Brief paper

A parametric periodic Lyapunov equation with application in semi-global stabilization of discrete-time periodic systems subject to actuator saturation

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1. Introduction

Periodic systems, as the simplest form of time-varying systems, have been attracting much attention during the past several decades. The interest in this type of system is motivated by several aspects, in particular, their many applications. For example, a multirate sampled-data system can be naturally represented as a periodic system (Chen & Francis, 1995; Longhi, 1994), many physical dynamics (such as those of a pendulum, Bittanti, Hernandez, & Zerbi, 1991) have a cyclic behavior can be modeled by periodic systems, and in some cases, periodic feedback is used to upgrade the performances of a time-invariant plant, resulting in a closed-loop system that is periodic. Because of the many applications of periodic systems, various control problems associated with them have been extensively studied in the literature. For example, structural property analysis (Bittanti & Colaneri, 1996), prediction (Bittanti, Colaneri, & De Nicolao, 1990), pole assignment (Kono, 1980; Lv, Duan, & Zhou, 2010; Varga, 2000), filtering (de Souza, 1991), fault detection (Zhang, Ding, Wang, & Zhou, 2005), switching (Xie & Wang, 2004), stabilizability and detectability (Bittanti & Bolzern, 1985) and robust control (Farges, Peaucelle, Arzelier, & Daafouz, 2007). For more results on this class of systems, see the recent monograph Bittanti and Colaneri (2008) and the references given there.

Actuator saturation is one of the most common and important nonlinearities existing in practical control systems since the capability of any physical actuator is limited. If its adverse effect is neglected, actuator saturation will not only result in performance degradation of the closed-loop system, but can also be the source of instability. For this reason, considerable attention has been paid to control systems with actuator saturation. It is well-recognized that for a time-invariant system subject to actuator saturation, when stabilization is considered, semi-global and global results are possible if and only if it is not exponentially unstable and is stabilizable in the absence of saturation. Systems with these properties are said to be asymptotically null controllable with bounded controls (ANCBC) (Sontag, 1984). For an ANCBC system, linear low gain feedback is an effective methodology that can be used to achieve semi-global results (Lin and Saberi, 1993; Lin, Saberi, & Stoorvogel, 1996; Lin, 1998). However, to achieve global results, nonlinear feedback is necessary (Sreedhar & Dooren, 1994; Sussmann, Sontag, & Yang, 1994; Teel, 1995). If a linear system is not ANCBC, only local results can be obtained. See Amato, Ariola, and Dorato (2001), Garcia and Tarbouriech (2007), Hu and Lin (2001), Kose and Jabbari (2003), Suarez, Alvarez-Ramirez, and Solis-Daun (1997) and the references therein.

However, control problems for periodic systems with actuator saturation have not been thoroughly studied in the literature. Only a few results are available. For example, it is shown in
Collado, Lozano, and Ailon (1999) that a discrete-time ANBC linear plant subject to actuator saturation can be semi-globally stabilized by periodic feedback via a lifted representation of the time-invariant plant; a local stabilization problem of an SISO discrete-time periodic systems with input constraint is solved in Colaneri, Kůčera, and Longhi (2003) by using a polynomial approach and polyhedra are used to characterize the invariant set; and controllability of periodically switched linear systems with saturating actuator is considered in Xie and Wang (2004).

In this paper, we will consider the semi-global stabilization problem of discrete-time linear periodic (DLP) systems subject to actuator saturation by extending the corresponding results for time-invariant systems. Our solution to the problem is based on the solution to a parametric discrete-time periodic Lyapunov equation (DPLE), which generalizes our recent results on time-invariant systems (Zhou, Lin, & Duan, 2009) to periodic systems. The extension is nontrivial due to the complexity of periodic systems. The low gain feedback approach based on the solution to the DPLE is then proposed to solve the semi-global stabilization problem. The advantages of the proposed method include the numerical reliability of the algorithm, the analytic solutions of the parametric DPLE and exact locations of the characteristic multipliers of the resulting closed-loop system. It should be pointed out that the DPLE based design methods developed for DLP systems in this paper can be readily used to solve other related constrained control problems for DLP systems such as semi-global output regulation, semi-global stabilization with magnitude and rate saturating actuator, and perfect regulation that have been well studied for discrete-time time-invariant systems (Lin, 1998).

The remainder of this paper is organized as follows. In Section 2, after a brief introduction to DLP systems, the problem considered in this paper is stated. Section 3 studies a class of discrete-time periodic Riccati equations which can be converted into a special case of periodic Lyapunov equations. Based on the properties to these equations, a solution to the semi-global stabilization problem is given in Section 4. A numerical example is worked out in Section 5 to demonstrate the effectiveness of the proposed method and Section 6 concludes the paper.

Notation. Throughout this paper, we will use standard notation. For a function $u$, we use $|u|_{\infty}$ to denote its $\infty$-norm. The sign function $\text{sign}(y)$ takes the value $+1$ if $y \geq 0$ and $-1$ if $y < 0$. The standard saturation function is defined as $\text{sat}(u) = \text{sign}(u) \min \{ |u|, 1 \}$.

The DLP system (1) is said to be controllable at time $t$, if there exists an $\omega \neq 0$, such that, for some $k$ and $\forall i \in \{ t - \omega + 1, \ldots, t \}$, there holds

$$\Phi_k^T(k) \eta = \eta, \quad R^T_{i-1} \Psi_i^T(k, i) \eta = 0.$$ 

The DLP system (1) is said to be controllable if it has no uncontrollable characteristic multiplier.

The DLP system (1) (or $(A_k, B_k)_{k=0}^{\omega-1}$) is said to be stabilizable if there exists an $\omega$-periodic matrix $(K_k)_{k=0}^{\omega-1}$ such that $(A_k + B_k K_k)_{k=0}^{\omega-1}$ is asymptotically stable.

The following standard result is helpful in our development.

**Lemma 2.** Assume that $(A_k, B_k)_{k=0}^{\omega-1}$ is controllable and $Q_k + \omega = Q_k \in P^{n \times n}, \forall k \in Z$. Then the following statements are equivalent.

1. The DLP system (1) $(A_k, B_k)_{k=0}^{\omega-1}$ is asymptotically stable.
2. The following forward-time DPLE

$$A_k P_k A_k^T - P_{k+1} = -B_k B_k^T, \quad \forall k \in Z,$$

has an $\omega$-periodic positive definite solution.
3. The following reverse-time DPLE

$$A_k^T P_{k+1} A_k - P_k = -Q_k, \quad \forall k \in Z,$$

has an $\omega$-periodic positive definite solution.

