Norm Vanishment and Its Applications in Constrained Control – Part II: the $L_2$ Case

Bin Zhou, Zongli Lin and Guang-Ren Duan

Abstract—In Part I of this paper, we introduced the notion of $L_\infty$-vanishment and established several kinds of characterizations of this notion. Based on this notion, $L_\infty$ low gain feedback was reconsidered and a systematic approach to $L_\infty$ low gain feedback design was proposed. In this second part of the paper, we consider the parallel notion of $L_2$-vanishment. We establish several characterizations of the $L_2$-vanishment property, based on which a new design approach referred to as the $L_2$ low gain feedback approach is developed. Just as the $L_\infty$ low gain design is useful in the control of linear systems subject to actuator magnitude saturation, the $L_2$ low gain feedback is instrumental in the control of systems with control energy constraints. As applications of $L_2$ low gain feedback, the problem of semi-global stabilization of linear systems with energy constraints and the problem of linear time-delay system with energy constraints are solved in this paper. The notion of $L_2$-vanishment and the resulting $L_2$ low gain feedback are also extended to nonlinear systems. As in the $L_\infty$ low gain feedback considered in Part I, a systematic approach is also developed for $L_2$ low gain feedback design in this second part of the paper. Finally, an example involving a linearized model of the relative motion with respect to another in a circular orbit around the Earth is used to illustrate the effectiveness of the results developed in this paper.

I. INTRODUCTION

In Part I of this paper [27], we introduced the notion of $L_\infty$-vanishment for a low gain feedback law and derived necessary and sufficient conditions under which a particular low gain feedback possesses the $L_\infty$-vanishment property. In this second part of the paper, we will consider another notion: $L_2$-vanishment. The notation used throughout this part is consistent with the notation we used in Part I.

We begin our discussion with an example.

Example 1: Consider a system in the form of

$$\dot{x}(t) = Ax(t) + Bu(t),$$

with $(A, B)$ given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

Since $(A, B)$ is asymptotically null controllable with bounded controls (ANCBC), the system (1) can be semi-globally stabilized by $L_\infty$ low gain feedback. Indeed, the following feedback gain (Equation (20) in Part I)

$$K(\varepsilon) = \begin{bmatrix} -\frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} & \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} & \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} \\ -\frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} & \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} & \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} \\ -\frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} & \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} & \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} \end{bmatrix}, \tag{3}$$

which assigns all the eigenvalues of $A - BK(\varepsilon)$ at $\{-\varepsilon, -\varepsilon, -\varepsilon\}$ is an $L_\infty$ low gain feedback as it satisfies all the conditions in Corollary 1 in Part I. It follows that $\lim_{\varepsilon \to 0+} \|K(\varepsilon)e^{(A - BK(\varepsilon))t}\|_{L_\infty} = 0$. Therefore, the peak value in the control signal (in the absence of input saturation)

$$u_\varepsilon(t) = K(\varepsilon)e^{(A - BK(\varepsilon))t}x_0, \quad \|x_0\| < \infty, \tag{4}$$

can be reduced to an arbitrarily low level by decreasing the value of $\varepsilon$. That is, the actuator saturation can be avoided by decreasing $\varepsilon$ for any bounded set of initial conditions. Now, let

$$U_\varepsilon(t) = K(\varepsilon)e^{(A - BK(\varepsilon))t}. \tag{5}$$

Then it is easy to compute that (see, for example, [1])

$$\int_0^\infty \|K(\varepsilon)e^{(A - BK(\varepsilon))t}\|_F^2 \, dt = \frac{g(\sqrt{\varepsilon})}{10\sqrt{\varepsilon}(\sqrt{\varepsilon^2 + 1})^7}, \tag{6}$$

where $g(s) = 4s^{24} - 6s^{23} + 3s^{22} + 12s^{16} - 20s^{15} + 28s^{14} + 40s^8 + 28s^7 + 33s^6 + 8$. It follows from (6) that $\lim_{\varepsilon \to 0+} \|U_\varepsilon(t)\|_{L_2} = \infty$, which, in view of (4), implies that there exists bounded initial condition $x_0$ such that $\lim_{\varepsilon \to 0+} \|u_\varepsilon(t)\|_{L_2} = \lim_{\varepsilon \to 0+} \|U_\varepsilon(t)\|_{L_2} = \infty$. This indicates that, if we decrease $\varepsilon$ so as to avoid input saturation, the control energy that used to steer the initial state to the origin approaches infinity.

The above example shows that although the $L_\infty$ norm of the resulting control signal can be made arbitrarily small by decreasing $\varepsilon$, there exists peaking phenomenon in the $L_2$ norm of the control signal as $\varepsilon$ approaches zero. The question then arises: for a linear system $(A, B)$, can we design a stabilizing feedback gain such that the $L_2$ norm of the resulting control signals also approaches zero as $\varepsilon$ does, namely,

$$\lim_{\varepsilon \to 0+} \|K(\varepsilon)e^{(A - BK(\varepsilon))t}\|_{L_2} = 0? \tag{7}$$

In other words, can we steer bounded state to the origin with a control input that has arbitrarily small energy? The answer to this problem is quite positive.

