( k \) approaches infinity). For the discrete-time periodic nonlinear system (3), if it is time-invariant, that is, \( \omega = 1 \), or the matrices \( A_k, B_k, F_k \) are constant, then the notion of a contractively invariant set is frequently used to estimate the domain of attraction. However, for a periodic system, such a notion is very conservative and instead the so-called notion of a periodic contractively invariant set is more useful.
Definition 2 (Blanchini & Ukkovitch, 1993). Consider the discrete-time dynamic system (4). A set $$\mathcal{X} \subseteq \mathbb{R}^n$$ is said to be an $$\omega$$-periodic invariant set for system (4) if $$x(0) \in \mathcal{X}$$ implies $$x(k) \in \mathcal{X}$$ for all $$k \in \mathbb{Z}$$. Moreover, if $$\mathcal{X} \subseteq \mathbb{R}^n$$ is said to be an $$\omega$$-periodic contractively invariant set for system (4) if it is an $$\omega$$-periodic invariant set and $$x(0) \in \mathcal{X} \Rightarrow \lim_{k \to \infty} \|x(k, x(0))\| = 0$$.

Definitions 1 and 2 clearly imply that an $$\omega$$-periodic contractively invariant set is inside the domain of attraction. In this paper, we will use it to estimate the domain of attraction for the discrete-time periodic nonlinear system (3).

Let $$\mathcal{X}$$ be the set of $$m \times m$$ diagonal matrices whose diagonal elements are either 1 or 0. Then there are $$2^m$$ elements in $$\mathcal{X}$$. Let each element of $$\mathcal{X}$$ be labeled as $$D_i$$, $$i \in [1, 2^m]$$. Moreover, denote $$D_i^\top = I_m - D_i$$.

Lemma 2 (Hu & Lin, 2001). Let $$u, v \in \mathbb{R}^n$$. Suppose that $$\|v\|_\infty \leq 1$$. Then $$\text{sat}(u) = \text{co}[D_i u + D_i^\top v : i \in [1, 2^m]]$$, where $$\text{co}[]$$ denotes the convex hull of a set.

3. Local stability and stabilization

3.1. Conditions for periodic invariant set and solutions to Problem 1

In this section, we will present a condition under which a periodic invariant set exists for the discrete-time periodic nonlinear system (3).

Theorem 1. If there exist matrices $$P_k \in \mathbb{P}^{m \times m}, k \in [0, \omega - 1]$$, and matrices $$H_k \in \mathbb{R}^{m \times n}, k \in [0, \omega - 1]$$, such that

$$
(E_k^\top)^T P_{k+1} E_k^\top - P_k < 0, \quad \forall i \in [1, 2^m],
$$

and

$$
\mathcal{E}(P_k) \subseteq \mathcal{X}(H_k), \quad k \in [0, \omega - 1],
$$

are satisfied, where $$E_k^\top = A_k + B_k (D_k F_k + D_k^\top H_k)$$ and $$P_\omega = P_0$$, then the periodic invariant set is a $$\omega$$-periodic contractively invariant set for the discrete-time periodic nonlinear system (3).

Proof. Associated with $$P_k$$ and $$H_k$$, $$k \in [0, \omega - 1]$$, we define two $$\omega$$-periodic matrices $$P_{k+\omega} = P_k \in \mathbb{P}^{m \times m}$$ and $$H_{k+\omega} = H_k \in \mathbb{R}^{m \times n}, \forall k \in \mathbb{Z}$$. We first show that under the conditions of this theorem, there holds

$$
x(k) \in \mathcal{E}(P_k) \Rightarrow x(k+1) \in \mathcal{E}(P_{k+1}), \quad \forall k \in \mathbb{Z}.
$$

Indeed, as (6) is satisfied, $$x(k) \in \mathcal{E}(P_k)$$ implies that $$\|H_k x(k)\|_\infty \leq 1$$. Then according to Lemma 2, we have

$$\text{sat}(P_k x(k)) = \text{co}[D_i F_i x(k) + D_i^\top H_k x(k) : i \in [1, 2^m]].
$$

Hence it follows from (5) that there exists a sufficiently small scalar $$\varepsilon > 0$$ such that

$$
\Delta_k \triangleq x^\top(k+1) P_{k+1} x(k+1) - x^\top(k) P_k x(k)
\leq \max_{i \in [1, 2^m]} [x^\top(k)((E_i^\top)^T P_{k+1} E_i^\top - P_k) x(k)]
\leq -\varepsilon \|x(k)\|^2.
$$

Therefore, $$x^\top(k+1) P_{k+1} x(k+1) \leq x^\top(k) P_k x(k) \leq 1$$, namely, $$x(k+1) \in \mathcal{E}(P_{k+1})$$.