2. Problem formulation

2.1. Introduction to discrete-time periodic linear systems

Consider the following DLP system:

$$x(k + 1) = A_k x(k) + B_k u(k), \quad \forall k \in Z,$$

where $x(k) \in \mathbb{R}^{n \times 1}$ and $u(k) \in \mathbb{R}^{m \times 1}$ are, respectively, the state vector and input vector, and the matrices $A_k \in \mathbb{R}^{n \times n}$ and $B_k \in \mathbb{R}^{n \times m}$ are $\omega$-periodic, where $\omega > 1$ is an integer, namely,

$$A_{k+\omega} = A_k, \quad B_{k+\omega} = B_k, \quad \forall k \in Z.$$
Problem 3. Consider the DLP system (3). For any a priori given (arbitrarily large) bounded set \( \Omega \subset \mathbb{R}^n \), find a state feedback law such that the closed-loop system is asymptotically stable at the origin with \( \Omega \) contained in the domain of attraction, that is,

\[
x(0) \in \Omega \Rightarrow \lim_{k \to \infty} \|x(k)\| = 0.
\]

where \( x(k) \) denotes the solution of the closed-loop system with initial condition \( x(0) \).

It is well-known that if \( \omega = 1 \), when the DLP system (3) becomes a discrete-time linear time-invariant system, Problem 3 is solvable if and only if \( (A_0, B_0) \) is stabilizable and all the eigenvalues of \( A_0 \) are located inside or on the unit circle. It is natural to conclude that the situation for general \( \omega \geq 2 \) will not be better than the time-invariant case. This is indeed the case as stated below.

Proposition 4. Problem 3 is solvable via state feedback only if the \( \omega \)-periodic matrix pair \( \{(A_k, B_k)\}_{k=0}^{\infty} \) is stabilizable and all the characteristic multipliers of \( A_k \) are located inside or on the unit circle.

Proof. The stabilizability of the \( \omega \)-periodic matrix pair \( \{(A_k, B_k)\}_{k=0}^{\infty} \) is clearly necessary. We need only to show that the second condition is also necessary. Let

\[
\begin{align*}
\chi^k(k) &= x(k\omega) \\
u^k(k) &= u^k(k\omega) = u^k(k\omega + 1) \cdots u^k(k\omega + \omega - 1)
\end{align*}
\]

for all \( k \in \mathbb{Z} \). Then \( \chi^k(k) \) satisfies the following lifted time-invariant system (see, e.g., Khargonekar & B"ulent Özgüür, 1994):

\[
\chi^k(k + 1) = \Phi_A \chi^k(k) + B^k \text{sat}(u^k(k)),
\]

where \( \Phi_A = \Phi_{A_0} \) and

\[
B^k = [A_{\omega - 1} A_{\omega - 2} \cdots A_1 B_0 \cdots A_{\omega - 1} B_{\omega - 2} B_{\omega - 1}].
\]

These two systems (5) and (3) are related in such a manner that if \( x(k) \) and \( u(k) \) are the state and the input vectors of system (5), respectively, then \( \chi^k(k) \) and \( u^k(k) \) defined in (4) satisfy the time-invariant system (5).

We prove the statement by contradiction. Assume that some characteristic multipliers of \( \{A_k\}_{k=0}^{\infty} \) are outside the unit circle and, given a bounded set \( \Omega \), for any \( x(0) \in \Omega \), there exists a control \( u^k(k) \) such that \( x(0) \in \Omega \Rightarrow \lim_{k \to \infty} \|x(k)\| = 0 \). Since \( x^0(0) = x(0) \), it follows that \( \chi^0(0) \in \Omega \Rightarrow \lim_{k \to \infty} \|\chi^k(k)\| = 0 \), namely, \( u^k(k) \) defined in (4) also solves the semi-global stabilization problem for the time-invariant system (5). This is impossible since it is well-known that the semi-global stabilization problem for the time-invariant system (5) is solvable if and only if all the eigenvalues of \( \Phi_A \) are inside or on the unit circle. A contradiction. The proof is completed. \( \square \)

3. A parametric DPLE and its properties

To introduce the parametric DPLE, we first introduce a lemma that collects some well-known results on optimal control of periodic systems (see, for example, Bittanti & Colaneri, 2008).

Lemma 5. Consider the DLP system (1) and the following optimization problem

\[
\inf_{u(k)} \left\{ \sum_{k=0}^{\infty} u^T(k) R_k u(k) \right\} \quad \text{s.t.} \quad \lim_{k \to \infty} x(k) = 0.
\]

(1) The optimization problem (7) has a stabilizing solution if and only if \( \{(A_k, B_k)\}_{k=0}^{\infty} \) is stabilizable and \( |\lambda_i(\Phi_A)| \neq 1, \forall i \in \mathcal{N} \).

(2) Under the condition in Item 1, the unique optimal solution is given by

\[
u^*(k) = -(R_k + B_k^T S_k + B_k)^{-1} B_k^T S_k + A_k x(k),
\]

where \( S_k, \forall k \in \mathbb{Z} \) is the maximal \( \omega \)-periodic solution to the following discrete-time periodic algebraic Riccati equation (DPARE)

\[
S_k = A_k^T S_k + A_k - A_k^T S_k + B_k \times (R_k + B_k^T S_k + B_k)^{-1} B_k^T S_k + A_k.
\]

(3) The DPARE (9) has a unique \( \omega \)-periodic positive definite solution \( S_k \in \mathbb{P}_{\times n}^+ \) if and only if \( \{(A_k, B_k)\}_{k=0}^{\infty} \) is controllable and \( |\lambda_i(\Phi_A)|_{\min} \geq 1 \). Moreover, such a solution is also the unique stabilizing solution.

(4) Assume that the DPARE (9) has a unique \( \omega \)-periodic solution \( S_k \in \mathbb{P}_{\times n}^+ \). Then \( H_k = S_k^{-1} \) satisfies the following forward-time DPLE:

\[
H_{k+1} - A_k H_k A_k^T = -B_k R_k^{-1} B_k^T, \quad \forall k \in \mathbb{Z}.
\]

(5) Assume that the DPARE (9) has a unique \( \omega \)-periodic positive definite solution \( S_k \in \mathbb{P}_{\times n}^+ \). Let \( \Phi_A \) be the monodromy matrix of the closed-loop system consisting of (1) and (8). Then the eigenvalues of \( \Phi_A \) are mirror images of those of \( \Phi_A \) with respect to the unit circle.

We now consider the following optimal control problem for the DLP system (1):

\[
\inf_{u(k)} \left\{ \sum_{k=0}^{\infty} \left( \prod_{i=0}^{k-1} (1 - \gamma_i) \right)^{-1} u^T(k) R_k u(k) \right\} \quad \text{s.t.} \quad \lim_{k \to \infty} x(k) = 0,
\]

which reduces to (7) if \( \gamma_k = 0, \forall k \in \mathbb{Z} \). Here \( \gamma \triangleq (\gamma_0, \gamma_1, \ldots, \gamma_{\omega-1}) \) with \( \gamma_k > 0 \). \( \forall k \in \mathbb{Z} \), is given. Based on Lemma 5, we are able to give the following corollary regarding the above optimization problem.

