This paper is concerned with stabilization of a linear system with distributed input delay and input saturation. Both constant and time-varying delays are considered. In the case that the input delay is constant, under the stabilizability assumption on an auxiliary system, it is shown that the system can be stabilized by state feedback for an arbitrarily large delay as long as the open-loop system is not exponentially unstable. In the case that the input delay is time-varying, but bounded, it is shown that the system can be stabilized by state feedback if the non-asymptotically stable poles of the open-loop system are all located at the origin. In both cases, stabilizing controllers are explicitly constructed by utilizing the parametric Lyapunov equation based low gain design approach recently developed. It is also shown that in the presence of actuator saturation and under the same assumptions on the system, these controllers achieve semi-global stabilization. Some discussions on the assumptions we impose on the system are given. A numerical example illustrates the effectiveness of the proposed stabilization approach.

1. Introduction

Time delay as a primary source of instability and performance degradation is frequently encountered in engineering, such as chemical systems and networked control systems (Gao, Chen & Lam, 2008; Hale & Verduyn, 1993). Since a time delay system is inherently an infinite dimensional system, it has been a challenging problem to design controllers for such a system. As a result, during the past several decades, a great deal of research results have been reported in the literature that deal with various problems that have been well studied for non-delayed systems (see Fiagbedzi & Pearson, 1987; Gu, 2003; Hale & Verduyn, 1993; Lam, Gao & Wang, 2005; Richard, 2003; Wu, He & She, 2004, and the references therein).

Distributed delay as one of the most frequently encountered delay types occurs in many practical systems (see, for example Fiagbedzi & Pearson, 1987; Hale & Verduyn, 1993). During the 70s, some authors (for example, MacDonald MacDonald, 1978) pointed out that in applications to biology the use of distributed delays often leads to models that are more tractable and more realistic than those with discrete delays. However, there are much fewer results available to check the stability of systems with distributed delays or to stabilize them. Recently, linear matrix inequality based approaches have been developed to deal with such systems (see Gu, 2003; Gu, Han, Luo & Niculescu, 2001; Kolmanovskii & Richard, 1997; Xu & Chen, 2004, and the references given there).

On the other hand, practical systems are also subject to input saturation. Like time delays, input saturation can also cause performance degradation and even instability of the closed-loop system if it is ignored in the controller design. For linear systems with input saturation, many results have been published in the past several decades on various control problems (see, for example Lin & Saberi, 1993; Sussmann, Sontag & Yang, 1994; Zhou, Li & Duan, 2008, and the references therein). Control design that takes into account input saturation has been recognized to be a difficult problem. It turns out that control design in the presence of both input saturation and time delays is even more difficult. During the past several years, some researchers have recognized this difficulty and a few results have been published in the literature,
(see, for example Lin & Fang, 2007; Mazenc, Mondié & Niculescu, 2004; Tarbouriech & da Silva, 2000, and the references therein). In particular, with the help of low gain feedback, especially, the parametric Lyapunov equation based low gain feedback, we have recently solved the semi-global stabilization problem for linear systems with both time delay and saturation in the input (Lin & Fang, 2007; Zhou, Lin & Duan, 2009, 2010a,b).

In this paper, we consider the problem of stabilization of linear systems with distributed input delay and input saturation. Linear systems with distributed delays have been extensively studied in the finite spectrum assignment framework. However, the resulting controller contains a distributed term and is hard to implement (see, for example Richard, 2003; VanAssche, Dambrine, Lafay & Richard, 1999). We will show in this paper that, under some reasonable assumptions, the systems can be stabilized either by static state feedback or observer based output feedback in the presence of arbitrarily large but bounded delay in the control input. Both constant delay and time-varying delay are considered. The stabilizing controllers are explicitly constructed by using the parametric Lyapunov equation based low gain feedback design approach we recently developed. The construction of these controllers involves only solving a linear matrix equation. We further show that these linear controllers can also achieve semi-global stabilization when the system is also subject to input saturation.

To show that the assumption imposed on the system is not restrictive, we use a scalar system to give some explanations, which we believe have its own value. A numerical example is worked out to show the effectiveness of the proposed stabilization approach. We would like to point out that distributed delay systems are much more difficult to deal with than their discrete delay counterparts considered in Zhou et al. (2009, 2010a,b). When the delay is constant, the stabilizing controllers involve integrations of the coefficient matrices and some key idea found in prediction based approach is required to establish the closed-loop stability. When the delay is time-varying, the construction of stabilizing controllers relies on much more involved matrix inequalities. Our results established in this paper complete our early results on this topic and help to identify some new problems that will be solved in the future work.

This paper is organized as follows. The problem formulation is given in Section 2. The global stabilization problem in the absence of input saturation and semi-global stabilization in the presence of input saturation are then solved in Sections 3 and 4, respectively. A discussion on the conservatism of the assumptions imposed on the system is presented in Section 5. In Section 6, a numerical example is worked out to show the effectiveness of the proposed approaches. Finally, this paper is concluded in Section 7.

Notation. The notation used in this paper is fairly standard. We use $A^t$, $\text{tr}(A)$, $\det(A)$ and $\text{rank}(A)$ to denote the transpose, the trace, the determinant and the rank of matrix $A$, respectively. For a square matrix $P$, $P > 0$ means that $P$ is positive definite. For a positive scalar $r$, let $\mathcal{V}_{r, T} = \mathcal{V}([-r, 0], \mathbb{R}^n)$ denote the Banach space of continuous vector-valued functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence, and $x_t = x(t + \theta), \forall \theta \in [-r, 0]$. Finally, for a stable transfer function $G(s)$, we use $\|G(s)\|_\infty$ to denote its $H_\infty$ norm, namely, $\|G(s)\|_\infty = \max_{\omega \in \mathbb{T}}|G(j\omega)|$ where $j = \sqrt{-1}$.

2. Problem formulation

Consider the following linear system with distributed input delay

$$\dot{x}(t) = Ax(t) + \int_0^t B(\sigma)u(t - \sigma)d\sigma,$$  \hspace{1cm} (1)

where $A \in \mathbb{R}^{n \times n}$ is the system matrix and $B(s) : [0, \tau] \rightarrow \mathbb{R}^{n \times m}$ is a piecewise continuous matrix function denoting the distributed control matrix of the system. The stabilization problem for system (1) is stated as follows.

**Problem 1 (Global Stabilization).** Consider system (1). For any given arbitrarily large but bounded delay $\tau$, find a controller $u(t)$ such that the resulting closed-loop system is asymptotically stable.

We also consider system (1) in the presence of input saturation,

$$\dot{x}(t) = Ax(t) + \int_0^t B(\sigma)\text{sat}(u(t - \sigma))d\sigma,$$  \hspace{1cm} (2)

where $\text{sat}(\cdot)$ is the standard vector-valued saturation functions defined as

$$\text{sat}(u) = \text{sat}(u_1) \text{sat}(u_2) \cdots \text{sat}(u_m)^T,$$  \hspace{1cm} (3)

with $\text{sat}(u_j) = \text{sign}(u_j) \max[1, |u_j|]$. Here, without loss of generality, we have assumed the unity saturation level. Non-unity saturation level can be absorbed by the matrix $B(\sigma)$ and the feedback gain. For system (2), the problem we are to solve in this paper is the following.

**Problem 2 (Semi-Global Stabilization).** Consider system (2). For any arbitrarily large but bounded delay and set $\Omega \subset \mathcal{V}_{\tau, T}$, find a controller $u(t)$ such that the resulting closed-loop system is asymptotically stable at the origin with $\Omega$ contained in the domain of attraction.

In this paper, we also present solutions to Problems 1 and 2 under the assumption that the delay in the systems (1) and (2) may be time-varying and even unknown, namely, we consider the following time-delay system both in the absence of input saturation

$$\dot{x}(t) = Ax(t) + \int_0^{\tau(t)} B(\sigma)u(t - \sigma)d\sigma,$$  \hspace{1cm} (4)

where $\tau(t) : \mathbb{R} \rightarrow \mathbb{R}^+$ is some scalar function representing the time-varying or unknown delay, and in the presence of input saturation

$$\dot{x}(t) = Ax(t) + \int_0^{\tau(t)} B(\sigma)\text{sat}(u(t - \sigma))d\sigma.$$  \hspace{1cm} (5)

Time-varying and/or unknown delay is clearly more challenging to deal with than constant and known delay. We will identify conditions under which these problems are solvable and explicitly construct controllers that solve them.

3. Solutions to the global stabilization problem

3.1. The case of constant delays

In this subsection, we present solutions to Problem 1 under the assumption that the delay $\tau$ is constant, namely, we consider system (1). To this end, we should impose some stabilizability assumptions on the open-loop system. Let

$$\mathcal{B} = \int_0^{\tau} e^{-\alpha(t)} B(\sigma)d\sigma.$$  \hspace{1cm} (6)

We assume in this paper that $(A, \mathcal{B})$ is stabilizable in the ordinary sense. An explanation of this assumption on $(A, \mathcal{B})$ is given as follows.