Now by using relation (7) recursively, we have $$x(0) \in \mathcal{E}(P_0) \Rightarrow x(1) \in \mathcal{E}(P_1) \Rightarrow \cdots \Rightarrow x(\omega) \in \mathcal{E}(P_\omega) = \mathcal{E}(P_0) \Rightarrow \cdots \Rightarrow x(\omega k) \in \mathcal{E}(P_\omega), \forall k \in \mathbb{Z}$$, namely, $$\mathcal{E}(P_\omega)$$ is an $$\omega$$-periodic invariant set for system (3) by definition. So the remaining task is to show that

$$
x(0) \in \mathcal{E}(P_0) \Rightarrow \lim_{k \to \infty} \|x(k)\| = 0.
$$

Choose a time-varying Lyapunov function $$V_k = x^\top(k) P_k x(k), \forall k \in \mathbb{Z}$$. Then

$$
\beta_{\min} \|x(k)\|^2 \leq V_k \leq \beta_{\max} \|x(k)\|^2,
$$

where $$\beta_{\min} = \min_{k \in [0, \omega - 1]} \{\lambda_{\min}(P_k)\}$$ and $$\beta_{\max} = \max_{k \in [0, \omega - 1]} \{\lambda_{\max}(P_k)\}$$. If $$x(0) \in \mathcal{E}(P_0)$$, then we get from (8) that

$$
V_{k+1} - V_k \leq -\frac{\varepsilon}{\beta_{\max}} x^\top(k) P_k x(k),
$$

that is, $$V_{k+1} \leq (\beta_{\max} - \varepsilon)/\beta_{\max} V_k, \forall k \in \mathbb{Z}$$. Clearly, as $$\varepsilon$$ is sufficiently small, we have $$(\beta_{\max} - \varepsilon)/\beta_{\max} \in (0, 1)$$. The above relation then implies, for all $$k \in \mathbb{Z},$$

$$
\|x(k)\|^2 \leq \frac{1}{\beta_{\min}} V_k \leq \frac{1}{\beta_{\min}} \left(\frac{\beta_{\max} - \varepsilon}{\beta_{\min}}\right)^k x^\top(0) P_0 x(0)
\leq \frac{\lambda_{\max}(P_0)}{\beta_{\min}} \left(\frac{\beta_{\max} - \varepsilon}{\beta_{\min}}\right)^k \|x(0)\|^2,
$$

from which (9) follows immediately. Hence, the proof is complete.

We next use the periodic invariant set $$\mathcal{E}(P_0)$$ given in Theorem 1 to solve Item 1 of Problem 1. Similar to the argument given in Hu and Lin (2001), the largeness of the ellipsoid $$\mathcal{E}(P_0)$$ with respect to a shape reference set $$\mathcal{X}_R \subseteq \mathbb{R}^n$$ can be measured by the scalar $$\alpha$$ which is the maximal number such that $$\alpha \mathcal{X}_R \subseteq \mathcal{E}(P_0)$$ is satisfied, where $$\alpha \mathcal{X}_R \triangleq \{\alpha x : x \in \mathcal{X}_R\}$$. Then a solution to Item 1 of Problem 1 can be obtained by solving the following optimization problem

$$
\sup_{P_\varepsilon > 0, \ P_\omega = P_0, H_k, \ k \in [0, \omega - 1]} \alpha
$$

s.t.

$$
\begin{align*}
\text{(a)} & \quad \alpha \mathcal{X}_R \subseteq \mathcal{E}(P_0), \\
\text{(b)} & \quad (E_k^\top)^T P_{k+1} E_k^\top - P_k < 0, \quad \forall i \in [1, 2^m], \\
\text{(c)} & \quad \mathcal{E}(P_\omega) \subseteq \mathcal{X}(H_k).
\end{align*}
$$