Corollary 6. Consider the DLP system (1). Then

(1) The optimization problem (11) has a stabilizing solution if and only if \( \{(A_k, B_k)\}_{k=0}^{\infty} \) is stabilizable and

\[
|\lambda_i(\Phi_A)| \neq 1, \forall i \in \mathcal{N}.
\]

(2) Under the condition in Item 1, the unique optimal solution to the optimization problem (11) is given by

\[
u^*(k) = -(R_k + B_k^T P_{k+1} + B_k)^{-1} B_k^T P_{k+1} A_k x(k),
\]

where \( P_k, \forall k \in \mathbb{Z} \), is the maximal \( \omega \)-periodic solution to the following parameteric DPARE

\[
(1 - \gamma_k) P_k = A_k^T P_{k+1} A_k - A_k^T P_{k+1} B_k \times (R_k + B_k^T P_{k+1} + B_k)^{-1} B_k^T P_{k+1} A_k.
\]

(3) The DPARE (14) has a unique \( \omega \)-periodic positive definite solution \( P_k \in \mathbb{P}_{\times n}^+ \) if and only if \( \{(A_k, B_k)\}_{k=0}^{\infty} \) is controllable and

\[
|\lambda_i(\Phi_A)|_{\min} > 1, \forall k \in \mathfrak{s} \end{mathfrak} \quad \text{and} \quad \prod_{k=0}^{\omega-1} (1 - \gamma_k) < |\lambda_i(\Phi_A)|_{\min}.
\]
(4) Assume that the DPARE (14) has a unique ω-periodic positive definite solution \( P_k \in \mathbb{P}^{n \times n} \). Then \( W_k = P_k^{-1} \) satisfies the following forward-time parametric DPE:

\[
W_{k+1} - \frac{1}{1 - \gamma_k} A_k W_k A_k^T = -B_k R_k^{-1} B_k^T, \quad \forall k \in \mathbb{Z}.
\]

(16)

(5) Assume that the DPARE (14) has a unique ω-periodic positive definite solution \( P_k \in \mathbb{P}^{n \times n} \). Let \( A_{\phi(x)}(y) \) be the monodromy matrix of the closed-loop system consisting of (1) and (13). Then the eigenvalues of \( \Phi_{A_{\phi(x)}(y)} \) are mirror images of those of \( \Phi_A \) with respect to the circle

\[
\Gamma(y) = \left\{ z \in \mathbb{C} : |z|^2 = \prod_{k=0}^{\omega-1} (1 - \gamma_k) \right\}.
\]

Consequently, \( \Phi_{A_{\phi(x)}(y)} \) is Schur stable if and only if

\[
\gamma_k < 1, \quad \forall k \in \mathbb{Z} \quad \text{and} \quad \prod_{k=0}^{\omega-1} (1 - \gamma_k) < |\lambda(\Phi_A)|_{\min}.
\]

\[
\text{Proof.} \quad \text{Consider a new state vector } x_n(k) \text{ and a new input vector } u_n(k) \text{ defined as}
\]

\[
x_n(k) = \frac{1}{\prod_{i=0}^{k} (\sqrt{\gamma} - \gamma_i)} x(k), \quad u_n(k) = \frac{1}{\prod_{i=0}^{k} (\sqrt{\gamma} - \gamma_i)} u(k),
\]

on which the original DLP system (1) can be written as

\[
x_n(k+1) = \frac{1}{\prod_{i=0}^{k+1} (\sqrt{\gamma} - \gamma_i)} x(k+1) + A_k x_n(k) + B_k u_n(k).
\]

on which the original DLP system (1) can be written as

\[
x_n(k+1) = \frac{1}{\prod_{i=0}^{k+1} (\sqrt{\gamma} - \gamma_i)} x(k+1) + A_k x_n(k) + B_k u_n(k)
\]

Clearly, as \( A_k, B_k \) and \( \gamma_k \) are \( \omega \)-periodic, we know that \( A_{nk} \) and \( B_{nk} \) are also \( \omega \)-periodic. Moreover, the optimization problem (11) can be equivalently written as

\[
\inf_{u_n(k)} \left\{ \sum_{k=0}^{\infty} u_n^T(k) R_k u_n(k) \right\} \quad \text{s.t.} \quad \lim_{k \to \infty} x_n(k) = 0
\]

which is a standard minimal energy control problem for the system (19).

1. According to Item 1 of Lemma 5, optimization problem (20) has a solution if and only if, for all \( i \in \mathcal{N} \),

\[
1 \neq \lambda_i(\Phi_{A_{nk}}) = \lambda_i \left( \prod_{k=0}^{\omega-1} \frac{A_k}{\sqrt{\gamma} - \gamma_{k+1}} \right) = \frac{\lambda_i(\Phi_A)}{\prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1}},
\]

which is (12), and \( \{A_{\phi(x)}(y), B_{\phi(x)}(y)\}_{i=0}^{\omega-1} \) is stabilizable, which is equivalent to \( \{A_k, B_k\}_{i=0}^{\omega-1} \) being stabilizable.

2. It follows from Item 2 of Lemma 5 that the unique solution to problem (20) is given by

\[
u_n^*(k) = - (R_k + B_{nk}^T P_{nk+1} B_{nk})^{-1} B_{nk}^T P_{nk+1} A_{nk} x_n(k),
\]

where \( P_{nk} \in \mathbb{P}^{n \times n}, \forall k \in \mathbb{Z} \), is the maximal \( \omega \)-periodic solution to the following DPARE

\[
P_{nk} = A_{nk}^T P_{nk+1} A_{nk} - A_{nk}^T P_{nk+1} B_{nk}
\]

\[
\times (R_k + B_{nk}^T P_{nk+1} B_{nk})^{-1} B_{nk}^T P_{nk+1} A_{nk}.
\]

(22)

This DPARE is equivalent to (14) by denoting \( P_k = P_{nk}/(1 - \gamma_k), \forall k \in \mathbb{Z} \). Consequently, the optimal control law in (21) reduces to (13).

3. It follows from Item 3 of Lemma 5 that the DPARE (22) has a unique \( \omega \)-periodic positive-definite solution \( P_k \in \mathbb{P}^{n \times n} \) if and only if

\[
1 < \frac{\lambda(\Phi_A)}{\prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1}} = \frac{\prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1}}{\gamma_k},
\]

which is (15) by noting that \( \gamma_k < 1 \) is necessary, and \( \{A_{nk}(y), B_{nk}(y)\}_{i=0}^{\omega-1} \) is controllable, which is equivalent to \( \{A_k, B_k\}_{i=0}^{\omega-1} \) being controllable.