**Remark 1.** Let $\lambda < 0$ be a fixed scalar. It is said that system (1) is $\lambda$-stabilizable if there exists a control $u$ such that

$$\lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{-\lambda t} |u(t)|dt = 0 \text{ and } \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{-\lambda t} x(t)dt = 0 \text{ (Olbrot, 1978)}.$$ Moreover, system (1) is said to be spectral assignable if it is $\lambda$-stabilizable for any real $\lambda$. It is shown in Olbrot (1978) that the system is $\lambda$-stabilizable (spectral assignable) if and only if $(A, \mathcal{B})$ is stabilizable (controllable) in the ordinary sense.
We also assume in this subsection that all the poles of A are on the closed left-half plane. Under such a condition, there exists a nonsingular matrix $T$ such that system (1) is transformed into the following one

$$
\begin{cases}
\dot{x}_\alpha(t) = A_\alpha x_\alpha(t) + \int_0^t B_\alpha(\sigma)u(t - \sigma)d\sigma, \\
\dot{x}_\beta(t) = A_\beta x_\beta(t) + \int_0^t B_\beta(\sigma)u(t - \sigma)d\sigma,
\end{cases}
$$

(7)

where $[x_\alpha^T(t) x_\beta^T(t)]^T = x_n(t) = T x(t)$ and

$$
A_n = T A T^{-1} = \begin{bmatrix} A_\alpha & 0 \\ 0 & A_\beta \end{bmatrix}, \quad B_n(\sigma) = T B(\sigma) = \begin{bmatrix} B_\alpha(\sigma) \\ B_\beta(\sigma) \end{bmatrix},
$$

in which $A_\alpha \in \mathbb{R}^{n_\alpha \times n_\alpha}$ is asymptotically stable and all the eigenvalues of $A_\alpha$ are on the imaginary axis. Clearly, $n_\alpha + n_\beta = n$. It is easy to verify that $(A, B)$ is stabilizable if and only if $(A_\alpha, B_\alpha)$ is controllable, where $A_\alpha = \int_0^t e^{-\beta\sigma}B_\beta(\sigma)d\sigma$. As $A_\beta$ is asymptotically stable, system (7) can be stabilized by state feedback if and only if the first subsystem in (7) can be stabilized by state feedback.

Based on the above explanation, we can impose, without loss of generality, the following assumption on system (1).

**Assumption 1.** All of the eigenvalues of $A \in \mathbb{R}^{n \times n}$ are on the imaginary axis and $(A, B)$ is controllable.

**Remark 2.** A system characterized by $(A, B)$ is known in the constrained control literature to be asymptotically null controllable with bounded controls (ANBC) if it is not exponentially unstable and is stabilizable. This property is necessary and sufficient for a linear system to be globally or semi-globally stabilizable in the presence of input saturation (Sussmann et al., 1994). On the other hand, it will be shown in Section 5 that, when the system has an exponentially unstable pole, it is not possible to achieve stabilization for an arbitrarily large delay even in the absence of input saturation. Restrictive as this may appear, Assumption 1 is satisfied by many real-world systems. For example, a linearized model of the relative motion in the orbit plane (the in-plane motion) of a spacecraft with respect to another in a circular orbit around the Earth (Ichikawa, 2008), and a chain of integrator systems which describes a wide range of systems such as inertia wheel pendulums (Ye, Wang & Wang, 2007).

A further discussion on the assumption on the pole locations of $A$ will be presented in Section 5. Here we give an explanation of the controllability of $(A, B)$.

**Remark 3.** If $B(\sigma)$ is a constant matrix, say, $B(\sigma) = B$, then $B = \int_0^t e^{-\beta\sigma}d\sigma B$. Since $\int_0^t e^{-\beta\sigma}d\sigma$ and $A$ commute, we have

$$
\text{rank}[B A A \cdots A^{n-1} B] = \text{rank}\left(\int_0^t e^{-\beta\sigma}d\sigma [B A B \cdots A^{n-1} B]\right).
$$

Therefore, $(A, B)$ is controllable if and only if $(A, B)$ is controllable and $\int_0^t e^{-\beta\sigma}d\sigma$ is nonsingular. Let $P A P^{-1} = J$, which is the Jordan canonical form of $A$ and has the diagonal elements $\lambda_i, i = 1, 2, \ldots, n$. Then

$$
\det\left(\int_0^t e^{-\beta\sigma}d\sigma\right) = \det\left(p^{-1}\left(\int_0^t e^{-\beta\sigma}d\sigma\right)p\right) = \det\left(\int_0^t e^{-\beta\sigma}d\sigma\right) = \prod_{i=1}^n \left(\frac{e^{\lambda_i t} - 1}{\lambda_i e^{\lambda_i t}}\right).
$$

Hence, $\int_0^t e^{-\beta\sigma}d\sigma$ is nonsingular if and only if $\frac{e^{\lambda_i t} - 1}{\lambda_i e^{\lambda_i t}} \neq 0, i = 1, 2, \ldots, n$, which is equivalent to

$$
\lambda_i \neq \pm \frac{2k\pi}{t}, \quad k = 1, 2, \ldots, i = 1, 2, \ldots, n.
$$

We now present our main results of this subsection as follows.

**Theorem 1.** Let **Assumption 1** be satisfied. Then for any given bounded delay $\tau$, there exists a $\gamma^* > 0$ such that the following family of linear state feedback

$$
u(t) = K(\gamma)x(t) = -\gamma^2 \lambda^T P(\gamma)x(t), \quad \forall \gamma \in [0, \gamma^*],
$$

solves Problem 1, where $P(\gamma) = P$ is the unique positive definite solution to the following algebraic Riccati equation

$$
A^T P + PA - PBR^2 P = -\gamma P.
$$

**Proof.** For simplicity, $K(\gamma)$ and $P(\gamma)$ will be denoted by $K$ and $P$ in the following. Rewrite the closed-loop system as follows

$$
\dot{x}(t) = Ax(t) + \int_0^t B(s)Kx(t - \sigma)d\sigma.
$$

(11)

We will adopt the idea found in the prediction or finite spectrum assignment based approach to simplify system (11) first (see, for example Artstein, 1982). Let $\chi(t) = x(t) + \psi(t)$ with

$$
\psi(t) = \int_{t-\tau}^t \left( \int_{t-\tau}^s e^{\alpha(t-\sigma)}B(\sigma)Kx(s)d\sigma \right) ds.
$$

Then by direct computation we get

$$
\dot{\psi}(t) = \int_0^t e^{-\alpha\tau}B(\sigma)Kx(t)d\sigma + \int_{t-\tau}^t \frac{d}{ds} \left( \int_{s}^t e^{\alpha(t-\sigma)}B(\sigma)Kx(s)d\sigma \right) ds = B(\sigma)Kx(t) + \int_0^t B(s)Kx(t - \sigma)d\sigma + A\psi(t).
$$

Then, by using (11), we have

$$
\dot{\chi}(t) = \dot{x}(t) + \dot{\psi}(t) = Ax(t) + BKx(t) + A\psi(t) = A\chi(t) + BKx(t) = (A + BK)\chi(t) - BK\psi(t),
$$

(12)

along which the time-derivative of the Lyapunov functional

$$
V_1(\chi(t)) = \chi^T(t)P(\chi(t))
$$

given by

$$
\dot{V}_1(\chi(t)) = -\gamma V_1(\chi(t)) - \chi^T(t)P(\gamma)\chi(t) + 2\chi^T(t)P(\gamma)^2\psi(t)
\leq -\gamma V_1(\chi(t)) - \chi^T(t)P(\gamma)\chi(t) + \chi^T(t)P(\gamma)\chi(t) + \psi^T(t)P(\gamma)\psi(t)
\leq -\gamma V_1(\chi(t)) + \gamma^2 V_1(\psi(t)),
$$

(13)

where we have used the ARE in (10) and Lemma 1 in Appendix A.2 on the properties of (10). Since $\chi(t) = x(t) + \psi(t)$, we get

$$
V_1(x(t)) = (x(t) + \psi(t))^T P(x(t) + \psi(t)) = V_1(x(t)) + 2\chi^T(t)P\psi(t) + V_1(\psi(t)),
$$

whose substitution in (13) gives

$$
\dot{V}_1(\chi(t)) \leq -\gamma V_1(x(t)) - 2\gamma x^T(\psi(t)) + (n - 1)\gamma^2 V_1(\psi(t)) + \gamma\frac{1}{2} V_1(x(t)) + 2\gamma V_1(\psi(t))
\leq -\gamma V_1(x(t)) + (n - 1)\gamma V_1(\psi(t)) + \gamma\frac{1}{2} V_1(x(t))
= -\gamma\frac{1}{2} V_1(x(t)) + (n + 1)\gamma V_1(\psi(t)).
$$

(14)
On the other hand, according to the Jensen inequality (Lemma 2 in Appendix A.2) and Lemma 1 and by denoting \( \Phi_i = \int_{t-i}^{t} S(\sigma) d\sigma \) and \( S(\sigma) = e^{-A\sigma} B(\sigma) \), we can compute

\[
\begin{align*}
\Phi(t)\Phi(t) & \leq \tau \int_{t-i}^{t} x^T(s) K^T \Phi_i e^{A(t-s)} Pe^{B(t-i)} \Phi_i Kx(s) ds \\
& \leq \tau \int_{t-i}^{t} e^{\omega(t-s)} x^T(s) K^T \Phi_i P \Phi_i Kx(s) ds,
\end{align*}
\]

where \( \omega \geq n - 1 \) is a constant. Notice that, for \( s \in [t - \tau, t] \), by using the Jensen inequality again,

\[
\begin{align*}
\Phi_i P \Phi_i & \leq (t - t + s) \int_{t-i}^{t} S(\sigma) P S(\sigma) d\sigma \\
& \leq (t - t + s) \mathrm{tr} \left( \int_{t-i}^{t} S(\sigma) P S(\sigma) d\sigma \right) I_m \\
& \leq (t - t + s) \tau \mathrm{tr}(P) \int_{t-i}^{t} tr(S(\sigma) S(\sigma)) d\sigma I_m \\
& \leq (t - t + s) \tau \mathrm{tr}(P) \int_{t-i}^{t} \|S(\sigma)\|^2 d\sigma I_m \\
& \leq (t - t + s) \tau \|S(\sigma)\|^2 I_m.
\end{align*}
\]