In fact, it is natural and important to consider control problems with energy constraints as practical systems can...
only be powered with finite energy. However, these problems have not received as much attention as control systems with magnitude saturation, which have been widely studied in the past several decades (see, for instance, [4, 6], [7], [9, [16], [17], [23], [24], [25] and the references therein). Only very recently has the null controllability with vanishing energy problem been considered in [18] and [8]. Roughly speaking, a system is null controllable with vanishing energy (NCVE) if any bounded initial state of it can be steered to the origin with arbitrarily small energy cost. According to the results obtained in [18] and [8], it is interesting to note that the condition for NCVE happens to be the same as conditions for asymptotically null controllability with bounded controls (see, for example, [12], [21] and [22]).

Parallel to the \(L_{\infty}\)-vanishment considered in Part I [27] of this paper, in this second part of the paper, we will consider \(L_2\)-vanishment and establish complete characterizations of this property. As applications of the notion of \(L_2\)-vanishment, we can introduce the concept of \(L_2\) low gain feedback. It will be shown that \(L_2\) low gain feedback plays a critical role in the design of control systems with energy constraints in the control input as the \(L_{\infty}\) gain feedback plays in the design of control systems with magnitude constraints on the control input. As an illustration of the use of \(L_2\) low gain feedback in energy constrained control, we show that semi-global stabilization of NCVE linear systems with energy constraint is solvable via \(L_2\) low gain feedback. Moreover, the idea of \(L_2\) low gain feedback proposed in the paper can even be extended to time-delay systems and nonlinear systems. Indeed, we are able to show in this paper that a class of linear systems with input delay and control energy constraints can also be semi-globally stabilized via \(L_2\) low gain feedback. However, the result obtained for nonlinear system is still rather preliminary and further study is required.

We should point out that considering peaking phenomena of control systems is not new in the literature. For example, ref. [3] considered the bounded peaking in the optimal linear regulator with cheap control. However, the feedback gain \(F_c\) considered in [3] is of a high gain type. From this point of view, the problem considered in this paper is dual to the problem considered in [3] as our feedback gain \(K(\varepsilon)\) in (7) is of a low gain type. To the best of our knowledge, this dual problem has not been fully studied in the literature, though some specific design have been proposed in [10] and [12]. Moreover, we would also like to point out that, although our problem may be viewed as the dual problem considered in [3], the method we are to use and the results we will obtain are totally different from those in [3].

\section{\(L_2\)-Vanishment and Its Characterizations}

For easy reference, we recall the following assumption from Part I [27].

\textbf{Assumption 1:} The matrix \(A(\varepsilon) : [0, 1] \to \mathbb{R}^{n \times n}\) is a continuous matrix function of \(\varepsilon\) and such that \(\lambda(A(\varepsilon)) \subset \mathbb{C}^{-}\), \(\forall \varepsilon \in (0, 1]\) and \(\lambda(A(0)) \subset \mathbb{C}^{0}\).

Basically, the above assumption means that the parameter matrix \(A(\varepsilon)\) should be asymptotically stable for all \(\varepsilon \in (0, 1]\) and \(A(0)\) should not be exponentially unstable. Moreover, it is clear that \(A(\varepsilon)\) is bounded over \(\varepsilon \in [0, 1]\).

We then give the following definition of \(L_2\)-vanishment which is dual to the \(L_{\infty}\)-vanishment studied in Part I.

\textbf{Definition 1:} Given \(S(\varepsilon) : [0, 1] \to \mathbb{R}^{m \times n}\) and \(A(\varepsilon) : [0, 1] \to \mathbb{R}^{n \times n}\). Let \(A(\varepsilon)\) satisfy Assumption 1. Then \((S(\varepsilon), A(\varepsilon))\) is said to be \(L_2\)-vanishing if \(\lim_{\varepsilon \to 0^+} \|S(\varepsilon) e^{A(\varepsilon) t}\|_{L_2} = 0\).

The following proposition is a dual to Proposition 1 in Part I.

\textbf{Proposition 1:} Given \(S(\varepsilon) : [0, 1] \to \mathbb{R}^{m \times n}\) and \(A(\varepsilon) : [0, 1] \to \mathbb{R}^{n \times n}\). Assume that \(A(\varepsilon)\) satisfies Assumption 1. Then (1). The matrix pair \((S(\varepsilon), A(\varepsilon))\) is \(L_2\)-vanishing only if \(\lim_{\varepsilon \to 0^+} \|S(\varepsilon)\| = 0.2\). If \((S(\varepsilon), A(\varepsilon))\) is \(L_2\)-vanishing, then for an arbitrary integer \(l \geq 1\), \((S(\varepsilon) A^l(\varepsilon), A(\varepsilon))\) is also \(L_2\)-vanishing.

However, the following proposition is weaker than the corresponding results for \(L_{\infty}\)-vanishment (Proposition 2 in Part I).