4. It follows from Item 4 of Lemma 5 that the unique solution to the DPARE (22) satisfies

\[
W_{nk+1} - A_{nk} W_{nk} A_{nk}^T = -B_{nk} R_k^{-1} B_k^T, \quad \forall k \in \mathbb{Z},
\]

which is (16) by noting that

\[
W_{nk} = P_{nk} = (1 - \gamma_k)^{-1} P_{nk}^{-1} = (1 - \gamma_k)^{-1} W_k
\]

5. It follows from Item 5 of Lemma 5 that the poles of

\[
\Phi_{A_{nk}(y)} = \prod_{k=0}^{\omega-1} A_{nk} \quad \text{s.t.} \quad \prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1} = \frac{\lambda(\Phi_A)}{\prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1}}
\]

(23)

and those of \( \Phi_{A_{nk}(y)} \) are mirror images of each other, namely, for each \( i \in \mathcal{N} \), there exists a \( j \in \mathcal{N} \) such that

\[
1 = \lambda_i(\Phi_{A_{nk}(y)}) \quad \lambda_j(\Phi_{A_{nk}(y)})
\]

\[
\prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1} = \prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1}
\]

\[
\lambda_i(\Phi_{A_{nk}(y)}) = \frac{\lambda_j(\Phi_A)}{\prod_{k=0}^{\omega-1} \sqrt{\gamma} - \gamma_{k+1}}.
\]

That is to say, the poles of \( \Phi_{A_{nk}(y)} \) and the poles of \( \Phi_A \) are mirror images of each other with respect to the circle (17). Hence, \( \Phi_{A_{nk}(y)} \) is Schur stable if and only if

\[
1 > \left| \lambda(\Phi_{A_{nk}(y)}) \right|_{\max} = \prod_{k=0}^{\omega-1} (1 - \gamma_k) \left| \lambda(\Phi_{A_{nk}(y)}) \right|_{\min}
\]

which is (18). The proof is finished. \( \Box \)

It follows from Item 1 of Corollary 6 that a convergence rate for the closed-loop system is guaranteed. For this reason, we may refer to (14) as the associated DPARE for the “minimal energy control with guaranteed convergence” problem.

Some remarks for Corollary 6 are in order.
Remark 7. There is a significant difference between Item 3 of Lemma 5 and Item 3 of Corollary 6. In Lemma 5, the positive definite solution $S_t$ is also the stabilizing solution for system (1), whereas in Corollary 6, the positive definite solution $P_t$ may not be the stabilizing solution for system (1). This difference is due to the fact that, in Corollary 6, we only know that the positive definite solution $P_{st}$ is the stabilizing solution for the auxiliary system (19).

Remark 8. Corollary 6 presents a simple method to stabilize a DLP system by solving a forward-time parametric DPLE in the form of (16), where $\gamma_k$ is such that $\gamma_k < 1$, $\forall k \in \delta$, and

\[
\prod_{k=0}^{\omega-1} (1 - \gamma_k) < \min \{ |\lambda_1 (P_t)|_{\min}, |\lambda_2 (P_t)|_{\min} \}.
\]

Though it can only deal with controllable and reversible systems, more general systems can be handled quite easily by using a similar technique as used in Sreedhar and Dooren (1994).

Remark 9. In Sreedhar and Dooren (1994), the following result was obtained: The periodic control law $u(k) = -F_kx(k), F_k = (I_m + B_k^TP_k^{-1} B_k)^{-1} B_k^TP_k^{-1} A_k,$ stabilizes the DLP system (1), where

\[
W_{k+\omega} = W_k \in P^{\omega \times \omega} \text{ solves the following reverse-time DPLE}
\]

\[
W_k - \frac{1}{\alpha} A_k W_{k+1} A_k^T = -2B_k^TP_k, \tag{24}
\]

in which $\alpha$ is chosen such that $0 < \alpha^\omega < \min \{ 1, |\lambda_1 (P_t)|_{\min}, |\lambda_2 (P_t)|_{\min} \}$. Moreover, all the characteristic multipliers of $[A_k - B_k F_k]_{\omega=0}^{\omega-1}$ lie within the $\alpha$-circle centered at the origin. We make the following observations on the relationship between our results in Corollary 6 and those of Sreedhar and Dooren (1994):

1. The scalars $\gamma_k$ in Corollary 6 are only assumed to be $\omega$-periodic while the scalar $\alpha$ is assumed to be a constant in (24). Therefore, our results are more general than those in Sreedhar and Dooren (1994);

2. We have explored the relations between DPLE (16) and DPARE (14) and present necessary and sufficient conditions for the existence of an $\omega$-periodic positive definite solution and stabilizing solution; and

3. In Corollary 6, we have discovered that the characteristic multipliers of the closed-loop system and those of the open-loop system are symmetric with respect to the circle $I(\gamma)$ defined in (17), while Sreedhar and Dooren (1994) only states that the characteristic multipliers of the closed-loop system are within a circle.

Finally, we notice that DPLE (16) in Corollary 6 is forward-time while DPLE (24) is reverse-time.

Based on Corollary 6, we are also able to show the monotonicity property of the unique $\omega$-periodic positive definite solution to DPARE (14).

**Theorem 10.** Let $\{(A_k, B_k)\}_{k=0}^{\omega-1}$ be controllable. Assume that (15) holds and $\gamma_k, \forall k \in \delta,$ are differentiable and monotonically increasing functions of a positive scalar $\epsilon \in (0, 1)$, namely, $\gamma_k = \frac{\epsilon^{\omega-1}}{1+\epsilon}$. Then the unique $\omega$-periodic positive definite solution $P_k(\gamma) \in P^{\omega \times \omega}, \forall k \in Z,$ to DPARE (14) is also differentiable and monotonically increasing with respect to $\epsilon$, i.e., for any $\epsilon \in (0, 1),

\[
\frac{dP_k(\gamma)}{d\epsilon} > 0, \quad \forall k \in \delta \Rightarrow \frac{dP_k(\gamma)}{d\epsilon} > 0, \quad \forall k \in Z. \tag{25}
\]

**Proof.** Let $Z_k = (R_k + B_k^TP_{k+1} (\gamma) B_k)^{-1}. Taking derivatives of both sides of (14) with respect to $\epsilon$ gives

\[
\frac{dP_k(\gamma)}{d\epsilon} = \frac{dP_{k+1}(\gamma)}{d\epsilon} A_k + \frac{d\gamma_k}{d\epsilon} P_k(\gamma)
\]

\[
- A_k^T \frac{dP_{k+1}(\gamma)}{d\epsilon} B_k Z_k B_k^T P_k(\gamma) A_k + \frac{d\gamma_k}{d\epsilon} P_k(\gamma)
\]

\[
- A_k^T P_{k+1} (\gamma) B_k Z_k B_k^T \frac{dP_{k+1}(\gamma)}{d\epsilon} A_k
\]

\[
+ A_k^T \frac{dP_{k+1}(\gamma)}{d\epsilon} B_k Z_k B_k^T P_{k+1} (\gamma) A_k.
\]

which can be rewritten as

\[
\frac{dP_k(\gamma)}{d\epsilon} = A_k^T \frac{dP_{k+1}(\gamma)}{d\epsilon} A_k - \frac{d\gamma_k}{d\epsilon} P_k(\gamma)
\]

\[
- A_k^T \frac{dP_{k+1}(\gamma)}{d\epsilon} B_k Z_k B_k^T P_{k+1} (\gamma) A_k
\]

\[
- A_k^T P_{k+1} (\gamma) B_k Z_k B_k^T \frac{dP_{k+1}(\gamma)}{d\epsilon} A_k
\]

\[
+ A_k^T \frac{dP_{k+1}(\gamma)}{d\epsilon} B_k Z_k B_k^T P_{k+1} (\gamma) A_k,
\]

where $A_{2k} (\gamma) = \frac{1}{\sqrt{1-\gamma_k}} A_k$. Let $A_{2k} (\gamma)$ be defined as

\[
A_{2k} (\gamma) = A_{2k} (\gamma) - B_k Z_k B_k^T P_{k+1}(\gamma) A_{2k} (\gamma).
\]