Hence, inequality (15) can be continued as

\[
\begin{align*}
\Phi_i & \leq (t - t + s) \int_{t-i}^{t} x^T(s) K^T \Phi_i Kx(s) ds \\
& \leq t^2 e^{\omega(t-s)} \int_{t-i}^{t} x^T(s) K^T \Phi_i Kx(s) ds \\
& \leq n \gamma^2 t^2 e^{\omega(t-s)} \int_{t-i}^{t} V_1(x(s)) ds.
\end{align*}
\]

Inserting the above inequality into (14) yields

\[
\dot{V}_1(\chi(t)) \leq -\frac{\gamma}{2} V_1(x(t)) + \phi_1^2(\tau) V_1(x(t)) \int_{t-i}^{t} V_1(x(s)) ds,
\]

where \( \phi = \phi(\tau, \gamma) \) is defined as

\[
\phi(\tau, \gamma) = n(n+1)\tau^2 e^{\omega(t-s)} \left( \int_{t-i}^{t} \|S(\sigma)\|^2 d\sigma \right).
\]

Since \( \tau \) is bounded and \( B(\sigma) \) is a piecewise continuous function in \( [0, \tau] \), we know that \( \phi(\tau, \gamma) \) is also bounded for any fixed \( \gamma \). Denote

\[
V_2(x_i) = \int_{t-i}^{t} \int_{t-i}^{t} V_1(x(l)) dlds.
\]

Then direct manipulation shows that

\[
\dot{V}_2(x_i) = \tau V_1(x(t)) - \int_{t-i}^{t} V_1(x(s)) ds.
\]

With this and in view of (17), the time derivative of the functional

\[
V(x_i) = V_1(\chi(t)) + \phi(\tau, \gamma) \psi^2 tr(P) V_2(x_i),
\]

along the trajectories of system (11) satisfies

\[
\dot{V}(x_i) \leq -\frac{\gamma}{2} V_1(x(t)) + \phi(\tau, \gamma) \psi^2 tr(P) V_1(x(t))
\]

\[
= -\frac{\gamma}{2} V_1(x(t))(1 - 2\phi(\tau, \gamma) \psi tr(P) \tau).
\]

Let \( \gamma^* = \gamma^*(\tau) \) be such that

\[
1 - 2\phi(\tau, \gamma) \psi tr(P) \tau \geq \frac{1}{2} \quad \forall \gamma \in (0, \gamma^*].
\]

The existence of such a \( \gamma^* \) is due to the fact that \( \lim_{\tau \to 0} \psi(t) = 0 \) (Lemma 1) and the boundedness of \( \phi(\tau, \gamma) \). It follows that

\[
\dot{V}(x_i) \leq -\frac{\gamma}{4} V_1(x(t)), \quad \forall \gamma \in (0, \gamma^*].
\]

We next use the Barbalat Lemma (Lemma 4 in Appendix A.2) to prove the stability of the system. Choose function \( E(t, x_i) \) as

\[
E(t, x_i) = \int_{t-i}^{t} V_1(x(s)) ds - V(x_i) + U(t),
\]

where \( U(t) \) is a non-decreasing function of \( t \). Then by using (22), the time derivative of \( E(t, x_i) \) along system (11) satisfies

\[
\dot{E}(t, x_i) = \dot{V}(x_i) + U(t) \leq -\frac{\gamma}{4} V_1(x(t)),
\]

from which it follows that \( V(x_i) + U(t) \leq V(x_0), \forall t \geq 0 \). Hence \( \lim_{t \to \infty} U(t) \) exists and is finite. Consequently, we know that \( \|x(t)\| \) is bounded for all \( t \geq 0 \). Since

\[
\|\dot{U}(t)\| \leq \frac{\gamma}{4} [2\dot{V}_1(x(t))]
\]

\[
\leq \frac{\gamma}{4} \left( \sup_{t \geq t} \|x(t)\| \right)^2 \|P\| \|A\| + \int_{t-i}^{t} \|B(\sigma)K\| d\sigma,
\]

it follows that \( \|\dot{U}(t)\| \) is also bounded. Therefore, \( \dot{U}(t) \) is uniformly continuous and we conclude from Barbalat’s Lemma that

\[
\lim_{t \to \infty} \dot{U}(t) = \lim_{t \to \infty} \frac{\gamma}{4} [2\dot{V}_1(x(t))]
\]

namely, \( \lim_{t \to \infty} \|x(t)\| = 0 \) in view of the fact that, for linear time invariant systems, attractivity implies asymptotic stability, the proof is complete. \( \square \)

Theorem 1 provides explicit solutions to Problem 1. To construct the controller in (9), we only need to solve a linear matrix equation in the form of (70) shown in Appendix A.2. We should point out that though the parameter \( \gamma^* \) can be computed based on (21), the resulting value may be much smaller than the true upper bound \( \gamma_{sup} \). In practice, we can apply the trial-and-error method to get a non-conservative estimate of \( \gamma_{sup} \). This can be done in the following steps: (1) Compute \( \gamma^* \) based on (21.2) and denote \( \gamma_2 = \gamma^* \). (2) Choose a large enough number \( \gamma_0 > \gamma^* \) (for example, \( \gamma_0 = 10\gamma^* \)) such that the closed-loop system is unstable (by simulation) with \( \gamma = \gamma_0 \). Let \( \gamma = \frac{1}{2}(\gamma_0 + \gamma_0) \) and check the stability of the closed-loop system by simulation. If the closed-loop system is asymptotically stable, then \( \gamma_2 \leftarrow \frac{1}{2}(\gamma_0 + \gamma_0) \); otherwise, \( \gamma_2 \leftarrow \frac{1}{2}(\gamma_0 + \gamma_0) \). (3) Repeat the third step until \( |\gamma_2 - \gamma_0| \leq \epsilon \), where \( \epsilon \) is a prescribed small number (for example, \( \epsilon = 10^{-6} \)). (5) Set \( \gamma_{sup} = \gamma_2 \).

In the following, we will introduce another approach to compute \( \gamma_{sup} \) for a special case. We consider the special case that the distributed control matrix \( B(t) \) in system (1) is of the form

\[
B(t) = C_k e^{\omega(t)} B_0, \quad t \in [0, \tau],
\]

in which \( C_k \in \mathbb{R}^{m \times q} \), \( A_0 \in \mathbb{R}^{q \times q} \), and \( B_0 \in \mathbb{R}^{q \times m} \) are some given matrices, and \( n_p \) is an integer. This indicates that \( B(t) \) is the impulse response of a strictly proper finite dimensional system characterized by \( (C_k, A_0, B_0) \) restricted on the interval \( [0, \tau] \). Then
the closed-loop system (11) is asymptotically stable if and only if its associated characteristic quasi-polynomial
\[ \Delta_c(s) = \det \left( sl_n - A - \int_0^t C_a e^{s(t-\tau)} B_B K e^{-s\sigma} d\sigma \right), \]
has all its roots on the open left-half plane (see, for example Hale & Verduyn, 1993). We let
\[ A_0 = \begin{bmatrix} A & C_B \\ B_B K & A_B \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0 & 0 \\ e^{\tau A} B_B K & 0 \end{bmatrix}. \]

**Proposition 1.** Let \( A_0 \) and \( A_\tau \) be denoted as above. Let
\[ \Delta^+ = \{ s : \det(s l_n + A_0 + A_\tau e^{-s\tau}) = 0, \text{ Re } s \geq 0 \}, \]
\[ \mathcal{D}^+ = \{ s : \det(s l_n + A_\tau) = 0, \text{ Re } s \geq 0 \}, \]
then all the roots of \( \Delta_c(s) = 0 \) are located on the open left-half plane if and only if
\[ \mathcal{D}^+ = \Delta^+. \tag{24} \]

**Proof.** Notice that
\[ \Delta_c(s) = \det(sl_n - A - C_B(sl - A_B)^{-1} \times \left( \int_0^t e^{-s(t-\tau)} d(sl_n + A_B s) \right) B_B K) = \det(sl_n - A + C_B(sl_n + A_B)^{-1} \times (e^{-s(t-\tau)} - 1) B_B K), \]
in view of which and Lemma 3 in Appendix A.2, we obtain
\[
\begin{align*}
\det(sl_n + A_\tau) \Delta_c(s) &= \det\left[ \begin{bmatrix} sl_n - A & -C_B \\ (e^{-s(t-\tau)} - 1) B_B K & sl_n - A_B \end{bmatrix} \right] \\
&= \det(sl_n - A_0 + A_\tau e^{-s\tau}). \tag{25}
\end{align*}
\]
We clearly get from (25) that all the roots of \( \Delta_c(s) = 0 \) are located on the open left-half plane if and only if (24) is satisfied. The proof is finished. \( \square \)

Notice that the last equation in (25) corresponds to the characteristic quasi-polynomial of the following pointed-delay system
\[ \dot{z}(t) = A_0 z(t) - A_\tau z(t - \tau). \tag{26} \]

The set of zeros on the closed right-half plane of system (26) or quasi-polynomial (25), say, \( \Delta^+ \), can be computed by the efficient software package DDE-BIFTOOL (Engelborghs, Luzyanina and Samaey, 2001). Therefore, the parameter \( \gamma_{\text{sup}} \) can be obtained by the trial-and-error method via computing \( \Delta^+ \) and testing (24).