\textbf{Proposition 2:} Assume that \(A(\varepsilon) : [0, 1] \to \mathbb{R}^{n \times n}\) satisfies Assumption 1. If \(A(0)\) is asymptotically stable, then \((S(\varepsilon), A(\varepsilon))\) is \(L_2\)-vanishing if and only if \(\lim_{\varepsilon \to 0^+} \|S(\varepsilon)\| = 0\).

\section{An Algebraic Characterizations of \(L_2\)-Vanishment}

We will present an algebraic characterization of \(L_2\)-vanishment in this subsection. The results are mainly based on the following technical lemma, which is a generalization of Lemma 2 in Part I. Let \(\mathcal{F} = \{f(\varepsilon) : [0, 1] \to \mathbb{R}^{n \times n}, \lim_{\varepsilon \to 0^+} f(\varepsilon) = 0\}\) exists. Clearly, \(f(\varepsilon) \in \mathcal{F}\) implies that \(\lim_{\varepsilon \to 0^+} f(\varepsilon) \geq 0\). Then the following result can be proven.

The proof is omitted due to space limitation.

\textbf{Lemma 1:} Let \(R_{i}(\varepsilon) : [0, 1] \to \mathbb{R}^{m \times n}, i \in 0, r - 1\), with \(r \geq 1\) a given integer, be some continuous matrix functions and \(f(\varepsilon) \in \mathcal{F}\). Denote \(W(\varepsilon, t) = \sum_{i=0}^{r-1} t^i R_{i}(\varepsilon) e^{-f(\varepsilon)t}\). Then \(\lim_{\varepsilon \to 0^+} \|W(\varepsilon, t)\|_{L_2} = 0\) if and only if \(\int^{\frac{1}{2}}(\varepsilon) f^{-1}(\varepsilon) R_{i}(\varepsilon) \in O^{m \times n}, \forall i \in 0, r - 1\).

Similar to the \(L_{\infty}\) case studied in Part I, we let \(T_{\lambda}(\varepsilon)\) be a matrix such that matrix \(A(\varepsilon)\) is transformed into its real Jordan from \(A_{\lambda}(\varepsilon) = T_{\lambda}(\varepsilon) A(\varepsilon) T_{\lambda}^{-1}(\varepsilon)\), with \(A_{\lambda}(\varepsilon) = \text{blkdiag}\{A_{\lambda}^{1}, A_{\lambda}^{2}, \ldots, A_{\lambda}^{n}\}\), where \(A_{\lambda}^{1}(\varepsilon) = f_{i}(\varepsilon) \in \mathbb{R}^{2m_{i} \times 2m_{i}}, i \in \overline{1, p}\), are given by

\begin{equation}
A_{\lambda}^{i}(\varepsilon) = \text{blkJordan}(O_{\omega_{i}(\varepsilon)} - f_{i}(\varepsilon) I_{2}),
\end{equation}

\begin{equation}
O_{\omega_{i}(\varepsilon)} = \begin{bmatrix}
0 & \omega_{i}(\varepsilon) \\
-\omega_{i}(\varepsilon) & 0
\end{bmatrix},
\end{equation}

with \(f_{i}(\varepsilon), \omega_{i}(\varepsilon) \in \mathcal{F}\) and \(A_{\lambda}^{n} = A_{\lambda}^{1}(g_{i}(\varepsilon)) \in \mathbb{R}^{n \times n}, i \in \overline{1, q}\), given by \(A_{\lambda}^{n} = \text{Jordan}(g_{i}(\varepsilon))\), in which \(g_{i}(\varepsilon) \in \mathcal{F}\). Clearly, we have \(\sum_{i=1}^{n_{1}} m_{i} + \sum_{i=1}^{n_{2}} n_{i} = n\). Consequently, we let \(S_{j}(\varepsilon) = S_{j}(\varepsilon) T_{\lambda}^{-1}(\varepsilon)\) and \(S_{j}(\varepsilon) = \left[ S_{j}^{1}, \ldots, S_{j}^{m_{1}}, S_{j}^{m_{2}}, \ldots, S_{j}^{m_{p}} \right]^{T}\) in which \(S_{j}^{1} = S_{j}^{1}(\varepsilon) \in \mathbb{R}^{m_{1} \times m_{1}}, i \in \overline{1, p}\), and \(S_{j}^{n} = S_{j}^{n}(\varepsilon) \in \mathbb{R}^{m_{n} \times m_{n}}, i \in \overline{1, q}\). Then results similar to those of Theorems 1 and 2 in Part I can be obtained.

\textbf{Theorem 1:} The matrix pair \((S(\varepsilon), A(\varepsilon))\) is \(L_2\)-vanishing if and only if \(f_{i}^{1}(\varepsilon) S_{j}^{1}(\varepsilon) A_{\lambda}^{i}(\varepsilon) \in \mathbb{R}^{m \times n}, i \in \overline{1, q}\).
Theorem 2: Assume that all the eigenvalues of $A(\varepsilon) \in \mathbb{R}^{n \times n}$ are $-f(\varepsilon)$ with $f(\varepsilon) \in \mathcal{F}$. Moreover, let $r$ be the maximal algebraic multiplicity of $f(\varepsilon)$ as the eigenvalue of $A(\varepsilon)$. Then $(S(\varepsilon), A(\varepsilon))$ is $L_2$-vanishing if and only if $j \in \mathbb{N}, i \in \mathbb{N}_0$.