Then the equation in (26) can be simplified as

\[
A_{2k} (\gamma) = A_{2k} (\gamma) - B_k Z_k B_k^T P_{k+1}(\gamma) A_{2k} (\gamma).
\]

We next show that the $\omega$-periodic matrix $\{(A_{2k} (\gamma))_{k=0}^{\omega-1}\}$ is Schur stable. In fact, by using (23), we get

\[
\Phi_{A_{2k} (\gamma)} = \prod_{k=0}^{\omega-1} A_{2k} (\gamma)
\]

\[
= \prod_{k=0}^{\omega-1} (A_{2k} (\gamma) - B_k Z_k B_k^T P_{k+1}(\gamma) A_{2k} (\gamma))
\]

\[
= \prod_{k=0}^{\omega-1} \left( I - \frac{1}{\sqrt{1-\gamma_k}} \right)
\]

\[
= \prod_{k=0}^{\omega-1} \left( I - \frac{1}{\sqrt{1-\gamma_k}} \right)
\]

\[
= \Phi_{A_{2k} (\gamma)}.
\]

According to the proof of Corollary 6 and Lemma 5, we know that $\Phi_{A_{2k} (\gamma)}$ is Schur stable. Thus $\Phi_{A_{2k} (\gamma)}$ is also Schur stable. Then, by Lemma 2, we know that DPLE (27) has a unique $\omega$-periodic matrix $dP_k(\gamma) \in P^{\omega \times \omega}.$ The proof is completed.

In the very special case that all the eigenvalues of $\Phi_k$ are located on the unit circle, the two conditions in (15) and (18) become

\[
\gamma_k < 1, \quad \forall k \in \delta, \quad \forall \epsilon \in (0, 1),
\]

\[
\prod_{k=0}^{\omega-1} (1 - \gamma_k) < 1, \quad \forall \epsilon \in (0, 1).
\]

In this special case, we have the following result, which is essential in constructing solutions to Problem 3.
Theorem 11. Assume that \( \{(A_k, B_k)\}_{k=0}^{n-1} \) is controllable, all the eigenvalues of \( \Phi_k \) are located on the unit circle and the \( \omega \)-periodic scalar \( |\gamma_k|_{k=0}^{\infty} \) is uniformly continuous with respect to \( \epsilon \) and is chosen such that (28) is satisfied. Let \( P_k(\gamma) \in P^{\times n} \) be the unique \( \omega \)-periodic positive definite solution to the parametric DPARSE (14). Then, if

\[
\frac{d\gamma_k(\epsilon)}{d\epsilon} > 0, \quad \forall k \in \mathbb{Z}, \quad \forall \epsilon \in (0, 1),
\]

and

\[
\prod_{k=0}^{n-1} \left( 1 - \lim_{\epsilon \to 0^+} \gamma_k(\epsilon) \right) = 1,
\]

are satisfied, then there holds

\[
\lim_{\epsilon \to 0^+} P_k(\gamma) = 0, \quad \forall k \in \mathbb{Z}.
\]

Proof. By Theorem 10, the limit of \( P_k(\gamma(\epsilon)) \) exists as \( \epsilon \) approaches zero. Let \( \lim_{\epsilon \to 0^+} P_k(\gamma(\epsilon)) = P_k \geq 0 \). Since \( |\gamma_k|_{k=0}^{\infty} \) is uniformly continuous with respect to \( \epsilon \), \( \lim_{\epsilon \to 0^+} \gamma_k(\epsilon) \), \( \forall k \in \mathbb{Z} \), exists. Denote

\[
\gamma_k(0) = \lim_{\epsilon \to 0^+} \gamma_k(\epsilon), \quad \forall k \in \mathbb{Z}.
\]

Then we clearly have \( \gamma_k(0) \leq 1 \) and \( \gamma_k(0) \neq 1, \forall k \in \mathbb{Z} \). Otherwise, if \( \gamma_k(0) = 1 \) for some \( k \), then it contradicts with (31). Taking limit on both sides of DPARSE (14) as \( \epsilon \) goes to zero yields

\[
(1 - \gamma_k(0)) P_k = A_k^T P_{k+1} A_k - A_k^T P_{k+1} B_k \times \left( R_k + B_k^T P_{k+1} B_k \right)^{-1} B_k^T P_{k+1} A_k,
\]

which is equivalent to

\[
P_k = A_k^T P_{k+1} A_k - A_k^T P_{k+1} B_k \left( R_k + B_k^T P_{k+1} B_k \right)^{-1} B_k^T P_{k+1} A_k,
\]

where

\[
A_k = \frac{A_k}{\sqrt{1 - \gamma_k(0)}}, \quad \forall k \in \mathbb{Z}.
\]

Since \( \{(A_k, B_k)\}_{k=0}^{n-1} \) is controllable, we know that \( \{(A_k, B_k)\}_{k=0}^{n-1} \) is also controllable. Also, as all the eigenvalues of \( \Phi_k \) are located on the unit circle, and

\[
\Phi_k = \prod_{k=0}^{n-1} \left( \frac{A_k}{\sqrt{1 - \gamma_k(0)}} \right) \left( \prod_{k=0}^{n-1} (1 - \gamma_k(0)) \right)^{\frac{1}{2}} \Phi_A,
\]

it follows from (31) that all the eigenvalues of \( \Phi_k \) are also located on the unit circle. However, it is well-known that the only nonnegative definite solution to (33) is zero (see, e.g., Bittanti & Colaneri, 2008; Bittanti, Colaneri, & De Nicolao, 1991). The proof is completed. \( \square \)

Remark 12. The functions \( \gamma_k(\epsilon) \) satisfying all the conditions in (28)-(29) and (30)-(31) are however not necessarily nonnegative for all \( k \) and \( \epsilon \in (0, 1) \). For example, if \( \omega = 2 \), we can choose

\[
\gamma_0(\epsilon) = -1 + \frac{1}{3} \epsilon \in \left( -1, -\frac{2}{3} \right],
\]

\[
\gamma_1(\epsilon) = \frac{1}{2} + \frac{1}{3} \epsilon \in \left[ \frac{1}{2}, \frac{5}{6} \right].
\]

Then \( \frac{d}{d\epsilon} \gamma_0(\epsilon) > 0, \frac{d}{d\epsilon} \gamma_1(\epsilon) > 0, \forall \epsilon \in (0, 1), (1 - \lim_{\epsilon \to 0^+} \gamma_0(\epsilon)) (1 - \lim_{\epsilon \to 0^+} \gamma_1(\epsilon)) = 1 \) and

\[
(1 - \gamma_0(\epsilon)) (1 - \gamma_1(\epsilon)) = 1 + \frac{1}{e^2} - \frac{1}{3} \left( 2 + \frac{1}{2} \right) \left( 1 - \epsilon \right)^2 < \left( 1 - \frac{1}{3} \right) < 1, \quad \forall \epsilon \in (0, 1),
\]

namely, all the conditions in (28)-(29) and (30)-(31) are satisfied. But \( \gamma_0(\epsilon) \) is negative for all \( \epsilon \in (0, 1) \).