Note that constraint (b) is equivalent to

$$
\begin{bmatrix}
-P_\varepsilon^{-1} & \mathbb{I} \\
E_k^\top P_\varepsilon^{-1} & -P_{k+1}^{-1}
\end{bmatrix} < 0, \quad \forall i \in [1, 2^m],
$$

and constraint (c) is equivalent to

$$
\begin{bmatrix}
1 & \mathbb{I} \\
P_\varepsilon^{-1} H_k & P_k^{-1}
\end{bmatrix} \geq 0, \quad \forall i \in [1, m].
$$

If the reference set is chosen as the ellipsoid, namely,

$$
\mathcal{X}_R = \mathcal{E}(R)
$$

where $$R$$ is a given positive definite matrix, then constraint (a) is equivalent to

$$
\begin{bmatrix}
1 & \mathbb{I} \\
\alpha^2 R & P_0^{-1}
\end{bmatrix} \geq 0.
$$

If the reference set is chosen as the polyhedron, namely,

$$
\mathcal{X}_R = \text{co}\{x_1, x_2, \ldots, x_l\}, \quad l \geq 1,
$$

where $$x_i \in \mathbb{R}^n, i \in [1, l]$$ are some given vectors, then constraint (a) is equivalent to

$$
\begin{bmatrix}
1 & \mathbb{I} \\
\alpha^2 x_i & P_0^{-1}
\end{bmatrix} \geq 0, \quad i \in [1, l].
Therefore, by denoting $Q_k = P_k^{-1}$, $G_k = H_k P_k^{-1}$, $\gamma = 1/\alpha^2$, $k \in \{0, \omega - 1\}$ and the $i$th row of $G_k$ by $G_{ki}$, the optimization problem (11) can be converted into

$$\inf_{Q_k > 0, \, Q_0, \, G_k, \, k \in \{0, \omega - 1\}} \gamma$$

s.t.

\[
\begin{align*}
& (a) \left[ \begin{array}{c} \gamma X \ \tilde{r} \\ I \end{array} \right] Q_0 > 0 \ \text{or} \ \left[ \begin{array}{c} \gamma \ \tilde{X} \\ I \end{array} \right] Q_0 > 0, \\
& (b) \left[ \begin{array}{c} -Q_k \\ J_k \end{array} \right] < 0, \\
& (c) \left[ \begin{array}{c} 1 \\ G_{k1}^T \end{array} \right] Q_k > 0,
\end{align*}
\]

which is an LMI-based optimization problem, where $I_k = A_k Q_k + B_k (D_k T_k Q_k + D_k^{-1} G_k)$, $j \in \{1, \ldots, m\}$, $i \in \{1, \omega - 1\}$ and $l \in \{1, \ldots, m\}$. The resulting maximal estimate of the domain of attraction can be recovered from $\varepsilon(P_0) = \varepsilon(Q_0^{-1})$.

Regarding the solution to Item 2 of Problem 1, we can solve the following LMI-based optimization problem

$$\inf_{Q_k > 0, \, Q_0, \, G_k, \, k \in \{0, \omega - 1\}} \gamma$$

s.t.

\[
\begin{align*}
& (a) \text{and (c) are in (14),} \\
& (b) \left[ \begin{array}{c} -Q_k \\ J_k \end{array} \right] < 0, \quad i \in \{1, 2^m\},
\end{align*}
\]

where $I_k = A_k Q_k + B_k (D_k T_k Q_k + D_k^{-1} G_k)$, and the resulting feedback gains can be computed from $F_k = Y_k Q_k^{-1}$, $k \in \{0, \omega - 1\}$ and the estimate of the domain of attraction is given by $\varepsilon(P_0) = \varepsilon(Q_0^{-1})$.

**Remark 1.** Similar to the argument given in Section 2.4 in Hu, Lin, and Chen (2002), we can show that if constraint (b) in (15) is replaced by the following simple

\[
\begin{align*}
& (b) \left[ \begin{array}{c} -Q_k \\ A_k Q_k + B_k G_k \end{array} \right] \leq 0,
\end{align*}
\]

then the optimal value of $\gamma$ in (15) will not change. In this case, the optimal gains $F_k$ will be recovered from $F_k = G_k Q_k^{-1}$, $k \in \{0, \omega - 1\}$ and this will generally lead to a very slow convergence rate of the resulting closed-loop system.