We then can show the relationship between the $L_2$-vanishment and $L_\infty$-vanishment studied in Part I.

Corollary 1: Let $A(\varepsilon)$ satisfy Assumption 1. If $(S(\varepsilon), A(\varepsilon))$ is $L_2$-vanishing, then $(S(\varepsilon), A(\varepsilon))$ is $L_\infty$-vanishing.

B. Lyapunov Inequality Characterization of $L_2$-Vanishment

In this subsection, we establish a Lyapunov inequality characterization of the $L_2$-vanishment property.

Theorem 3: Let $A(\varepsilon)$ satisfy Assumption 1. Then the following statements are equivalent. 1) The matrix pair $(S(\varepsilon), A(\varepsilon))$ is $L_2$-vanishing. 2) The unique positive semi-definite solution $L(\varepsilon)$ to the Lyapunov matrix equation $A^T(\varepsilon)L(\varepsilon) + L(\varepsilon)A(\varepsilon) = -S^T(\varepsilon)S(\varepsilon)$ satisfies $\lim_{\varepsilon \to 0^+} L(\varepsilon) = 0$. 3) There exist a scalar $\varepsilon^* \in (0, 1)$, a constant scalar $\kappa > 0$ and a matrix $P(\varepsilon) : [0, \varepsilon^*) \to \mathbb{R}^{n \times n}$, which is positive semi-definite for all $\varepsilon \in (0, \varepsilon^*)$, such that

$$A^T(\varepsilon)P(\varepsilon) + P(\varepsilon)A(\varepsilon) \leq -\kappa S^T(\varepsilon)S(\varepsilon),$$

for all $\varepsilon \in [0, \varepsilon^*)$ and $\lim_{\varepsilon \to 0^+} P(\varepsilon) = 0$ are satisfied.

We next introduce the notion of $L_2$-finiteness, which is dual to $L_\infty$-finiteness discussed in Part I.

Definition 2: Given $S(\varepsilon) : (0, 1] \to \mathbb{R}^{m \times n}$ and $A(\varepsilon) : [0, 1] \to \mathbb{R}^{n \times n}$. Let $A(\varepsilon)$ satisfy Assumption 1. Then $(S(\varepsilon), A(\varepsilon))$ is said to be $L_2$-finite if $\lim_{\varepsilon \to 0^+} \|S(\varepsilon)e^{Ah(\varepsilon)}\|_{L_2} < \infty$.

Results similar to those in Theorems 1, 2 and 3 can be obtained. We only show the corresponding result to Theorem 2 without a proof.

Theorem 4: Assume that all the eigenvalues of $A(\varepsilon)$ are $-f(\varepsilon)$ with $f(\varepsilon) \in \mathcal{F}$. Moreover, let $r$ be the maximal algebraic multiplicity of the eigenvalue $f(\varepsilon)$. Then $(S(\varepsilon), A(\varepsilon))$ is $L_2$-finite if and only if

$$f(-\varepsilon^* + \frac{1}{2})S(\varepsilon)(A(\varepsilon) + f(\varepsilon)I_n)^i \in \mathbb{R}^{m \times n}(\varepsilon), i \in (0, r - 1).$$

C. Extension of $L_2$-Vanishment to Nonlinear Systems

We take a new look at Definition 1. Consider the following family of linear systems

$$\dot{x}(t) = A(\varepsilon)x(t), \quad y(t) = S(\varepsilon)x(t),$$

where $x(0) = x_0 \in \mathbb{R}^n, (S(\varepsilon), A(\varepsilon))$ is defined in Definition 1 and $\varepsilon \in (0, 1]$. Notice that

$$\|y(t)\|_{L_2} = \left(\int_0^\infty \|S(\varepsilon)e^{A(\varepsilon)t}x_0\|^2 dt\right)^{\frac{1}{2}}.$$  

It follows from Definition 1 that $(S(\varepsilon), A(\varepsilon))$ is $L_2$-vanishing if and only if the $L_2$ norm of the output of system (11) for an arbitrary bounded initial condition $x_0 \in \mathbb{R}^n$ approaches zero as $\varepsilon$ does. This observation implies the possibility of extending the definition of $L_2$-vanishment for matrix pair $(S(\varepsilon), A(\varepsilon))$ to nonlinear systems.

Definition 3: Consider the following family of nonlinear systems

$$\dot{x}(t) = A(\varepsilon)x(t), \quad y(t) = S(\varepsilon)x(t),$$

where $x(0) = x_0 \in \mathbb{R}^n, A(\varepsilon, x) : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to $\varepsilon$ and globally Lipschitz with respect to $x, S(\varepsilon, x) : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $\varepsilon \in (0, 1]$. Assume that for arbitrary $\varepsilon \in (0, 1]$, the system (12) is globally asymptotically stable. Then the system (12) is said to be $L_2$-vanishing if $\|x_0\| \leq D < \infty \Rightarrow \lim_{\varepsilon \to 0^+} \|y(t)\|_{L_2} = 0$.