At the end of this subsection, we present the following result for solving the parametric DPLE (16) and the parametric DPARSE (14). The proof is standard and is omitted for brevity.

Lemma 13. The \( \omega \)-periodic matrix \( W_k(\gamma) \) is the solution to the parametric DPARSE (16) if and only if \( W_0 \) satisfies the following parametric discrete-time Lyapunov matrix equation

\[
W_0 - \Phi_k W_0 \Phi_k^T = -U(\gamma),
\]

where \( \Phi_k \) and \( U(\gamma) \) are, respectively, given by

\[
\Phi_k = \prod_{i=0}^{n-1} \left( \frac{A_k}{\sqrt{1 - \gamma_i}} \right),
\]

\[
U(\gamma) = \sum_{k=0}^{n-1} \left( A_k (\gamma) B_{k-1} R_{k-1}^{-1} B_{k-1}^T + B_{k-1} R_{k-1}^{-1} B_{k-1}^T \right)
\]

with \( A_k(\gamma) = \prod_{i=0}^{n-1} A_i(\gamma) \sqrt{1 - \gamma_i} \), and \( W_k, k = \omega - 1, \omega - 2, \ldots, 1 \), satisfy the recursive relations:

\[
W_k = \frac{1}{1 - \gamma_k} A_k - W_{k-1} A_k R_{k-1}^{-1} B_{k-1}^T - B_{k-1} R_{k-1}^{-1} B_{k-1}^T,
\]

\[
U(\gamma) = \sum_{k=0}^{n-1} \left( A_k (\gamma) B_{k-1} R_{k-1}^{-1} B_{k-1}^T + B_{k-1} R_{k-1}^{-1} B_{k-1}^T \right)
\]

or, equivalently, \( p_k, k = \omega - 1, \omega - 2, \ldots, 1 \), satisfy the recursive relations:

\[
p_k = \frac{1}{1 - \gamma_k} A_k (p_{k+1} - p_k + B_k Z_k B_k^T + p_{k+1}) A_k,
\]

\[
p_\omega = p_0 = W_0^{-1}
\]

in which \( Z_k = (R_k + B_k^T p_k + B_k) \). The above lemma shows that to solve the parametric DPLE (16) and the parametric DPARSE (14), we need only to solve a linear discrete-time Lyapunov equation with coefficient matrices of dimensions \( n \times n \).

4. Solutions to Problem 3

According to Proposition 4, we should assume that \( \{(A_k, B_k)\}_{k=0}^{n-1} \) is stabilizable and all the characteristic multipliers of \( \{A_k\}_{k=0}^{n-1} \) are located inside or on the unit circle. Then by applying the periodic real Schur decomposition (Bojanic, Golub, & Dooren, 1992; Varga, 1997), we can assume, without loss of generality, that the original DLP system (3) has the following structure:

\[
A_k = \begin{bmatrix} A_k^1 & A_k^2 \\ 0 & A_k^3 \end{bmatrix}, \quad B_k = \begin{bmatrix} B_k^1 \\ B_k^2 \end{bmatrix}, \quad \forall k \in \mathbb{Z},
\]

in which \( \{(A_k^1, B_k)\}_{k=0}^{n-1} \) is controllable, all the characteristic multipliers of \( \{A_k^1\}_{k=0}^{n-1} \) are on the unit circle and all the characteristic multipliers of \( \{A_k^1\}_{k=0}^{n-1} \) are strictly inside the unit circle. Since the stable part \( \{A_k^1\}_{k=0}^{n-1} \) does not affect the solvability of the problem, namely, we are able to stabilize the subsystem \( \{(A_k^2, B_k)\}_{k=0}^{n-1} \) without destabilizing the stable subsystem \( \{(A_k^3, B_k)\}_{k=0}^{n-1} \), we can further assume that the following is true.
Assumption 14. All the characteristic multipliers of \([A_k]_{i=0}^{\omega-1}\) are on the unit circle and \([A_k, B_k]_{i=0}^{\omega-1}\) is controllable.

For simplicity, we define the following notation associated with DPARE (14):
\[
F_k(\gamma) = (R_k + B_k^TP_{k+1}(\gamma)B_k^{-1}B_k^TP_{k+1}(\gamma)A_k),
\]
and \(A_{zk}(\gamma) = A_k - B_kF_k(\gamma), \quad \forall k \in \mathbb{Z}.
\]
Hereafter, we will suppress the independent variable \(\gamma\) for simplicity. For example, \(P_k(\gamma)\) will be denoted by \(P_k\). Then it is easy to show that
\[
A_{zk}^TP_{k+1}A_{zk} - P_k = -\gamma P_k - F_k^T R_k F_k.
\]

The solutions to Problem 3 can then be stated as follows.

Theorem 15. Consider the DLP system (3). Let Assumption 14 hold. Let \(W_k \in \mathbb{P}^{n 	imes n}, \forall k \in \mathbb{Z}\) be the \(\omega\)-periodic solution to the parametric DPLE
\[
W_{k+1} + \frac{1}{1 - \gamma_k} A_k W_k A_k^T = -B_k R_k^{-1} B_k^T, \quad \forall k \in \mathbb{Z},
\]
where \(\gamma_{k+\omega} = \gamma_k = \gamma(\varepsilon) : (0, 1] \to \mathbb{R}, \forall k \in \mathbb{Z}\), are \(\omega\)-periodic functions that are uniformly continuous with respect to \(\varepsilon\) and such that (28)-(29) and (30)-(31) are satisfied. Then the following family of periodic linear feedback control laws
\[
u(k) = -F_k(\gamma) x(k), \quad k \in \mathbb{Z},
\]
solves Problem 3, where \(F_k(\gamma)\) is defined in (38) with \(P_{k+1} = W_{k+1}^{-1}, \forall k \in \mathbb{Z}\). Namely, for any \(\varepsilon > 0\) that is small enough, the resulting closed-loop system is asymptotically stable at the origin with the set \(\Omega\) contained in the domain of attraction. Moreover, for any \(k \geq 0\) and \(\varepsilon \in (0, \varepsilon^*]\), there holds
\[
\|x(k)\| \leq \sqrt{\frac{X_{\min}(\Omega)}{X_{\max}(\Omega)}} \prod_{j=0}^{k-1} (1 - \gamma_j) \|x(0)\|.
\]

Proof. Rewrite the closed-loop system as
\[
x(k+1) = A_k x(k) - B_k \text{sat}(F_k x(k)), \quad \forall k \geq 0.
\]
Let \(\varepsilon^*_1 \in (0, 1]\) be such that
\[
\Omega \subseteq \varepsilon^*(P_0), \quad \forall \varepsilon \in (0, \varepsilon^*_1].
\]
The existence of \(\varepsilon^*_1\) is due to the fact that \(\lim_{\varepsilon \to 0^+} \text{sat}(F_0 x(k)) = 0\) and the boundedness of \(\Omega\).