**Remark 2.** By recognizing the LMIs’ characterization of stability and stabilization of discrete-time periodic systems (see, e.g., de Souza and Trofino (2000)), it is not difficult to verify that the optimization problem (14) (problem (15)) is well posed with compatible conditions if and only if the closed-loop system (3) is stable (the open-loop system (1) is stabilizable) in the absence of actuator saturation, which is the assumption that will be imposed on the closed-loop system (3) (the open-loop system (1)) throughout the paper.

In the above, we have assumed that the period of the periodic invariant set is the same as the period of the plant. However, this is not necessary. In fact, the period of the periodic invariant set can be any multipliers of $\omega$, say, $\omega p$, where $p \geq 1$ is an integer. In this case, Theorem 1 can also be used to solve Items 1–2 of Problem 1. In this case, we need only to replace $\omega$ in the system (3) by $\omega p$. This method is called periodic enhancement. We notice that a similar technique is also used in Hu and Lin (2000) to enlarge the domain of attraction for time-invariant plants with input saturation. Periodic enhancement can increase the size of the estimate of the domain of attraction as stated in the following proposition.

**Proposition 1.** Let $\gamma^*(\omega)$ denote the optimal value of the optimization problem (14). Then $\gamma^*(\omega p) \leq \gamma^*(\omega)$, $\forall p \geq 1$.

**Proof.** Let the optimal value $\gamma^*(\omega)$ be achieved in optimization problem (14) with the $\omega$-periodic matrices $Q_k > 0$, $G_k$, $k \in \{0, \omega - 1\}$. Then the optimization problem (14) where $\omega$ is replaced by $\omega p$ has a feasible solution $\gamma = \gamma^*(\omega)$ and $Q_k > 0$, $G_k$, $k \in \{0, \omega p - 1\}$ with $Q_{k+i\omega} = Q_k$, $Q_{k+i\omega} = G_k$, $i \in \{0, p\}$, $k \in \{0, \omega - 1\}$. This completes the proof.

The effectiveness of periodic enhancement will be illustrated by examples given in Section 5. However, it should be pointed out that the cost paid for using this approach is the computational complexity. Also, the numerical examples will show the limitation of this approach: when $p$ is large enough, a large increment of $p$ can only lead to a very small increment of the size of the periodic invariant set $\varepsilon(P_0)$.

3.2. Applications to time-invariant system with input saturation

Consider a discrete-time linear time-invariant system subject to input saturation

$$x(k+1) = Ax(k) + Bu(k), \quad \forall k \in \mathbb{Z},$$

with periodic feedback $u(k) = F_k x(k)$, $F_{k+i\omega} = F_k$, $\forall k \in \mathbb{Z}$, where the period $\omega \geq 1$ is a given integer. The resulting closed-loop system reads

$$x(k+1) = Ax(k) + Bu(k), \quad k \in \mathbb{Z},$$

which is also a discrete-time periodic system and is a special case of (3). Then applying Theorem 1 on system (17) yields the following corollary.

**Corollary 1.** If there exist matrices $P_k \in \mathbb{R}^{m \times n}$, $k \in \{0, \omega - 1\}$, and matrices $H_k \in \mathbb{R}^{m \times n}$, $k \in \{0, \omega - 1\}$, such that

\[
(E_k^T)^T P_{k+1} E_k - P_k < 0,
\]

and (6) are satisfied, where $E_k^T = A + B (D_k T_k + D_k^{-1} H_k)$ and $P_{\omega} = P_0$, then $\varepsilon(P_0)$ is contained in the domain of attraction for the discrete-time periodic system (17).

Similar to the development made in Section 3.1, the problem of enlarging the domain of attraction for the closed-loop system (17) can be handled by solving the following optimization problem

$$\inf_{Q_k > 0, \, Q_0, \, G_k, \, k \in \{0, \omega - 1\}} \gamma$$

s.t.

\[
\begin{align*}
& (a) \text{and (c) are in (14),} \\
& (b) \left[ \begin{array}{c} -Q_k \\ J_k \end{array} \right] < 0, \quad i \in \{1, 2^m\},
\end{align*}
\]

where $I_k = A_k Q_k + B_k (D_k T_k Q_k + D_k^{-1} G_k)$.