With this, Theorem 3 can be extended as follows.

Theorem 5: The system in (12) is $L_2$-vanishing if there exist a scalar $\varepsilon^* > 0$ and a function $V(\varepsilon, x) : [0, \varepsilon^*] \times \mathbb{R}^n \to \mathbb{R}_0^+, V(\varepsilon, 0) = 0$ such that $\|x\| \leq D < \infty \Rightarrow \lim_{\varepsilon \to 0^+} V(\varepsilon, x) = 0$ and $V(\varepsilon, x(t)) \leq -\kappa(\varepsilon)\|S(\varepsilon, x(t))\|^2$ where $\kappa(\varepsilon) : [0, \varepsilon^*] \to \mathbb{R}_0^+$ is bounded over $\varepsilon \in [0, \varepsilon^*]$.

III. APPLICATIONS

In this section, we present several applications of the theory of $L_2$-vanishment developed in Section II.

A. $L_2$ Low Gain Design for Control Energy Constrained Linear Systems

Consider a linear system in the following form

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, are, respectively, the state and input vectors. Let $x(t, x_0, u)$ denote the solutions of (13) with initial condition $x_0$ and input $u$. We recall the following definition of null controllability with vanishing energy for system (13). Denote

$$L_2(T, \mathbb{R}^m) = \left\{ f(t) : [0, T] \to \mathbb{R}^m \mid \int_0^T \|f(t)\|^2 dt < \infty \right\}.$$  

Definition 4: ([8]) The system (13) (or the matrix pair $(A, B)$) is said to be null controllable with vanishing energy (NCVE) if for each initial $x(0) = x_0$, there exists a sequence of pairs $(T_N, u_N), 0 \leq T_N < \infty, u_N \in L_2(T_N, \mathbb{R}^m)$ such that $x(T_N, x_0, u_N) = 0$ and $\lim_{N \to \infty} \int_0^{T_N} \|u_N(t)\|^2 dt = 0$.

Regarding the criterion for null controllability with vanishing energy, we recall the following condition from [18] in which the results are developed for infinite dimensional linear systems.

Lemma 2: The linear system (13) is NCVE if and only if $(A, B)$ is controllable in the ordinary sense and all the eigenvalues of $A$ are located in the closed left-half $s$-plane.

It follows from the above lemma that the conditions for null controllability with vanishing energy happens to be the conditions for asymptotically null controllability with bounded controls ([22]).
Definition 5: \( (\mathcal{L}_2, \text{Low Gain Feedback}) \) Assume that \((A, B) \in (\mathbb{R}^{n \times n}, \mathbb{R}^{n \times m}) \) is NCVE. A stabilizing feedback gain \( K(\varepsilon) : [0, 1] \to \mathbb{R}^{m \times n} \) is said to be an \( \mathcal{L}_2 \) low gain feedback if \( (K(\varepsilon), A - BK(\varepsilon)) \) is \( \mathcal{L}_2 \)-vanishing, namely, \( \lim_{\varepsilon \to 0^+} \|K(\varepsilon) e^{(A-BK(\varepsilon))t}\|_{\mathcal{L}_2} = 0 \).

The following corollary can be immediately obtained by using the results developed in Section II.

Corollary 2: The ARE based low gain (See Lemma 3 in Part I) is an \( \mathcal{L}_2 \) low gain.

To illustrate the application of the \( \mathcal{L}_2 \) low gain feedback, we consider the following \( \mathcal{L}_2 \) semi-global stabilization problem. Let \( U_E = \{ u(t) : [0, \infty) \to \mathbb{R}^m | \int_0^\infty \|u(t)\|^2 \, dt \leq 1 \} \). (14)

Problem 1: \((\mathcal{L}_2, \text{Semi-global Stabilization})\) Consider the linear system (13) which is NCVE. For any given bounded set \( \Omega \subset \mathbb{R}^n \), find a control \( u(t) \in U_E \) such that the closed-loop system is locally asymptotically stable with \( \Omega \) contained in the domain of attraction.

Theorem 6: Assume that \((A, B) \) is an \( \mathcal{L}_2 \) low gain feedback for the matrix pair \((A, B)\), then there exists an \( \varepsilon > 0 \) such that \( u(t) = -K(\varepsilon)x(t) \), \( \forall \varepsilon \in (0, \varepsilon^*] \) solves Problem 1.

Remark 1: With the help of Theorem 5, the \( \mathcal{L}_2 \) low gain design principle given in Definition 5 can be readily extended to nonlinear systems. We believe that \( \mathcal{L}_2 \) low gain feedback for nonlinear systems by adopting the idea of Slow-Peaking Dilation principle in [20] is a good study project in the future.

B. Semi-global Stabilization of Time Delay Linear System with Energy Constraints

In this subsection, we use the idea used in proving Theorem 5 for nonlinear systems to study the semi-global stabilization problem of time-delayed linear system with energy constraints.