The boundedness of the innermost integral is due to the fact that \(\lim_{\varepsilon \to 0+} (\cdot) = 0\) and the boundedness of \(\Omega\).

For arbitrary integer \(k\), there exist two integers \(k_1 \geq 0\) and \(k_2 \in [0, \omega - 1]\) such that \(\omega k_1 + k_2 \leq k\). Then according to (29), we have
\[
\prod_{j=0}^{k-1} (1 - \gamma_j) \leq \prod_{j=0}^{k-1} (1 - \gamma_j) = \prod_{j=k}^{k_2} (1 - \gamma_j), \quad k < \omega
\]
and
\[
\prod_{j=0}^{k-1} (1 - \gamma_j) \leq \prod_{j=k_1}^{k_2} (1 - \gamma_j), \quad k < \omega.
\]
Hence, the closed-loop system can always be written as (47) for all \(k \geq 0\). The asymptotic stability then follows from the fact that \([A_k - B_k F_{k+1}]_{i=0}^{\omega-1}\) is asymptotically stable.

where \(\Delta(\varepsilon)\) is defined as
\[
\Delta(\varepsilon) = \max \left\{ \frac{1}{\Delta(\varepsilon)} \right\} \|x(0)\|.
\]

Obviously, as \(\gamma(\varepsilon)\) are uniformly continuous functions of \(\varepsilon\) and \(\varepsilon > 0\) is true, we know that \(\Delta(\varepsilon)\) is bounded for all \(\varepsilon \in (0, 1]\). Rewrite DPARE (14) as
\[
(1 - \gamma_k) P_k = A_k^T P_{k+1}^{-1} + B_k R_k^{-1} B_k^T A_k,
\]
from which we get
\[
Y_k = P_{k+1} A_k P_k^{-1} A_k^T P_{k+1}^{-1} (1 - \gamma_k).
\]

Let the \(i\)-th row of the matrix \((I_k + B_k R_k^{-1} B_k^T A_k)^{-1}\) be \(r_{ki}, \forall i \in \{1, 2, \ldots, m\} \subseteq M, k \in \mathbb{Z}\). Then by using (45) we have
\[
\delta \left( \frac{1}{\Delta(\varepsilon)} P_0 \right) \subseteq \mathcal{L}(F_0),
\]

\(\varepsilon^*_1\) be such that
\[
\Delta(\varepsilon) \max_{i \in \mathbb{M}} \{r_{ki} Y_k r_i^T\} \leq 1.
\]

Let \(\varepsilon^*_2\) be such that
\[
\Delta(\varepsilon) \max_{i \in \mathbb{M}} \{r_{ki} Y_k r_i^T\} \leq 1.
\]

is satisfied for all \(\varepsilon \in (0, \varepsilon^*_2]\). By (44) and (46), we know that, for arbitrary initial condition \(x(0) \in \Omega\), the following relation
\[
x(k + 1) = (A_k - B_k F_k) x(k), \quad \forall k \in (0, \varepsilon^*_2],
\]
with \(k = 0\). Next we will show that the actuator in system (43) will never saturate for all \(k \geq 0\). Indeed, by using the relation (39) and system (47), we have
\[
x^T(k + 1) P_{k+1} x(k) = x^T(k + 1) P_{k+1} \text{sat}(F_k x(k))
\]

That is to say
\[
x^T(k) P_k x(k) \leq (1 - \gamma_{k-1}) x^T(k - 1) P_k x(k - 1)
\]
\[
\leq \prod_{j=0}^{k-1} (1 - \gamma_j) x^T(0) P_0 x(0).
\]

Therefore, it follows from (50) and (46) that
\[
x(k) \in \varepsilon^*_1 \left( \frac{1}{\Delta(\varepsilon)} P_0 \right) \subseteq \mathcal{L}(F_0), \quad \forall k \geq 0, \varepsilon \in (0, \varepsilon^*_2].
\]

Hence the closed-loop system can always be written as (47) for all \(k \geq 0\). The asymptotic stability then follows from the fact that \([A_k - B_k F_{k+1}]_{i=0}^{\omega-1}\) is asymptotically stable.
Finally, we show (42). Using (49), we get
\[ \lambda_{\min}(P_k) \|x(k)\|^2 \leq x^T(k) P_k x(k) \]
\[ \leq \sum_{i=0}^{k-1} (1 - \gamma_i) x^T(i) P_0 x(i) \]
\[ \leq \sum_{i=0}^{k-1} (1 - \gamma_i) \lambda_{\max}(P_0) \|x(i)\|^2, \]
which is (42). The proof is completed. \(\square\)

The following remarks are in order.

**Remark 16.** We give some explanations on the proof of Theorem 15. According to Remark 12, the functions \(\gamma_k(\epsilon)\) satisfying all the conditions in (28)-(29) and (30)-(31) are not necessarily non-negative, namely, we may have \(\gamma_k(\epsilon) < 0, \quad \exists k^* \in \mathbb{R}, \forall \epsilon \in (0, 1].\) As a result, it follows from (48) that
\[ \Delta V(x(k^*), k^*) = x^T(k^*) (-\gamma_k P_k - F_k^T(\gamma) R_k F_k(\gamma)) x(k^*) > 0, \]
where \(x(k^*) \in \mathcal{N}(k^*) \Delta \{ x : F_k(\gamma)x = 0, x \neq 0 \} \) and \(V(x(k), k) = x^T(k) P_k(k) x(k).\) Therefore, the function \(V(x(k), k)\) is not non-increasing for all \(k\) and is not thus a Lyapunov function for the closed-loop system (47). For this reason, we introduce the scalar \(\Delta(\epsilon)\) to bound the possible increase of \(V(x(k), k)\) along the trajectories of the closed-loop system.

**Remark 17.** It follows from (42) that
\[ \|x(k)\| \leq \sqrt{\lambda_{\max}(W_k(\gamma)) \lambda_{\min}(W_0(\gamma))} \beta(\gamma) \|x(0)\|, \quad \forall k \in \mathbb{Z}, \tag{51} \]
where
\[ \beta(\gamma) = \sum_{i=0}^{k-1} \left( \frac{1}{\sqrt{1 - \gamma_i}} \right). \]

Inequality (51) indicates that \(\beta(\gamma)\) is the convergence rate of the state of the closed-loop system sampled at time \(ko, \forall k \in \mathbb{Z}.\) It is thus clear that if \(\Omega\) is large (thus \(\gamma^*\) is small), the convergence of the closed-loop system will be slow.