It is well-known that periodic feedback can improve performances of time-invariant plants (see Bittanti and Colaneri (2008) and Khargonekar and Ozugler (1994)). Usually, periodic feedback can be used to enlarge the domain of attraction for linear time-invariant system with input saturation (Hu & Lin, 2000). However, unlike the approach in Hu and Lin (2000) where the lifting technique is used, Corollary 1 can be applied to the system directly without utilizing the lifting technique.

We can also use Corollary 1 to design the matrices $F_k$, $k \in \{0, \omega - 1\}$ such that the domain of attraction of system (17) is maximized. In view of Remark 1, by solving the following optimization problem

$$\inf_{Q_k > 0, \, Q_0, \, G_k, \, k \in \{0, \omega - 1\}} \gamma$$

s.t.

\[
\begin{align*}
& (a) \text{and (c) are in (14),} \\
& (b) \left[ \begin{array}{c} -Q_k \\ A_k Q_k + B_k G_k \end{array} \right] \leq 0,
\end{align*}
\]
the optimal feedback gains $F_k$ can be determined by $F_k = G_k Q_k^{-1}$, $k \in \{0, \omega - 1\}$.

Periodic feedback is clearly more expensive than time-invariant feedback. If only time-variant feedback in the form of $u(k) = F(x(k))$ is designed for the open-loop system (16), the closed-loop system is given by

$$x(k + 1) = Ax(k) + B satire(Fx(k)), \quad k \in \mathbb{Z}. \tag{19}$$

Then applying Theorem 1 to system (19) gives the following corollary.

**Corollary 2.** If there exist an integer $\omega \geq 1$, matrices $P_k \in \mathbb{R}^{n \times n}$, $k \in \{0, \omega - 1\}$, and matrices $H_k \in \mathbb{R}^{n \times m}$, $k \in \{0, \omega - 1\}$, such that

$$(E_k')^T P_{k+1} E_k' - P_k < 0, \quad i \in \{1, 2^m\}, \tag{20}$$

and (6) are satisfied, where $E_k' = A + B(D_k F_k + D_k^- H_k)$ and $P_{\omega} = P_0$, then $\eta(P_0)$ is contained in the domain of attraction for the closed-loop nonlinear system (19).

Notice that if we set $P_k = P$, $H_k = H$, $k \in \{0, \omega - 1\}$, in (20) and (6), then Corollary 2 reduces to Theorem 1 in Hu et al. (2002). Relations (21) clearly imply that Corollary 2 is less conservative than the results in Hu et al. (2002). Our numerical example to be shown in Section 5 will also validate this observation. Similar to (18), by solving the following optimization problem

$$\inf \{ \text{s.t.} \left\{ \begin{array}{l} -Q_k \preceq \begin{bmatrix} 0 & 0 \\ 0 & -Q_{k+1} \end{bmatrix} < 0, \quad i \in \{1, 2^m\} \end{array} \right. \right\}, \tag{22}$$

where $Q_k = A Q_k + B(D_k F_k + D_k^- G_k)$, $k \in \{0, \omega - 1\}$, and $\gamma$,

Clearly, these four kinds of matrices are also $\omega$-periodic.

**Theorem 2.** If there exist matrices $P_k \in \mathbb{R}^{(n+m) \times (n+m)}$, $k \in \{0, \omega - 1\}$, diagonal positive definite matrices $S_k \in \mathbb{R}^{n \times n}$, $k \in \{0, \omega - 1\}$, and diagonal positive definite matrices $G^+ \in \mathbb{R}^{n \times n}$ and $G^- \in \mathbb{R}^{n \times n}$ such that

$$\Theta_k = \begin{bmatrix} \Theta_{1k} & \Theta_{2k} \\ \Theta_{1k}^T & \Theta_{2k}^T \end{bmatrix} < 0, \quad k \in \{0, \omega - 1\}, \tag{23}$$

are satisfied, where $\Theta_{yk}$ are given by

$$\Theta_{1k} = -Q_k^+ G_k^+ - \Theta_k,$$

$$\Theta_{2k} = G_k^+ G_k^+ G_k^+ F_k + \Theta_k^T \Theta_k + G_k^+ G_k^+ - \Theta_k^T \Theta_k G_k^+ G_k^+,$$

$$\Theta_{22k} = \Theta_{2k}^T + G_k^+ G_k^+ G_k^+ F_k + \Theta_k^T \Theta_k - 2 \Theta_k^T \Theta_k G_k^+ G_k^+ - 2 S_k,$$

with $\mathcal{P}_k = \mathcal{P}_0$, $S_k = S_0$, and $\mathcal{P}_k = P_k + \Theta_{2k}^T G^- \Theta_k$, $\forall k \in \{0, \omega\}$, then the discrete-time periodic nonlinear system (3) is globally asymptotically stable at the origin.