**Remark 4.** The stability of the closed-loop system (11) and the stability of the pointed-delay system (26) can also be related as follows: Let \( x_0 = \int_{\tau}^t e^{s(t-\tau)} B_B K x(s) ds \). Then
\[ \dot{x}_0(t) = B_B K x(t) - e^{\tau A} B_B K x(-\tau) + A_B x_0(t), \tag{27} \]
and the closed-loop system (11) can be written as
\[ \dot{x}(t) = A x(t) + C_B x_0(t). \tag{28} \]

Systems (27) and (28) can be written as (26) by denoting \( \varsigma = [x^T, x_0^T]^T \). However, the stability of the closed-loop system (11) and the stability of the pointed-delay system (26) are not equivalent due to (25). In fact, if \( A_B \) contains unstable eigenvalues, system (26) will never be stable even if system (11) is stable. Nevertheless, according to our earlier development, we can always use (24) to test the stability of system (11) according to the locations of zeros on the closed right-half plane of system (26), say, to test (24). Finally, we point out that, though a similar technique was used in Ozbay, Bonnet and Clairambault (2008) to test stability of distributed delay system, the above phenomenon was not discovered there.

At the end of this subsection, we give a brief discussion on the output feedback solution to Problem 1. Without loss of generality, we assume that system (1) is already in the form of (7) and has an output in the form of
\[ y(t) = [C_0 C_1] x_n(t) \triangleq C_x x_n(t), \]
where \( (A_n, C_n) \) is detectable. We construct the following observer based controller
\[
\begin{align*}
\dot{z}(t) &= A_n z(t) + \int_0^t B_n(\sigma) u(t - \tau) d\sigma \\
+ L_n(y(t) - C_n z(t)), \\
u(t) &= [K_n(\gamma) 0] z(t) \triangleq K_n(\gamma) z(t),
\end{align*}
\]
where \( L_n \) is such that \( A_n - L_n C_n \) is Hurwitz and \( K_n(\gamma) \) is the feedback gain designed for system
\[ \dot{x}_o(t) = A_o x_o(t) + \int_0^t B_o(\sigma) u(t - \tau) d\sigma, \tag{30} \]
based on the approach in Theorem 1.

**Proposition 2.** Assume that \( \gamma^* \) is such that system (30) with \( u(t) = K_n(\gamma) x_0(t) \) is asymptotically stable for all \( \gamma \in (0, \gamma^*) \). Then the closed-loop system consisting of (7) and (29) is also asymptotically stable for all \( \gamma \in (0, \gamma^*) \).

**Proof.** By denoting \( z(t) - x_n(t) = e(t), \) the closed-loop system reads
\[
\begin{align*}
\dot{x}_o(t) &= A_o x_o(t) + \int_0^t B_o(\sigma) K_n(\gamma) x_n(t - \sigma) d\sigma \\
&+ \int_0^t B_o(\sigma) K_{o}(\gamma) e(t - \sigma) d\sigma, \\
\dot{e}(t) &= (A_n - L_n C_n) e(t),
\end{align*}
\]
which can be re-written as
\[ \dot{\xi}(t) = A_{cn} \xi(t) + \int_0^t A_t(\sigma) \xi(t - \sigma) d\sigma, \]
by denoting \( \xi(t) = [x_0^T(t) e(t)^T] \) and
\[ A_{cn} = \begin{bmatrix} A_n & 0 \\ 0 & A_n - L_n C_n \end{bmatrix}, \quad A_t(\sigma) = \begin{bmatrix} B_o(\sigma) K_n(\gamma) & 0 \\ 0 & 0 \end{bmatrix}. \]
The characteristic quasi-polynomial of the above system is
\[ 0 = \Delta(s) \triangleq \det\left( sl_n - A_{cn} - \int_0^t A_t(\sigma) e^{-s\sigma} d\sigma \right) \\
= \det\left( sl_n - A_n - \int_0^t B_o(\sigma) e^{-s\sigma} d\sigma \right) \\
\times \det(sl_n - (A_n - L_n C_n)) \\
= \det\left( sl_n - A_0 - \int_0^r B_o(\sigma) K_n(\gamma) e^{-s\sigma} d\sigma \right) \\
\times \det(sl_n - A_n) \\
= \Delta_o(s) \det(sl_n - A_n) \det(sl_n - (A_n - L_n C_n)). \]
We notice that $\Delta_\sigma(s)$ is exactly the characteristic quasi-polynomial of system (30) with $u(t) = K_\sigma(y) x_\sigma(t)$. Since all the roots of $\Delta_\sigma(s) = 0$ have negative real parts for all $\gamma \in (0, \gamma^*]$, we conclude that all the roots of $\Delta(s) = 0$ also have negative real parts for all $\gamma \in (0, \gamma^*]$. The proof is complete. \[ \square \]

### 3.2. The case of time-varying delays

We assume in this subsection that all the poles of $A$ are on the closed left-half plane and those poles on the imaginary axis are located at the origin. Under such a condition, similarly to the development in Section 3.1, we can assume, without loss of generality, that all the poles of $A$ are zero. Moreover, associated with system (4), we define the following time-varying matrix

$$ B'(t) = \int_0^{t(\tau)} B(\sigma) d\sigma. \quad (31) $$

**Theorem 2.** Assume that all the eigenvalues of $A$ are zero and $0 < \tau(t) \leq \tau_{\text{max}} \in (0, \infty)$. If there exists a positive scalar $\alpha < \infty$ and a constant matrix $B \in \mathbb{R}^{m \times m}$ such that

$$ \alpha(\psi(\tau)B^T + B' B^T) \geq \gamma^* B^T, \quad \forall \tau \in \mathbb{R}, \quad (32) $$

and $(A, B)$ is controllable, then there exists a $\gamma^* = \gamma^*(\tau_{\text{max}}, \alpha) > 0$ such that the following family of linear state feedback laws

$$ u(t) = K(\gamma) x(t) = -\alpha \gamma^* P(\gamma) x(t), \quad \forall \gamma \in (0, \gamma^*], \quad (33) $$

solves Problem 1 for system (4), where $P(\gamma)$ is the unique positive definite solution to the following algebraic Riccati equation

$$ A^T P + PA - P B^T B = -\gamma P. \quad (34) $$

**Proof.** The closed-loop system consisting of (4) and (33) can be written as

$$ \dot{x}(t) = A x(t) + \int_0^{t(\tau)} B(\sigma) K(\gamma) x(t - \sigma) d\sigma \quad (35) $$

$$ = \left( A + \int_0^{t(\tau)} B(\sigma) d\sigma \right) K(\gamma) x(t) + \int_0^{t(\tau)} B(\sigma) K(\gamma) x(t - \sigma) d\sigma $$

$$ = \left( A + B'(t) K(\gamma) \right) x(t) + \int_0^{t(\tau)} B(\sigma) K(\gamma) \Delta_\sigma d\sigma $$

$$ \triangleq \left( A + B'(t) K(\gamma) \right) x(t) + \psi(t), \quad (36) $$

where $\Delta_\sigma = x(t - \tau) - x(t), \sigma \leq \tau(t) \leq \tau_{\text{max}}$. By using (32) and (34), we can compute

$$ (A + B'(t) K(\gamma))^T P + P(A + B'(t) K(\gamma)) $$

$$ = (A - \alpha B'(t) B^T)^T P + P(A - \alpha B'(t) B^T) P $$

$$ = A^T P + PA - \alpha P B'(t) B^T + \alpha B^T B P $$

$$ \leq A^T P + PA - \alpha \gamma^* B^T B P $$

$$ = -\gamma P. \quad (37) $$

The time derivative of the Lyapunov function $V(x(t)) = x^T(t) P x(t)$ along the trajectories of system (36) can be evaluated as follows

$$ \dot{V}(x(t)) = 2 \psi(t) x^T(t) P x(t) + 2 \psi(t) (A + B'(t) K(\gamma))^T P x(t) $$

$$ \leq \psi(t) x^T(t) P x(t) + \psi(t) (A + B'(t) K(\gamma))^T P x(t) $$

$$ \leq -\frac{1}{\gamma} \psi V(x(t)) + \frac{2}{\gamma} V(\psi(t)). \quad (38) $$

where we have used (37). By using Lemma 1 and the Jensen inequality in Lemma 2, both in Appendix A.2, we can get

$$ V(\psi(t)) \leq \tau(t) \int_0^{t(\tau)} \Delta_\sigma^T K^T Z(\sigma) K \Delta_\sigma d\sigma $$

$$ \leq \alpha^2 \tau(t) \int_0^{t(\tau)} \Delta_\sigma^T P \alpha \gamma^* \Delta_\sigma \Delta_\sigma d\sigma $$

$$ \leq \alpha^2 \gamma \tau(t) \int_0^{t(\tau)} \Delta_\sigma^T P \Delta_\sigma d\sigma $$

$$ \leq \alpha^2 \psi(\tau) \gamma \tau_{\text{max}} \int_0^{t(\tau)} \Delta_\sigma^T P \Delta_\sigma d\sigma, \quad (39) $$