Consider the following linear system with input delay
\[
\dot{x}(t) = Ax(t) + Bu(t - \tau),
\]
where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are respectively the state and input vectors, and \( \tau > 0 \) is a known constant scalar representing the delay in the control input. Assume that the matrix pair \((A, B)\) is NCVE, namely, all the poles of \( A \) are on the closed left-half s-plane and \((A, B)\) is stabilizable. The problem we are interested in is as follows:

Problem 2: \((\mathcal{L}_2, \text{Semi-global Stabilization of Input-delayed Linear Systems})\) Consider the linear time-delayed system (15) where \( \tau > 0 \) is a known constant. For any given bounded set \( \Omega \subset \mathbb{U}_{n, \tau} \), find a control \( u(t) \in U_E \) which is defined in (14) such that the closed-loop system is locally asymptotically stable with \( \Omega \subset \mathbb{U}_{n, \tau} \) contained in the domain of attraction.

We can then establish the following state feedback solution to Problem 2. The proof is rather involved and omitted due to space limitation.

Theorem 7: Consider the linear system (15) with an arbitrarily large but bounded delay \( \tau \). Assume that all the eigenvalues of \( A \) are on the imaginary axis and \((A, B)\) is controllable. Then there exists an \( \varepsilon^* \in (0, 1] \) such that the family of linear state feedback laws
\[
u(t) = -B^T P(\varepsilon)e^{A^T}x(t), \quad \forall \varepsilon \in (0, \varepsilon^*],
\]
where \( P(\varepsilon) \) is the unique positive definite solution to the parametric ARE
\[
A^T P + PA - PBB^T P = -\varepsilon P,
\]
solves Problem 2.

C. On Lyapunov Inequality Characterization of Eigenstructure Assignment Based Low Gain Feedback

The frequently used eigenstructure assignment based low gain design (see, for example, [12]) is both an \( \mathcal{L}_\infty \) low gain and an \( \mathcal{L}_2 \) low gain. In this subsection, we will present a simpler Lyapunov inequality characterization of this eigenstructure assignment based low gain feedback and show that it is indeed both an \( \mathcal{L}_\infty \) and an \( \mathcal{L}_2 \) low gain.

We first recall the eigenstructure assignment based low gain design from [12]. Assume that \((A, B)\) is ANCBC. Then there exists two nonsingular matrices \( T_0 \in \mathbb{R}^{n \times n} \) and \( T_1 \in \mathbb{R}^{m \times m} \) such that \( A_S = T_0^{-1} A T_0 \) and \( B_S = T_1^{-1} B T_1 \) is a stable matrix, \( B_{S2}, B_{S1} \) and \( B_{S3} \) are some matrices with appropriate dimensions, and \((A_l, B_l)\) are given by \( A_l = \text{blkdiag}[A_1, A_2, \ldots, A_l] \) and
\[
B_l = \begin{bmatrix}
B_1 & B_{1, 2} & \cdots & B_{1, l} \\
0 & B_2 & \cdots & B_{2, l} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_l
\end{bmatrix},
\]
with each matrix pair \((A_i, B_i) \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}\) being controllable and all the eigenvalues of \( A_l \) on the imaginary axis. Moreover, each matrix pair \((A_l, B_l)\) in (19) takes the controllability canonical form.

Let \( K_i(\varepsilon), i \in \overline{1, l} \), be the unique feedback gain such that the eigenvalues of \((A_l - B_l K_i(\varepsilon))\) are placed at \(-\varepsilon + \lambda(A_i), i \in \overline{1, l}\). Then the eigenstructure assignment based low gain feedback can be constructed as \( K_{EA}(\varepsilon) = T_1 K_S(\varepsilon) T_0^{-1} \text{blkdiag} \{K_1(\varepsilon), 0 \} \) where \( K_1(\varepsilon) = \text{blkdiag} \{K_1(1) \}, K_2(1) \), \( \ldots, K_l(1) \}. \)

Theorem 8: Let \( P_l(\varepsilon), i \in \overline{1, l} \), be the unique positive definite solutions to the series of parametric AREs
\[
A_i^T P_l(\varepsilon) + P_l(\varepsilon) A_i - \rho_i^l P_l(\varepsilon) B_i B_i^T P_l(\varepsilon) = -\varepsilon P_l(\varepsilon) \]
where \((A_i, B_i)\) are determined in (19). Define \( P_l(\varepsilon) \) as \( P_l(\varepsilon) = \text{blkdiag} \{\rho^{-1} P_l(\varepsilon), \ldots, \rho P_l(\varepsilon), P_l(\varepsilon) \} \). Let \( P_0 > 0 \) be the unique solution to the Lyapunov matrix equation
\[
A_0^T P_0 + P_0 A_0 = -I_m. \]
Then there exist two scalars \( \rho^* \in (0, 1] \) and \( \varepsilon^* \in (0, 1] \) such that \( P_l(\varepsilon) > 0, \forall \rho \in (0, \rho^*] \) and
\[
(A_S - B_S K_S(\varepsilon))^T P_S(\varepsilon) + P_S(\varepsilon) (A_S - B_S K_S(\varepsilon)) \leq -\frac{\varepsilon}{2} P_S(\varepsilon) - \frac{\rho^*}{2} K_S^T(\varepsilon) K_S(\varepsilon),
\]
(20)
is satisfied for all \( \varepsilon \in (0, \varepsilon^*], \) where \( P_\varepsilon (\varepsilon) = \text{blkdiag}\{P_1 (\varepsilon), P_0 (\varepsilon)\}. \) Moreover, \( \lim_{\varepsilon \to 0} P_\varepsilon (\varepsilon) > 0 \) and \( \lim_{\varepsilon \to 0} P_\varepsilon (\varepsilon) = 0. \)