**Proof.** Define a new state vector as

$$\eta(k) \triangleq \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix} \triangleq \begin{bmatrix} x(k + 1) \\ y(k) \end{bmatrix}.$$ Then by using the state equation (3), we get

$$\eta(k + 1) = \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix} = \begin{bmatrix} x(k + 1) \\ y(k) \end{bmatrix}.$$ Therefore, if $\eta(k + 1) < x(k + 1)$, then

$$\eta(k + 1) = \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix} = \begin{bmatrix} x(k + 1) \\ y(k) \end{bmatrix}.$$ Thus, the system $\eta(k + 1) = \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix}$ is globally asymptotically stable at the origin.

4. Global stability and solutions to Problem 1

In this section, by using the special properties of the saturation nonlinearity and applying the idea in Park and Kim (1998) and Zhou, Duan, and Lam (2010), we will present a solution to Item 3 of Problem 1. First, associated with the discrete-time periodic nonlinear system (3), we define

$$\Theta_{k} = \begin{bmatrix} A_k & 0 \\ F_k & 0 \end{bmatrix}, \quad \mathcal{P}_k = \begin{bmatrix} B_k & 0 \\ F_k & 0 \end{bmatrix}.$$ Clearly, these four kinds of matrices are also $\omega$-periodic.

**Theorem 2.** If there exist matrices $P_k \in \mathbb{R}^{(n+m) \times (n+m)}$, $k \in \{0, \omega - 1\}$, diagonal positive definite matrices $S_k \in \mathbb{R}^{n \times n}$, $k \in \{0, \omega - 1\}$, and diagonal positive definite matrices $G^+ \in \mathbb{R}^{n \times n}$ and $G^- \in \mathbb{R}^{n \times n}$ such that

$$\Theta_k = \begin{bmatrix} \Theta_{1k} & \Theta_{2k} \\ \Theta_{1k}^T & \Theta_{2k}^T \end{bmatrix} < 0, \quad k \in \{0, \omega - 1\}, \tag{23}$$

are satisfied, where $\Theta_{yk}$ are given by

$$\Theta_{1k} = \begin{bmatrix} A_k & 0 \\ F_k & 0 \end{bmatrix}, \quad \mathcal{P}_k = \begin{bmatrix} B_k & 0 \\ F_k & 0 \end{bmatrix}.$$ Clearly, these four kinds of matrices are also $\omega$-periodic.

With $\mathcal{P}_k = P_k + \Theta_{2k}^T G^- \Theta_k$, $\forall k \in \{0, \omega\}$, then the discrete-time periodic nonlinear system (3) is globally asymptotically stable at the origin.

**Proof.** Define a new state vector as

$$\eta(k) \triangleq \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix} \triangleq \begin{bmatrix} x(k + 1) \\ y(k) \end{bmatrix}.$$ Then by using the state equation (3), we get

$$\eta(k + 1) = \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix} = \begin{bmatrix} x(k + 1) \\ y(k) \end{bmatrix}.$$ Therefore, if $\eta(k + 1) < x(k + 1)$, then

$$\eta(k + 1) = \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix} = \begin{bmatrix} x(k + 1) \\ y(k) \end{bmatrix}.$$ Thus, the system $\eta(k + 1) = \begin{bmatrix} x(k + 1) \\ F_k x(k) \end{bmatrix}$ is globally asymptotically stable at the origin.