where $\psi(\tau) = \max_{\gamma \in (0, \gamma^*]} \psi(\tau_{\text{max}})$ and $Z(\gamma) = B^T(\gamma) P B(\gamma)$. Taking integrals on both sides of (35) from $t - \sigma$ to $t$ results in

$$ -\Delta_\sigma = x(t) - x(t - \sigma) = \int_{t - \sigma}^{t} x(s) d\sigma $$

$$ = \int_{t - \sigma}^{t} A x(s) d\sigma + \int_{t - \sigma}^{t} \int_{t - \sigma}^{t} B(\xi) K(\gamma) x(s - \xi) d\xi d\sigma, $$

from which it follows that

$$ \Delta_\sigma^T P \Delta_\sigma \leq 2 \left( \int_{t - \sigma}^{t} A x(s) d\sigma \right)^T P \left( \int_{t - \sigma}^{t} A x(s) d\sigma \right) $$

$$ + 2 \left( \int_{t - \sigma}^{t} \int_{t - \sigma}^{t} B(\xi) K(\gamma) x(s - \xi) d\xi d\sigma \right)^T P $$

$$ \leq 2 \sigma \int_{t - \sigma}^{t} \int_{t - \sigma}^{t} x^T(s) A^T P A x(s) d\sigma + 2 \sigma \tau_{\text{max}} $$

$$ x^T(s - \xi) A^T P B(\xi) K(\gamma) x(s - \xi) d\xi d\sigma $$

$$ \leq 6 n^2 \gamma^2 \sigma \int_{t - \sigma}^{t} V(x(s)) d\sigma + 2 \alpha^2 \psi(\tau) \gamma \tau_{\text{max}} $$

$$ \times \int_{t - \sigma}^{t} \int_{t - \sigma}^{t} A^T P A x(s) d\sigma + 2 \sigma \tau_{\text{max}} $$

$$ \times \int_{t - \sigma}^{t} \int_{t - \sigma}^{t} x^T(s - \xi) A^T P B(\xi) K(\gamma) x(s - \xi) d\xi d\sigma $$

where we have used Lemma 1 and the Jensen inequality in Lemma 2. Let $\mu > 1$ be given and assume that

$$ V(x(t - s)) < \mu V(x(t)), \quad \forall \gamma \in [-2 \tau_{\text{max}}, 0]. \quad (41) $$

Then it follows from (39) that

$$ \int_0^{t(\tau)} \Delta_\sigma^T P \Delta_\sigma d\sigma $$

$$ \leq 6 n^2 \gamma^2 \int_0^{t(\tau)} \sigma \int_{t - \sigma}^{t} V(x(s)) d\sigma d\sigma + 2 \alpha^2 \psi(\tau) \gamma \tau_{\text{max}} $$

$$ \times \int_{t - \sigma}^{t} \sigma \int_{t - \sigma}^{t} V(x(s - \xi)) d\xi d\sigma $$

$$ \leq \mu (6 n^2 \gamma^2 + 2 \alpha^2 \psi(\tau) \gamma \tau_{\text{max}}) \int_0^{t(\tau)} \sigma^2 d\sigma V(x(t)) $$

$$ \leq \frac{\tau_{\text{max}}^3}{3} \mu (6 n^2 \gamma^2 + 2 \alpha^2 \psi(\tau) \gamma \tau_{\text{max}}) V(x(t)). \quad (42) $$
whose substitution in (38) and (39) gives
\[
\dot{V}(x(t)) \leq -\frac{1}{2} \gamma V(x(t)) + 2\alpha \varphi(\gamma) T_{\max} \int_0^{\tau(t)} \Delta^T_\gamma P \Delta_\sigma d\sigma \\
\leq -\frac{1}{2} \gamma \varphi(\gamma) V(x(t)),
\]
(43)
where
\[
\varphi(\gamma) = 1 - \frac{4}{3} \gamma^2 \mu(\varphi) h(\gamma) + 2\alpha^2 \varphi(\gamma) \mu(\gamma^2).
\]
Since \(\lim_{\gamma \to 0^+} \varphi(\gamma) = 0\) and \(\tau_{\max} < \infty\), there exists a \(\gamma^* = \gamma^*(\tau_{\max}, \alpha) > 0\) such that \(\varphi(\gamma) > \frac{1}{2}\), \(\forall \gamma \in (0, \gamma^*)\). Therefore, we get from (43) that
\[
\dot{V}(x(t)) \leq -\frac{1}{2} \gamma V(x(t)), \quad \forall \gamma \in (0, \gamma^*].
\]
The stability of the closed-loop system then follows from the Razumikhin stability theorem (Theorem 6 in Appendix A.2). The proof is complete. \(\square\)

**Theorem 2** involves the inequality in (32). In what follows, we identify a few situations when this inequality can be satisfied.

**Corollary 1.** Consider the following linear system with distributed input delay
\[
\dot{x}(t) = Ax(t) + B(\sigma) u(t - \sigma) d\sigma,
\]
(44)
where \(\tau > 0\) is a constant that can be arbitrarily large but bounded. Assume that all the eigenvalues of \(A\) are zero and \((A, B)\) is controllable, where
\[
\mathfrak{B} = \int_0^\tau B(\sigma) d\sigma.
\]
Then there exists a \(\gamma^* = \gamma^*(\tau) > 0\) such that the following family of linear state feedback laws
\[
u(t) = K(\gamma)x(t) = -ß^T(\gamma) x(t), \quad \forall \gamma \in (0, \gamma^*)]
\]
solves Problem 1 for system (44), where \(P(\gamma)\) is the unique positive definite solution to the following algebraic Riccati equation
\[
A^T P + PA - PBB^T P = -\gamma P.
\]
**Proof.** According to Theorem 2, the associated matrix \(B'\) defined in (31) for system (44) is given by \(B' = B^T(t) h(\gamma)\).
\[
\mathfrak{B} = \int_0^\tau B(\sigma) d\sigma.
\]
Therefore, inequality (32) is satisfied with \(\alpha = 1\) and \(\mathfrak{B} = B'\). The proof is complete. \(\square\)

The advantage of Corollary 1 over Theorem 1 is that the coefficient matrix \(\mathfrak{B}\) in Corollary 1 does not involve the matrix exponent of matrix \(A\) and is simpler to compute.

**Corollary 2.** Consider the following linear system with distributed input delay
\[
\dot{x}(t) = Ax(t) + B \int_0^\tau h(\sigma) u(t - \sigma) d\sigma,
\]
(45)
in which all the eigenvalues of \(A\) are zero, \((A, B)\) is controllable, \(0 < \tau(t) \leq \tau_{\max} \in (0, \infty)\), and \(h(t)\) is a scalar function and such that
\[
0 < h_{\min} \leq \int_0^\tau h(\sigma) d\sigma < \infty
\]
(46)
Then there exists a scalar \(\gamma^* = \gamma^*(\tau_{\max}, h_{\min}) > 0\) such that the following family of linear state feedback laws
\[
u(t) = K(\gamma)x(t) = -\frac{1}{2h_{\min}} B^T P x(t), \quad \forall \gamma \in (0, \gamma^*)
\]
solves Problem 1 for system (45), where \(P = P(\gamma)\) is the unique positive definite solution to the following algebraic Riccati equation
\[
A^T P + PA - PBB^T P = -\gamma P.
\]
**Proof.** According to Theorem 2, the associated matrix \(B'\) defined in (31) for system (44) is given by \(B' = B^T(t) h(\gamma)\).
\[
\mathfrak{B} = \int_0^\tau B(\sigma) d\sigma.
\]
Therefore, inequality (32) is satisfied with \(\alpha = 1\) and \(\mathfrak{B} = B'\). The proof is complete. \(\square\)

**Remark 5.** Corollaries 1 and 2 are two cases where the assumptions in (32) are satisfied. There are other cases that also satisfy this assumption. For example, consider a system in the form of (4) with \(\tau(t) = 1 + \cos^2 t\) and
\[
B(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{-\gamma t} \end{bmatrix}.
\]
Such a system is neither of the particular case in Corollary 1 because \(\tau(t)\) is time-varying nor of the particular case in Corollary 2. However, if we compute
\[
B'(t) = \begin{bmatrix} 1 + \cos^2 t & 0 & 0 \\ 0 & 0 & 1 - e^{-(1+\cos^2 t)} \end{bmatrix},
\]
we find that the assumption in (32) is fulfilled with
\[
\alpha = \frac{1}{2(1 - e^{-1})}, \quad \mathfrak{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

4. Solutions to the semi-global stabilization problem

We first present a solution to Problem 2 under the assumption that the delay in the system is constant and exactly known, namely, we consider system (2).