The following corollary follows immediately from Theorem 8, Theorem 2 and from Theorem 3 in Part I.

**Corollary 3:** The eigenstructure assignment based low gain is both an \( L_\infty \) low gain and an \( L_2 \) low gain.

Theorem 8 gives elegant Lyapunov inequality characterization of the eigenstructure assignment based low gain design. With this characterization, the ARE based low gain design and the eigenstructure assignment based low gain design can be unified under a Lyapunov inequality framework, even though they have been developed separately. More specifically, (20) shows that low gain feedback resulting from the EA method satisfy the conditions in Item 3 of Theorem 2 and the conditions in Theorem 3 in Part I.

Another contribution of Theorem 8 is that it provides a very simple Lyapunov function of the closed-loop system with system matrix \( A_S - B_S K_S (\varepsilon) \). This Lyapunov function plays important roles in solving many related control problems. For example, the positive definite matrix \( P_\varepsilon (\varepsilon) \) can be used to design low-and-high gain feedback ([14]), robust semi-global/global stabilization ([15]), gain scheduling control ([11]), and time-delayed feedback ([12] and [13]).

**IV. A SYSTEMATIC APPROACH TO \( L_2 \) LOW GAIN FEEDBACK DESIGN**

In this section, we use the method in Part I [27] to develop a systematic approach for \( L_2 \) low gain feedback design. As the development is quite similar to that in Part I for \( L_\infty \) low gain feedback, we only show the main result. Also, for the same reason given in Part I, for an NCVE matrix pair \( (A, B) \in (\mathbb{R}^{m \times n}, \mathbb{R}^{m \times n}) \), the \( L_2 \) low gain we considered belongs to the following set

\[ K_{\varepsilon}(\varepsilon) = \left\{ K(\varepsilon) \in \mathbb{R}^{m \times m} \mid A(\varepsilon) = -\varepsilon + \lambda(\varepsilon) \right\} \]

Let \( \mathcal{N}(s) \in \mathbb{R}^{m \times n}[s] \) and \( \mathcal{D}(s) \in \mathbb{R}^{m \times m}[s] \) be a pair of right coprime polynomials such that \((A - \sigma I) \mathcal{N}(\sigma) = BD(\sigma)\). Let \( \mathcal{N}_e(s) = \mathcal{N}(s - \varepsilon) \triangleq \sum_{i=0}^{\omega} N_i (\varepsilon) s^i \) and \( \mathcal{D}_e(s) = D(s - \varepsilon) \triangleq \sum_{j=0}^{\omega} D_j (\varepsilon) s^j \) where \( \omega \) is some positive scalar. Then it follows from Theorem 6 in Part I that the set \( K_{\varepsilon}(\varepsilon) \) can be characterized as

\[ K_{\varepsilon}(\varepsilon) = \left\{ K(\varepsilon) = W(\varepsilon) V^{-1}(\varepsilon) \mid \det(V(\varepsilon)) \neq 0 \right\} \]

in which \( (V(\varepsilon), W(\varepsilon)) \) is given by \( V(\varepsilon) = \sum_{i=0}^{\omega} N_i (\varepsilon) Z A_j^i, W(\varepsilon) = \sum_{j=0}^{\omega} D_j (\varepsilon) Z A_j^i \) where \( A_j^i \) is similar to \( A \) and \( Z = Z(\varepsilon) : [0, 1] \rightarrow \mathbb{R}^{m \times m} \) is an arbitrary parametric matrix and is continuous in \( \varepsilon \) representing the degree of the freedom in the solution.

Let \( A_j \) be given by \( A_j = \text{blkdiag}\{A_{j1}^{\omega}, A_{j2}^{\omega}, \ldots, A_{jn}^{\omega}\} \}, \) where \( A_{j1}^{\omega} = A_{j1}^{\omega}(\lambda) \in \mathbb{R}^{2m \times 2m}, \lambda_i > 0, i \in \bar{1}, \bar{p}, \) are in the form of (8) with \( O_{\omega j} \) defined as (9) where \( \omega_j (\varepsilon) = \omega_j > 0, \) and \( A_{j1}^{\omega} = A_{j1}^{\omega}(\mu_i) \in \mathbb{R}^{n \times m}, \mu_i > 0, i \in \bar{1}, q, \) are given by \( A_{j1}^{\omega}(\mu_i) = \text{Jordan}(\mu_i). \) Consequently, we let \( W(\varepsilon) = \left[ W_{j1}^{m} \cdots W_{j1}^{m} \right] \), in which \( W_{j1}^{m} = W_{j1}^{m}(\varepsilon) \in \mathbb{R}^{m \times 2m}, i \in \bar{1}, \bar{p}, \) and \( W_{j1}^{m} = W_{j1}^{m}(\varepsilon) \in \mathbb{R}^{m \times n}, i \in \bar{1}, q. \) Let

\[ Z_{\varepsilon} = \{ Z(\varepsilon) \in \mathbb{R}^{m \times n} \mid \text{rank}(V(\varepsilon)) = n, \forall \varepsilon \in [0, a]\}. \]