On the other hand, the saturation function $\text{sat}(\cdot)$ belongs to the sector $[0, 1]$, namely.

$$2sat^2(y(k + 1))S_{k+1} + (y(k + 1) - \text{sat}(y(k + 1))) \geq 0,$$ where $S_k = S_k$, $k \in \{0, \omega - 1\}$ being as defined in this theorem. By using the above two inequalities, a tedious algebraic manipulation gives

$$\Delta V_k \triangleq V_{k+1} - V_k \leq \xi^T(k) \Theta_{k+1} \xi(k), \tag{26}$$

where $\Theta_{k+1} = \Theta_k$, $k \in \{0, \omega - 1\}$ given in (23) and $\xi^T(k) = [\text{sat}(y(k + 1))]$. Therefore, if (23) is satisfied for all $k \in \{0, \omega - 1\}$, it follows from (26) that there exists a sufficiently small number $k > 0$ such that $\Delta V_k \leq -k \|y(k + 1)\|^2$, $k \in \mathbb{Z}$. According to the Lyapunov stability theorem for time-varying systems, the nonlinear system (25) is asymptotically stable at the origin (Khalil, 1996), namely, $\lim_{k \to \infty} \|x(k + 1) - y(k + 1)\| = 0$, which implies that $\lim_{k \to \infty} \|x(k)\| = 0$. That is, the nonlinear system (3) is globally asymptotically stable and the proof is thus complete.

Note that the inequalities in (23) are linear in all the unknowns and can be efficiently solved by the LMI solver in Matlab environment. Moreover, similar to the development made in Section 3, the periodic enhancement technique can also be used here to reduce the conservatism in the result, but the details are omitted for brevity.
5. Numerical illustration

In this section, we will present three numerical and practical examples to illustrate the effectiveness of the obtained results.

**Example 1.** Consider a periodic discrete-time linear system in the form of (1) with \( \omega = 2 \),

\[
A_0 = \begin{bmatrix} -1.4 & 0.95 \\ -1.2 & -1.44 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1.2 & 0.65 \\ 1.41 & 0.94 \end{bmatrix},
\]

and \( B_0^T = [-1.0, 0.8], B_1^T = [0.4, -0.5] \). Direct computation gives \( \lambda(A_1A_0) = (3.0214, -2.1355) \), namely, the open-loop system is unstable. We design a periodic state feedback control law by the periodic LQR approach (Bittanti & Colaneri, 2008) as

\[
F_0 = [0.2887 \ 1.3961], \quad F_1 = [2.0809 \ -1.1317].
\]

We use Theorem 1 to estimate the domain of attraction of the closed-loop system in the form of (3). By choosing the reference set \( \mathcal{X}_R = \mathcal{E}(I_2) \) and solving the optimization problem (14), we get \( \alpha = 0.3980 \) and

\[
P_0 = \begin{bmatrix} 5.8100 & 1.5548 \\ 1.5548 & 1.5075 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 2.1287 & -1.9458 \\ -1.9458 & 6.4693 \end{bmatrix}.
\]

These two ellipsoids \( \mathcal{E}(P_0) \) and \( \mathcal{E}(P_1) \) are plotted in Fig. 1. To illustrate that \( \mathcal{E}(P_0) \) is a periodic invariant set for the closed-loop system, we plot the trajectories of the closed-loop system with two initial conditions \( x_0 = [0.4553, -0.7609]^T \) and \( x_0 = [-0.3787, -0.1225]^T \) which lie on the boundary of \( \mathcal{E}(P_0) \). For these two initial conditions, the states starting at \( \mathcal{E}(P_0) \) leave it and enter \( \mathcal{E}(P_1) \) at the first step. At the second step, they all return to \( \mathcal{E}(P_0) \) and remain there for ever. Consequently, they converges to the origin asymptotically. The control profile with initial condition \( x_0 = [-0.3787, -0.1225]^T \) is given in Fig. 2, from which it can be seen that the actuator is saturated during the convergence of the system state.

**Example 2.** Consider the input-saturated digital control system resembling the key features of the angular position regulation for a satellite where the three axes are completely decoupled, so that one can concentrate on only one axis and implement three decoupled controllers (Massimetti, Zaccarian, Hu, & Teel, 2009). The system is time-invariant and is in the form of (19) with (see Massimetti et al. (2009))

\[
A = \begin{bmatrix} 2 & -0.5 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.125 \\ 0 \end{bmatrix}.
\]