**Theorem 3.** Let Assumption 1 hold. Then the family of linear state feedback laws (9), where \(P(\gamma)\) is the unique positive definite solution to the parametric Riccati Eq. (10), solves Problem 2, namely, for any arbitrarily large but bounded set \(\Omega \subset \mathbb{R}_+\), there exists a \(\gamma'(\Omega) > 0\) such that, for any \(\gamma \in (0, \gamma'(\Omega)]\), the closed-loop system is asymptotically stable at the origin with \(\Omega\) contained in the domain of attraction.
Proof. We first consider the closed-loop system in the absence of input saturation. Notice that
\[
\begin{align*}
\dot{u}(t)u(t) &= (\chi(t) - \psi(t))^T K^T K (\chi(t) - \psi(t)) \\
&\leq 2\gamma^2(\chi(t) + \psi(t))^T K^T K (\chi(t) + \psi(t)) \\
&\leq 2\gamma(\chi(t) + \psi(t))^T P_K (\chi(t) + \psi(t)),
\end{align*}
\]
where we have used Lemma 1. By applying the inequality in (16), we further obtain
\[
\begin{align*}
\dot{u}(t)u(t) &\leq \frac{2\gamma^2\phi(\tau, \gamma)^T \tau(P)}{n + 1} \int_{t-\tau}^t V_1(x(s))ds \\
&\quad + 2\gamma \phi(\gamma)\tau \int_{t-\tau}^t V_1(x(s))ds,
\end{align*}
\]
where \(V_1(\chi(t))\) is defined in the proof of Theorem 1. Let
\[
W(\chi) = V(\chi(t)) + \frac{\gamma(\gamma)^T \tau(P)}{n + 1} \int_{t-\tau}^t V_1(x(s))ds,
\]
where \(V(x(t))\) is defined in (19). Then in view of (20), we can compute
\[
\begin{align*}
\dot{W}(\chi(t)) &= \dot{V}(\chi(t)) + \frac{\gamma(\gamma)^T \tau(P)}{n + 1} \int_{t-\tau}^t V_1(x(s))ds \\
&\quad + \phi(\gamma) \int_{t-\tau}^t \phi(\gamma) \tau \int_{t-\tau}^t V_1(x(s))ds,
\end{align*}
\]
where
\[
\dot{\gamma}(\gamma) = 1 - \phi(\tau, \gamma) \tau \int_{t-\tau}^t \dot{V}_1(x(s))ds.
\]
Since \(\lim_{\gamma \to 0^+} P(\gamma) = 0\), there exists a \(\gamma_1^* \in (0, \gamma^*)\), where \(\gamma^*\) is as defined in Theorem 1, such that \(\dot{\gamma}(\gamma) \geq \frac{1}{2}\), \(\gamma \in (0, \gamma_1^*)\). Consequently, we have
\[
\dot{W}(\chi(t)) \leq -\frac{\gamma}{4} V_1(x(t)), \quad \forall \gamma \in (0, \gamma_1^*].
\]
That is to say, \(W(\chi)\) is a non-increasing function, namely, \(W(\chi(t)) \leq W(\chi(0))\), \(\forall t \geq 0\). \(\forall \gamma \in (0, \gamma^*)\). Therefore, it follows from (49) that
\[
\begin{align*}
\dot{u}(t)u(t) &\leq \frac{2\gamma^2\phi(\gamma)^T \tau(P)}{n + 1} \int_{t-\tau}^t V_1(x(s))ds \\
&\quad + 2\gamma V_1(x(t)) \\
&\leq \frac{2\gamma^2\phi(\gamma)^T \tau(P)}{n + 1} \int_{t-\tau}^t V_1(x(s))ds \\
&\quad + 2\gamma V_1(x(t)) + \phi(\gamma) \gamma^2 \tau \int_{t-\tau}^t V_1(x(s))ds,
\end{align*}
\]
where \(V_2(\chi(t))\) is defined in (18). Since \(W(\chi)\) is bounded, we know that \(W(\chi(t)) \leq W(\chi(0))\), \(\forall t \geq 0\), \(\forall \gamma \in (0, \gamma^*)\). We thus conclude that there exists a \(\gamma_2^* \in (0, \gamma_1^*]\) such that
\[
\|u(t)\|_{\infty} \leq 1, \quad \forall \gamma \in (0, \gamma_2^*], \quad \forall t \geq 0,
\]
namely, the input saturation can be avoided for all control signals \(u(t)\), \(\forall t \geq 0\). Since
\[
\|u(t)\|_{\infty} = \|\tau(\gamma)\|_{\infty} \leq \|K\|_{\infty} \|\psi(t)\|_{\infty},
\]
holds for any \(t \in [\tau, 0]\) and \(\psi \in \Omega\), and \(\Omega\) is bounded, there exists a \(\gamma_3^* \in (0, \gamma_2^*]\) such that
\[
\|u(t)\|_{\infty} \leq 1, \quad \forall t \in [-\tau, 0], \quad \forall \gamma \in (0, \gamma_3^*].
\]
By combining (51) and (53) we can see that there exists a \(\gamma_4^*\) such that, for any initial condition \(\psi \in \Omega\), there holds
\[
\|u(t)\|_{\infty} \leq 1, \quad \forall t \geq -\gamma_4^*, \quad \forall \gamma \in (0, \gamma_4^*].
\]
namely, the closed-loop system will never saturate and will remain a linear one. The stability result then follows from Theorem 1. This completes the proof. \(\square\)

By using a similar technique as used in the proof of Theorem 2.7 in Zhou et al. (2010b), we can also show that the feedback controller in the form of (29) will achieve semi-global stabilization for system (2). The details are omitted for brevity.

We next establish a solution to Problem 2 for the delay system in (5) where the delay is allowed to be time-varying and/or unknown.

Theorem 4. Consider the input saturated delay system in (5). Assume that all the conditions in Theorem 2 are satisfied for this system. Then the family of linear state feedback laws (33) semi-globally stabilizes system (5) at the origin, i.e., for any a priori given bounded set \(\Omega \subset \mathbb{R}_{n, \tau_{max}}\), where \(\tau_{max}\) is the upper bound of \(\tau(t)\), there exists a \(\gamma^* > 0\) such that, for any \(\gamma \in (0, \gamma^*)\), the closed-loop system is asymptotically stable at the origin with \(\Omega\) contained in the domain of attraction.

Proof. Rewrite the closed-loop system comprising of (5) and (33) as follows
\[
\dot{x}(t) = Ax(t) + \int_0^{\tau(t)} B(\gamma) sat(K(\gamma)x(t - \gamma))d\gamma.
\]

For an initial condition \(\psi \in \Omega\), let the initial condition with initial time \(t = \tau_{max}\), namely, \(\psi_{\tau_{max}} \in \mathbb{R}_{n, \tau_{max}}\) be determined by
\[
\psi_{\tau_{max}}(\gamma) = \begin{cases} 
\psi(t = \gamma_{\tau_{max}} + \theta), & \forall \theta \in [-2\tau_{max}, \tau_{max}], \\
\psi(t = \gamma_{\tau_{max}}), & \forall \theta \in [-\tau_{max}, 0].
\end{cases}
\]

where \(x(t), \forall t \in [0, \tau_{max}]\) is the solution to system (54) over \(t \in [0, \tau_{max}]\) with initial condition \(\psi \in \Omega\). Denote
\[
\Omega' \triangleq \{\psi_{\tau_{max}}(\gamma), \forall \theta \in [-2\tau_{max}, 0] | \psi \in \Omega \subset \mathbb{R}_{n, \tau_{max}}\}
\]
As \(\Omega\) is bounded, it can be readily shown that \(\Omega'\) is also bounded. Denote \(\rho(\gamma) = \frac{\gamma}{\gamma^2}\), and the ith column of matrix \(\rho\) as \(\rho_i\), \(i = 1, 2, \ldots, m\). Then it can be readily shown that, by using Lemma 1,
\[
\rho_i^T P(\gamma)(\frac{P(\gamma)}{\rho(\gamma)})^{-1} P(\gamma) \rho_i \alpha^2 \leq 1, \quad i = 1, 2, \ldots, m,
\]
which implies that (see, for example Hu & Lin, 2001)
\[
\begin{align*}
\rho_i^T x(t + \theta) \left( \frac{P(\gamma)}{\rho(\gamma)} \right) x(t + \theta) &\leq 1, \quad \forall \theta \in [-2\tau_{max}, 0].
\end{align*}
\]
By noting that, with the definition of \(\mathcal{M}(\rho(\gamma), 2\tau_{max})\) given in (72),
\[
x_i \in \mathcal{M}(\rho(\gamma), 2\tau_{max})
\]
\[
\Rightarrow \rho_i^T x(t + \theta) \left( \frac{P(\gamma)}{\rho(\gamma)} \right) x(t + \theta) \leq 1, \quad \forall \theta \in [-2\tau_{max}, 0].
\]

It follows from (57) and (58) that
\[
x_i \in \mathcal{M}(\rho(\gamma), 2\tau_{max})
\]
\[
\Rightarrow \|\alpha \rho_i^T P(\gamma)x(t + \theta)\|_{\infty} \leq 1, \quad \forall \theta \in [-2\tau_{max}, 0].
\]
system at \( t = 0 \) can be written as (35). Consequently, following the procedure used in the proof of Theorem 2, we can show that
\[
V(x(t + \theta)) < \mu V(x(t)), \quad \forall \theta \in [-2\tau_{\text{max}}, 0]
\]
\[
\Rightarrow \dot{V}(x(t)) < \frac{1}{4} \gamma V(x(t)), \quad \forall \gamma \in (0, \gamma^*_\theta],
\]
where \( \mu > 1 \) and \( \gamma^*_\theta = \min\{\gamma^*, \gamma^*_1\} \) with \( \gamma^* \) determined in Theorem 2. Then, by virtue of the Razumikhin Stability Theorem (Theorem 6 in Appendix A.2), \( \mathcal{M}(\rho(\gamma), 2\tau_{\text{max}}) \) is an invariant set for the closed-loop system (54), which is asymptotically stable at the origin with \( \mathcal{M}(\rho(\gamma), 2\tau_{\text{max}}) \supseteq \Omega' \) contained in the domain of attraction. The proof is complete. \( \square \)

Corollaries similar to Corollaries 1 and 2 can also be easily obtained for semi-global stabilization of system (5). The details are omitted for brevity.