**Remark 18.** From the proof of Theorem 15 we can see that, for a given bounded set of initial states, the controller is designed such that the actuator will not saturate, namely, the closed-loop system operates in a linear region. This is the key of low gain feedback (Lin, 1998).

Notice that using Theorem 15 to solve Problem 3, we need only to solve a DPLE in the form of (16) which is linear in all the unknowns. If the parameter \(\gamma_k(\epsilon)\) is prescribed, such a DPLE can be solved numerically. In fact, there are many numerical algorithms for solving such kinds of linear matrix equations (see Varga (1997) and the references therein). If we want to keep \(\gamma_k(\epsilon)\) as variables and need analytical solutions, we can use the method given in Lemma 13. Since only a linear matrix equation is involved, the closed-form solutions can be obtained by symbolic computation software such as Maple.

Combining Proposition 4 and Theorem 15 gives the following result which shows a necessary and sufficient condition for the solvability of Problem 3.

**Proposition 19.** Consider the DLP system (3). Problem 3 is solvable if and only if \(\{A_k, B_k\}_{k=0}^{\infty}\) is stabilizable and all the characteristic multipliers of \(\{A_k\}_{k=0}^{\infty}\) are located inside or on the unit circle.

**5. An illustrative example**

Consider a DLP system in the form of (3) with \(\omega = 2\) and
\[
A_0 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix},
\]
\[
A_1 = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Direct manipulation shows that \(\{A_k, B_k\}_{k=0}^{\infty}\) is controllable and the characteristic multipliers of \(\{A_k\}_{k=0}^{\infty}\) are \(\{1, 1, 1\}.\) According to Lemma 13, by setting \(\gamma_0 = \epsilon, \gamma_1 = 4 - \epsilon \in (0, 1]\) and \(R_0 = R_1 = I_2,\) we solve the parametric DPLE (40) to obtain
\[
W_0(\epsilon) = \begin{bmatrix} -\mu_1(\epsilon) (\epsilon - 1)^3 & -\epsilon(\epsilon - 1)^2 \mu_2(\epsilon) & (1 - \epsilon)^3 \\ \epsilon^2(\epsilon - 2)^4 & -\mu_2(\epsilon) (\epsilon - 1)^3 & \epsilon^2(\epsilon - 2)^4 \\ -\epsilon^2(\epsilon - 2)^4 & (-\epsilon - 1)^2 & \epsilon^2(\epsilon - 2)^4 \end{bmatrix},
\]
\[
W_1(\epsilon) = \begin{bmatrix} W_1^{11}(\epsilon) & 0 & -\mu_2(\epsilon) (\epsilon - 1) \\ 0 & W_1^{22}(\epsilon) & -\epsilon(\epsilon - 1)^2 \\ -\epsilon(\epsilon - 1)^2 & -\epsilon(\epsilon - 1)^2 & W_1^{11}(\epsilon) \end{bmatrix},
\]
where \(\mu_1(\epsilon) = \epsilon^4 - 5\epsilon^3 + 7\epsilon^2 - 4\epsilon + 2, \mu_2(\epsilon) = \epsilon^4 - 6\epsilon^3 + 10\epsilon^2 - 7\epsilon + 3\) and \(W_1^{11}(\epsilon)\) and \(W_1^{22}(\epsilon)\) are given by
\[
\begin{bmatrix} W_1^{11}(\epsilon) & 0 & -\mu_2(\epsilon) (\epsilon - 1) \\ 0 & W_1^{22}(\epsilon) & -\epsilon(\epsilon - 1)^2 \\ -\epsilon(\epsilon - 1)^2 & -\epsilon(\epsilon - 1)^2 & W_1^{11}(\epsilon) \end{bmatrix}.
\]

Hence, according to Theorem 15, the following family of periodic linear state feedback laws solves Problem 3 for the given system:
\[
u(k) = -F_k(\epsilon) x(k), \quad \forall k \in \mathbb{Z}, \epsilon \in (0, 1),
\]
where the 2-periodic matrix \(\{F_k\}_{k=0}^{\infty}\) is given by (Box 1) in which
\[
F_0(\epsilon) = \begin{bmatrix} \epsilon^{13} - 15\epsilon^{12} + 98\epsilon^{11} - 367\epsilon^{10} + 874\epsilon^9 \\ -1398\epsilon^8 + 1573\epsilon^7 - 1325\epsilon^6 + 896\epsilon^5 \\ -498\epsilon^4 + 225\epsilon^3 - 83\epsilon^2 + 21\epsilon - 3 \end{bmatrix},
\]
\[
F_1(\epsilon) = \begin{bmatrix} \epsilon^{13} - 15\epsilon^{12} + 98\epsilon^{11} - 367\epsilon^{10} + 874\epsilon^9 \\ -1398\epsilon^8 + 1573\epsilon^7 - 1325\epsilon^6 + 896\epsilon^5 \\ -498\epsilon^4 + 225\epsilon^3 - 83\epsilon^2 + 21\epsilon - 3 \end{bmatrix},
\]
\[
F_2(\epsilon) = \begin{bmatrix} \epsilon^{13} - 15\epsilon^{12} + 98\epsilon^{11} - 367\epsilon^{10} + 874\epsilon^9 \\ -1398\epsilon^8 + 1573\epsilon^7 - 1325\epsilon^6 + 896\epsilon^5 \\ -498\epsilon^4 + 225\epsilon^3 - 83\epsilon^2 + 21\epsilon - 3 \end{bmatrix}.
\]
Theorem 10

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system consisting of Fig. 1.

References


Box I.


Fig. 1. The 2-norms of the states and control signals of the closed-loop periodic system consisting of (3) and (52) for different values of $\varepsilon$.

Direct manipulation shows that

$$\lambda \left( (A_0 - B_0F_0 (\varepsilon)) (A_1 - B_1F_1 (\varepsilon)) \right) = \left( (1 - \varepsilon)^2, (1 - \varepsilon)^2, (1 - \varepsilon)^2 \right)$$

$$= (1 - \gamma_0)(1 - \gamma_1)\lambda (A_0A_1),$$

which validates the results obtained in Theorem 10.

For a given initial condition $x(0) = [3 3 - 3] ^{T}$, and for three different values of $\varepsilon$, the 2-norms of the states and control signals of the closed-loop system consisting of (3) and (52) are recorded in Fig. 1, from which we clearly see that the peak value of the control signals decreases as $\varepsilon$ does. It is this property that implies semi-global stabilization.

6. Conclusions

This paper dealt with the problem of semi-global stabilization for DLP systems subject to actuator saturation. To solve this problem we first considered a parameterized DPLE whose properties were carefully studied in the paper. Based on these properties, the semi-global stabilization problem was solved by constructing a class of parameterized feedback gains. The advantages of a DPLE-based solution include the numerical reliability of the algorithm and the allowance of obtaining analytical solutions and exact locations of the characteristic multipliers of the closed-loop system. A numerical example was presented to illustrate the proposed approach.

References


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