A feedback gain \( F \) designed by using a pole placement approach is given by \( F = [-12.6292 \ 3.8478] \). We use different methods to estimate the domain of attraction for this system by choosing the reference set \( \mathcal{X}_R = \mathcal{E}(I_2) \) where \( x_1 = [21 \ 71]^T \) and \( x_2 = [11.4 \ 7.6]^T \). The maximal value of \( \alpha \) such that \( \alpha \mathcal{X}_R \subseteq \mathcal{E}(P) \) for some \( P \succ 0 \) by using the methods in Hu et al. (2002) and Cao and Lin (2003) are \( \alpha_q = 0.0304 \) and \( \alpha_d = 0.0317 \), respectively. By solving the optimization problem (22), the computed \( \alpha_q = \alpha_q(\omega) \) for different values of \( \omega \) are recorded at the upper part of Fig. 3. With \( \omega = 20 \), we have \( \alpha(20) = 0.0327 \), which is larger than that computed by the methods in Hu et al. (2002) and Cao and Lin (2003). The resulting estimates of the domain of attraction by these three approaches are plotted at the lower part of Fig. 3, which indicates that our method is competitive with the saturation dependent approach given in Cao and Lin (2003) which is shown to be less conservative than the results in Hu et al. (2002).

**Example 3.** We consider the lossy Mathieu differential system with actuator saturation in the form of (see Zhou and Hagiwara (2002))

\[
\dot{x}(t) = Ax(t) + B(t)\text{sat}(u(t)), \quad y(t) = Cx(t),
\]

where

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & -2\xi \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 - 2\beta \cos(\omega_0t) \end{bmatrix}.
\]
Figure 4. Estimation of the domain of attraction for the lossy Mathieu differential system with input saturation in Example 3.

\[ C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \]

in which \( \xi = 0.2 \) represents the damping ratio, \( \beta = 0.2 \), and \( \omega_h = 2 \) is the pumping frequency (Zhou and Hagihara, 2002). It follows that system (27) is a continuous-time periodic system with period \( T = \pi \). To apply our results, a corresponding discrete-time model in the form of (1) will be obtained from discretization of the continuous-time periodic system (27) with sampling time \( T_s = \frac{\pi}{n} \), where \( \omega \) is some integer representing the period of the discrete-time model (1). The coefficient matrices are determined by \( A_k = e^{A\tau_k} \) and

\[ B_k = \int_{t_k}^{(k+1)t_k} e^{A(k+1)t_\tau - \tau} B(\tau) d\tau, \]

for \( k = 0, 1, 2, \ldots, \omega - 1 \) (see Bittanti and Colaneri (2008)). It is well-known that the larger the value of \( \omega \), more accurately that the discretized system matches the continuous one (Varga, 2008).

The controller is assumed to be the static output feedback \( u(t) = ky(t) \) with \( k = 0.5 \) (see Zhou and Hagihara (2002)). By choosing the reference set as \( \mathcal{D}_k = \mathcal{D}(L_k) \) and the period as \( \omega = 35 \) (i.e., the sampling period is \( T_s = \frac{\pi}{35} \)), the maximal estimate of the invariant ellipsoid \( \mathcal{E}(P_k) \) for the discretized model is shown in Figure 4. For a couple of initial conditions on the boundary of the ellipsoid \( \mathcal{E}(P_0) \), the resulting trajectories of the closed-loop continuous-time system are also recorded in Figure 4. It is seen that such an estimation of domain of attraction is quite nonconservative since some trajectories of the closed-loop system skim along the inner boundary of \( \mathcal{E}(P_0) \) and converge to the origin eventually.

6. Conclusion

In this paper we have investigated stability and stabilization of discrete-time periodic linear systems with input saturation. In the first place, we have studied local stability and stabilization by applying the periodic invariant ellipsoid to estimate the domain of attraction. It has been shown that both the problem of enlarging the domain of attraction and the problem of designing a periodic feedback law such that the domain of attraction of the closed-loop system is enlarged can be resolved by solving an LMI based optimization problem. Moreover, the conservatism in the approach can be further reduced by the so-called periodic enhancement technique. Secondly, we examined global stability of discrete-time periodic nonlinear systems. We have explored the properties of the saturation nonlinearity and introduced a parameter dependent periodic Lyapunov function to establish the global stability conditions. Finally, an interesting topic for future research is to extend the results with the domain of attraction given in piecewise-linear constraints for time-invariant plants to periodic systems.

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References

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