5. Discussions on the assumptions on the system

In this section, we will give a discussion on Assumption 1 imposed on the time delay system (1) with constant delays. To this end, we first introduce the following result, which also has its own value. The proof is provided in Appendix A.1.

**Theorem 5.** Consider the following scalar distributed delay system
\[
\dot{x}(t) = \alpha x(t) - k \int_0^t e^{-\beta \theta} x(t - \theta) d\theta, \quad (60)
\]
where \( \alpha > 0 \) and \( \beta > 0 \). Then we have the following statements:

1. If \( \alpha > \beta \), then there exists a \( k \) such that system (60) is asymptotically stable only if
\[
\frac{\tau \beta}{e^{\beta \tau} - 1} > 1 - \frac{\beta}{\alpha}, \quad (61)
\]
2. If \( \alpha = \beta \), then system (60) is asymptotically stable if and only if
\[
\frac{\alpha \beta}{1 - e^{-\tau \beta}} < k < \frac{\pi^2}{\tau^2} + \frac{\alpha \beta}{1 + e^{-\tau \beta}}, \quad (62)
\]
3. If \( \beta > \alpha \), then there exists a \( k \) such that system (60) is asymptotically stable if
\[
\tau > \frac{1}{2\beta} \ln \left(1 + \frac{4\alpha \beta}{(\beta - \alpha)^2}\right). \quad (63)
\]

From Theorem 5 we can make the following observation.

1. If \( \alpha > \beta \), then it follows from (61) that
\[
e^{\beta \tau} < 1 + \frac{1}{1 - \frac{\beta}{\alpha} \tau \beta}, \quad (64)
\]
Notice that the following equation
\[
e^x = 1 + \frac{1}{1 - \frac{\beta}{\alpha} x}
\]
has a unique positive real solution for any \( \beta < \alpha \). Denote such solution by \( x^* \). Then (64) is satisfied if and only if \( \tau < \frac{\pi}{\beta} \). This indicates that in this case system (60) is not asymptotically stable for any feedback gain \( k \) if the delay is large enough.

2. If \( \alpha = \beta \), then we know from (62) that there exists a feedback gain \( k \) such that system (60) is asymptotically stable if and only if
\[
\frac{\alpha \beta}{1 - e^{-\tau \beta}} < \frac{\pi^2}{\tau^2} + \frac{\alpha \beta}{1 + e^{-\tau \beta}},
\]
which is equivalent to
\[
e^{\beta \tau} > 1 + \frac{2\alpha \beta \tau^2}{\pi^2} = 1 + \left(\frac{2\alpha}{\pi^2 \beta}\right) (\beta \tau)^2 = 1 + \frac{2}{\pi^2} (\beta \tau)^2.
\]
The above inequality holds true for all \( \beta > 0 \), namely, for any delay \( \tau > 0 \), there exists \( k \) such that system (60) is asymptotically stable.

3. If \( \beta > \alpha \), it follows from Item 3 of Theorem 5 that there exists a feedback gain \( k \) such that system (60) is asymptotically stable for all \( \tau \) satisfying (63). On the other hand, if \( k > \alpha \beta \), by continuity of roots of Eq. (65), which has no roots in the closed left-half plane, there exists a sufficiently small number \( \tau^* \) such that system (60) is asymptotically stable for all \( \tau < \tau^* \). But it is not clear whether there exists a feedback gain \( k \) such that system (60) is asymptotically stable for \( \tau \) that does not satisfy (63) and is larger than \( \tau^* \).

Summarizing the above discussions we conclude that Assumption 1 is not a conservative one. For example, if \( A = \alpha > 0 \), which means that the open-loop system is unstable, and \( B(\sigma) = e^{-\sigma} \), \( 0 < \beta < \alpha \), then it follows from Item 1 of the above discussion that system (1) is not stabilizable for sufficiently large delay \( \tau \), while we have shown in Theorem 1 that this system can be stabilized for any \( \tau \) if Assumption 1 is satisfied. On the other hand, Assumption 1 may also be restrictive in some cases. For example, if \( A = \alpha > 0 \) and \( B(\sigma) = e^{-\sigma} \), \( -e^{-\sigma} \), then according to Item 2 of the above discussion, system (1) is stabilizable for any delay \( \tau > 0 \). Since we do not impose any special structure on \( B(\sigma) \) in this paper, the condition in Assumption 1 is reasonably unrestricted. Moreover, in all the cases discussed above, the feedback gain \( k \) cannot be made as small as possible for any fixed \( \tau > 0 \), which means that semi-global stabilization of system (60) cannot be achieved. But we have shown in Theorem 3 that semi-global stabilization is always achievable for any delay \( \tau > 0 \) under Assumption 1.

In a word, Assumption 1 is not restrictive if no further assumption is imposed on \( B(\sigma) \), namely, for a general \( B(\sigma) \), Assumption 1 is necessary for the existence of a solution to Problem 1 for any delay \( \tau \). But if we know more information of \( B(\sigma) \), such an assumption may be relaxed at least in the scalar case, namely, if \( B(\sigma) = -e^{-\sigma} \), then Problem 1 has solutions for any delay \( \tau > 0 \). However, the analysis turns out to be very complicated even in the scalar case and it is not clear whether a similar conclusion can be drawn for more general systems.

6. A numerical example

We consider a linear system with distributed delay in the form of (1) with
\[
A = \begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}, \quad B(\sigma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} e^{-\sigma}, \quad B = Be^{-\sigma},
\]
where \( \beta \geq 0 \). Clearly, the open-loop system is marginally unstable. Direct computation gives
\[
\det(A \sigma) = -\alpha^2 \left(\frac{\beta}{\omega^2 + \omega^2} + 1\right) (e^{-\beta \tau} - \cos(\omega \tau))^2 + \sin^2(\omega \tau))^2.
\]
Hence \((A, \sigma)\) is controllable for all \( \beta > 0 \) and \( \tau > 0 \). If \( \beta = 0 \), then based on Remark 3, we conclude that \((A, \sigma)\) is controllable if and only if
\[
\omega \neq \frac{2k\pi}{\tau}, \quad k = 1, 2, \ldots,
\]
since \((A, B)\) is controllable.
Fig. 1. State evolution of the closed-loop system for some $\gamma \in (0, 0.742)$.

Fig. 2. Control signals of the closed-loop system for some $\gamma \in (0, 0.742)$.

To demonstrate the effectiveness of the proposed approach, we choose $\omega = 2$, $r = 1$ and $\beta = \frac{1}{2}$. Then feedback gains can be computed according to (9). For example, if we choose $\gamma = 0.5$, then we obtain

$$K = \begin{bmatrix} -0.1779 & -0.2630 & -0.4419 & -0.6871 \\ 0.3929 & -0.4549 & 1.8728 & -2.3818 \end{bmatrix}.$$  

By simulation, we find that the resulting closed-loop system is asymptotically stable for all $\gamma \in (0, 0.742)$. For different values of $\gamma \in (0, 0.742)$ and by choosing the initial condition $x(\theta) = [1 - 1 - 1]^{T}$, $\theta \in [-r, 0]$, the state trajectories and the control signals of the closed-loop system are recorded in Figs. 1 and 2, respectively. It follows that the peak values of the control signals decrease as $\gamma$ decreases, which implies semi-global stabilization. But at the same time the peak values in the state signals increase, which indicates that smaller value of $\gamma$ leads to worse transient performances of the closed-loop system. Finally, following the procedures indicated in the proof of Theorem 3, the parameter $\gamma^*$ is computed as $\gamma^* = 0.11$.

We have also made an interesting observation from the simulation. Though the closed-loop system is unstable if $\gamma > 0.742$, it is asymptotically stable if $\gamma \in (0.985, 1.6815)$. In fact, as the distributed control matrix $B(\sigma)$ in this example is of the form (23), for $\gamma = 0.985$, $\gamma = 1.3$ and $\gamma = 1.6815$, the rightmost roots of the characteristic quasi-polynomial of the closed-loop system are respectively computed as $\lambda_{\max} = 0$, $\lambda_{\max} = -0.1989 \pm 0.3692j$ and $\lambda_{\max} = \pm 0.5469j$ by using the software package DDE-BIFTOOL (Engelborghs et al., 2001). The fact can also be observed in Fig. 3 where the state trajectories for some $\gamma \in [0.985, 1.6815]$ are recorded. Moreover, if $\gamma > 1.6815$, we find that the system cannot be asymptotically stable anymore. This phenomenon indicates that Theorem 1 is conservative in providing conditions on $\gamma$ such that the closed-loop system is asymptotically stable. One of our future works is to find more accurate conditions on $\gamma$ guaranteeing stability of the closed-loop system. However, our initial efforts have indicated that this is not an easy task.

7. Conclusions

This paper has studied the stabilization problem for linear systems subject to distributed input delay and input saturation. Both time-varying delay and constant delays are considered. Under the assumption that the open-loop system is not exponentially unstable and a stabilizability assumption on an auxiliary system, we have shown that the system is globally stabilizable by state feedback for arbitrarily large but bounded delays. The stabilizing controllers were explicitly constructed by using the parametric Lyapunov equation based low gain feedback design approach we recently developed. Moreover, in the presence of input saturation, we have further shown that these linear controllers achieve semi-global stabilization. We have also discussed the implications of the assumption imposed on the system. A numerical example has been worked out to illustrate the effectiveness of the established approach.