Then similar to Theorem 7 in Part I, by using Theorem 1, we have the following result on the design of \( L_2 \) low gain feedback.

**Theorem 9:** Let \( K(\varepsilon) \in K_{\varepsilon}(\varepsilon). \) Then \( K(\varepsilon) \) is an \( L_2 \) low gain for the matrix pair \((A, B)\) in the sense of Definition 5 if and only if there exists a matrix \( Z(\varepsilon) \in Z_{\varepsilon} \) for some \( \varepsilon^* > 0 \) such that

\[ \left( \varepsilon + \lambda_i \right)^{-(j + \frac{i}{2})} W_{j1}^{m}(\varepsilon, \lambda_i) I_2 \in \mathbb{O}^{m \times 2m}(\varepsilon), j \in \bar{0}, m_i - 1, i \in \bar{1}, p \]

and \( \left( \varepsilon + \lambda_i \right)^{-(i + \frac{j}{2})} W_{j1}^{m}(\varepsilon, \lambda_i) I_{2n} \in \mathbb{O}^{m \times n}(\varepsilon), j \in \bar{0}, m_i - 1, i \in \bar{1}, q. \)

Also, the following result is parallel to Theorem 8 in Part I.

**Theorem 10:** Assume that the matrix \( A \) has a single eigenvalue \(-\lambda \leq 0\) and the maximal algebraic multiplicity of the eigenvalue is \( r \). Let \( K(\varepsilon) \in K_{\varepsilon}(\varepsilon) \), where \( A_j = A \). Then \( K(\varepsilon) \) is an \( L_2 \) low gain for the matrix pair \((A, B)\) in the sense of Definition 5 if and only if there exists a matrix \( Z(\varepsilon) \in Z_{\varepsilon} \) for some \( \varepsilon^* > 0 \) such that \( \left( \varepsilon + \lambda \right)^{-(i + \frac{j}{2})} W(\varepsilon, \lambda^i) = \mathbb{O}^{m \times n}(\varepsilon), i \in \bar{0}, r - 1. \)

**V. A NUMERICAL EXAMPLE**

Consider a linear system

\[ \dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & -2\omega & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \]

where \( A, B \) is as in (21) and \( \tau \) is some known and bounded positive scalar. As \((A, B)\) is NCVE, it follows from Theorem 7 that the system (22) can be semi-globally stabilized with bounded energy by linear state feedback.

According to (16), the state feedback controller is given by \( u(t) = -B^T P(\varepsilon) e^{A^T x(t)}. \) For simulation, we choose the initial condition \( x(\theta) = x_0, \theta \in [-\tau, 0], \) where \( x_0 = \begin{bmatrix} 2 & -1 & 2 & -1 \end{bmatrix}^T. \) For different values of the low gain parameter \( \varepsilon \), the \( L_2 \) norms of the input signals are plotted in Fig. 1, from which we clearly see that \( \|u(t)\|_{\varepsilon} \) approaches zero as \( \varepsilon \) does, namely, the \( L_2 \) semi-global stabilization problem for system (22) can be solved. Specially, for \( \varepsilon = 0.15 \) and \( \varepsilon = 0.1 \), the state evolution and control signals are shown in Fig. 2.

**VI. CONCLUSIONS**

In parallel to Part I [27], in this second part of the paper, we have introduced and studied a new notion, the \( L_2 \) vanishment in low gain feedback design. A series of necessary and sufficient conditions for guaranteeing this property
Fig. 1. $L_2$ norm of the control signal for different $\epsilon$.  

Fig. 2. State evolution and control signals of the time-delayed system (22) for different values of $\epsilon$.  

were established. Based on this new notion, a new low gain design approach referred to as the $L_2$ low gain feedback was developed. Unlike the $L_\infty$ low gain feedback considered in Part I for magnitude constraints on the control signal, the $L_2$ low gain feedback plays an important role in control of systems with constraints on the energy of the control input. As applications of this new design methodology, the problems of semi-global stabilization of linear systems with actuator energy constraints and semi-global stabilization of linear time-delayed systems with energy constraints were solved. It is expected that this new design approach can be used to deal with many other control problems with energy constraints. Moreover, we have also extended the notion of $L_2$-vanishment and $L_2$ low gain feedback design to nonlinear systems. However, the results obtained here for nonlinear systems are rather preliminary and further study on the topic is under way.

REFERENCES