Appendix

A.1. The Proof of Theorem 5

Proof of Item 1. The proof of this item can be found in Ozbay et al. (2008). Here we provide a proof for completeness. The characteristic equation of system (60) is given by

$$0 = s - \alpha + k \int_0^r e^{-\beta \rho} e^{-s \theta} d\theta = s - \alpha + \frac{k(1 - e^{-r(s+\beta)})}{s + \beta}.$$
or equivalently,
\[
0 = 1 + \frac{k(1 - e^{-\tau(s + \beta)})}{(s - \alpha)(s + \beta)} \equiv 1 + H(s). \tag{65}
\]

According to the Nyquist criterion, all the roots of Eq. (65) have negative real parts if and only if \(H(j\omega)\) encircles \(-1\) once in the counter clockwise direction, which is equivalent to (Ozbay et al., 2008)

\[
\begin{aligned}
H(0) &< -1, \\
\left. \frac{d}{d\omega} \mathcal{L}H(j\omega) \right|_{\omega = 0} &> 0, \\
-1 &< H(j\omega_0),
\end{aligned}
\tag{66}
\]

where \(\omega_1 > 0\) is the smallest \(\omega\) such that \(\mathcal{L}H(j\omega) = -\pi\). Notice that

\[
\mathcal{L}H(j\omega) = \mathcal{L} \left( \frac{k(1 - e^{-\tau(s + \beta)})}{(s + \beta)(s - \alpha)} \right) = \arctan \frac{\omega}{\alpha} - \arctan \frac{\omega}{\beta} + \arctan \left( \frac{\sin \omega \tau}{e^{\beta \tau} - \cos \omega \tau} \right) - \pi,
\tag{67}
\]

from which it follows that

\[
\frac{d}{d\omega} \mathcal{L}H(j\omega) = \frac{\alpha}{\alpha^2 + \omega^2} - \frac{\beta}{\beta^2 + \omega^2} + \frac{\tau (e^{\beta \omega} \cos \omega \tau - 1)}{(e^{\beta \omega} - \cos \omega \tau)^2 + \sin^2 \omega \tau}.
\]

Hence, we can compute

\[
0 < \left. \frac{d}{d\omega} \mathcal{L}H(j\omega) \right|_{\omega = 0} = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{\tau}{e^{\beta \omega} - 1} \iff \frac{\beta \tau}{e^{\beta \omega} - 1} > 1 - \frac{1}{\alpha},
\]

which is (61). \(\square\)

**Proof of Item 2.** According to (65), we have

\[
H(0) < -1 \iff -k(1 - e^{-\tau \beta}) < -1 \iff k > \frac{\alpha \beta}{1 - e^{-\tau \beta}},
\]

which is the left inequality in (62). Since \(\alpha = \beta\), we know from (67) that the minimal positive solution to equation \(\mathcal{L}H(j\omega) = -\pi\) is \(\omega_1 \tau = \pi\). Consequently, we get

\[
-1 < H(j\omega_0) = \frac{k(1 - e^{-\tau(j\omega_0 + \beta)})}{(j\omega_0 + \beta)(j\omega_0 - \alpha)} = \frac{k(1 + e^{-\tau \beta})}{-\tau^2 - \alpha \beta}
\]

which is the right inequality in (62). \(\square\)

**Proof of Item 3.** Notice that the characteristic Eq. (65) can be written as \(G(s) e^{-\tau(s + \beta)} = 1\), where

\[
G(s) = \frac{k}{(s - \alpha)(s + \beta)} + k \equiv \left( \frac{k}{k - n} \right) \frac{1}{\frac{1}{k - \alpha^2} + \frac{m}{k - \alpha} + 1},
\]

in which \(m = \beta - \alpha > 0\) and \(n = \alpha \beta\). Assume that \(k > n\). Then \(G(s)\) is a stable transfer function and by the small gain theorem, the overall system is asymptotically stable if \(\|G(s)e^{-\tau(s + \beta)}\|_{\infty} < 1\), which is equivalent to \(e^{\tau \beta} > \|G(s)\|_{\infty}\), or

\[
\tau > \frac{1}{\beta} \ln(\|G(s)\|_{\infty}). \tag{68}
\]

We next compute \(\|G(s)\|_{\infty}\). By setting \(T = \frac{1}{\sqrt{\|n\|}}\) and \(\xi = \frac{m}{\tau} \sqrt{\|k\|}\), we can write

\[
\|G(s)\|_{\infty} = \frac{k}{k - n} \left\| \frac{1}{T^2 + 2\xi T + 1} \right\|_{\infty}.
\]

Then by letting \(\frac{d}{d\xi} \|G(j\omega)\|_{\infty} = 0\) we get the unique nonzero solution

\[
\omega_{\text{opt}} = \frac{1}{T} \sqrt{1 - 2\xi^2}, \quad 0 < \xi \leq \frac{1}{\sqrt{2}}.
\]

Notice that \(0 < \xi \leq \frac{1}{\sqrt{2}}\) if and only if \(k \geq n + \frac{m^2}{\tau}\). Consequently, we can compute

\[
\|G(s)\|_{\infty} = |G(j\omega_{\text{opt}})| = \frac{k}{k - n} \frac{1}{2\xi \sqrt{1 - \xi^2}} = \frac{1 + \frac{4n}{m^2} \xi^2}{2\xi \sqrt{1 - \xi^2}},
\]

where we have used \(k = n + \frac{m^2}{\tau}\). If \(\xi = \frac{m}{\tau} \sqrt{\|k\|} > \frac{1}{\sqrt{2}}\), namely, \(n < k < n + \frac{m^2}{\tau}\), then

\[
\|G(s)\|_{\infty} = \frac{k}{k - n} = 1 + \frac{n}{k - n}. \tag{69}
\]

To get a lower bound of \(\tau\) according to (68), in the following, we should find a \(k\) such that \(\|G(s)\|_{\infty}\) is minimized. If \(k \geq n + \frac{m^2}{\tau}\), then by setting

\[
\frac{d}{d\xi} \|G(s)\|_{\infty} = \frac{1}{2\xi \sqrt{1 - \xi^2}} - \frac{m^2}{m^2 \xi \sqrt{1 - \xi^2}} + \frac{4n}{m^2} \xi^2 = 0,
\]

we obtain the unique nonzero solution

\[
\xi^* = \sqrt{\frac{m^2}{4n + 2m^2}} \in \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),
\]

which corresponds to \(k^* = 2n + \frac{m^2}{\tau}\). Hence, we conclude that

\[
\min_{k \geq n + \frac{m^2}{\tau}} \{ \|G(s)\|_{\infty} \} = \|G(s)\|_{\infty} |_{k^* = \xi^*} = \sqrt{1 + \frac{4n}{m^2}}.
\]

If \(n < k < n + \frac{m^2}{\tau}\), then it follows from (69) that

\[
\min_{k \in (n, n + \frac{m^2}{\tau})} \{ \|G(s)\|_{\infty} \} = \|G(s)\|_{\infty} |_{k = n + \frac{m^2}{\tau}} = 1 + \frac{2n}{m^2}.
\]

Summarizing above we get

\[
\min_{k \geq n} \|G(s)\|_{\infty} = \min_{k \in (n, n + \frac{m^2}{\tau})} \{ \|G(s)\|_{\infty} \} = \sqrt{1 + \frac{4n}{m^2}}.
\]

The proof is finished by inserting the above inequality into (68). \(\square\)

**A.2. Some technical results**

In this Appendix, we recall some existing basic results that are needed in establishing the results of this paper. We first recall the following results from Zhou, Duan and Lin (2008); Zhou et al. (2010b) regarding properties of solutions to the parametric Riccati Eq. (10).

**Lemma 1.** Assume that the matrix pair \((A, B) \in \mathbb{R}^{n \times n}, \mathbb{R}^{n \times m}\) is controllable and all the poles of \(A\) are on the imaginary axis. Then the
Moreover, \( P(\gamma) = \gamma_1 P(\gamma) + \gamma_2 P(\gamma) \) is strictly increasing and \( p(s) > s \). If there is a continuous function \( \theta(t) \) that satisfies the integrals in the following are well-defined, then

\[
\left( \int_{\gamma_1}^{\gamma_2} \omega^\theta(\beta) d\beta \right) Q \left( \int_{\gamma_1}^{\gamma_2} \omega(\beta) d\beta \right) \leq (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^\tau(\beta) Q \omega(\beta) d\beta.
\]

The second technical lemma is the so-called Jensen Inequality.

**Theorem 6** (Gu, 2000) For any positive definite matrix \( Q > 0 \), two scalars \( \gamma_2 \) and \( \gamma_1 \) with \( \gamma_2 \geq \gamma_1 \), and a vector valued function \( \omega : [\gamma_1, \gamma_2] \rightarrow \mathbb{R}^n \) such that the integrals in the following are well-defined, then

\[
\left( \int_{\gamma_1}^{\gamma_2} \omega^\theta(\beta) d\beta \right) Q \left( \int_{\gamma_1}^{\gamma_2} \omega(\beta) d\beta \right) \leq (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^\tau(\beta) Q \omega(\beta) d\beta.
\]

The fourth result is the Razumikhin Stability Theorem.

**Lemma 4.** Let \( t \mapsto U(t) \) be a differentiable function with a finite limit as \( t \to \infty \). If \( \hat{U}(t) \) is uniformly continuous (or \( \hat{U}(t) \) is bounded), then \( \lim_{t \to \infty} \hat{U}(t) = 0 \).

**References**


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