Global Stabilization of the Double Integrator System With Saturation and Delay in the Input

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Abstract—This paper is concerned with the problem of global stabilizing a double integrator system subject to actuator saturation and delay in the input. Two solutions to the problem are proposed by using linear state feedback which are parameterized in a single scalar $\gamma$. The first solution is delay-dependent in the sense that the delay information is directly used in the feedback design. The second solution is delay-independent as the delay information is not directly used in the feedback. Ranges of the value of the free parameter $\gamma$ are established within which the closed-loop system is globally asymptotically stable. A numerical example is used to illustrate the effectiveness of the proposed approach.

Index Terms—Actuator saturation, global stabilization, linear feedback, nonlinear systems, time-delay.

I. INTRODUCTION

SATURATION nonlinearity is in engineering systems. Every practical physical actuator has a limit on the magnitude of output it can deliver. Ignoring the saturation nonlinearity in control design will degrade the system performances of the resulting closed-loop system when the saturation occurs and may even lead to instability (see, for example, [35]). Because of the significant influence of saturation nonlinearity on engineering systems, much research has been devoted to deal with saturation nonlinearity (see, for example, [4], [15], [16], [18], and [22]). The research topics include, among others, global stabilization [28], [36], [45], regulator design [2], semi-global stabilization [24], estimation of domain of attraction [17], quantized feedback [4], practical stabilization [9], input-output stabilization [7].

On the other hand, control of linear systems in the presence of time-delay, especially input delay, has also been attracting continuous attention for several decades. Many problems, such as stability analysis and stabilization [6], [26], [39], adaptive stabilization [27], $H_\infty$ filtering [5], [42], $H_\infty$ control [40], absolute stability [13], [43], and switching [21], have been studied in the literature. The delays in the control signals arise from a variety of sources such as the signal transmission delay when the system is controlled over a network, the physical transport delay that exists in a chemical plant, and the computational delay when the control algorithm is complicated and/or the measured signals are required to be filtered before being fed into the control algorithm. In fact, the analysis and design of control of systems that take into account delays in the control input is a classical problem. The well-known Smith-Predictor approach was developed for this very problem [33].

Recently, several authors have started to consider control problems for systems that are subject to both actuator saturation and time delay. For example, estimation of domain of attraction for linear time-delay systems with input saturation is considered in [3]. In [29], it is shown that a chain of integrators with arbitrary large input delay can be globally stabilized by saturated feedback and a nested nonlinear controller is proposed. The result is then extended in [41] to a class of linear systems that is null controllable by bounded controllers (ANCBC). For an oscillator system with input saturation and input delay, both [8] and [30] present linear controllers that globally stabilize the system for an arbitrary large delay. However, for general ANCBC linear systems with input delay and actuator saturation, especially for those systems with repeated poles on the imaginary axis, only semi-global stabilization can be achieved [23].

In this paper, we are interested in global stabilization of the double integrator system in the presence of both actuator saturation and input delay. The double integrator system commonly arises in applications. For example, double integrators represent single-degree-of-freedom translational or rotational motion [31], single-axis spacecraft rotation [19], rotary crane motion [11] and the dynamics of a spacecraft [20]. As a result, control of the double integrator system with actuator saturation has been of interest in the literature and many results have been reported. The minimum-time, minimum-fuel and quasi-time-fuel-optimal controllers are, respectively, considered in [1] and [20], the global stabilization problem is researched in [37]. The input-to-state stability and external stability problems are studied in [7], [32], [34], and [38], and the finite time stabilization problem is considered in [14]. An interesting comparison among various control design for the double integrator system can be found in [31]. There are also results on the control of the double integrator system with input delay in the literature (see, for example, [25]).

In this paper, we will propose two new solutions to the problem of global stabilization for the double integrator system with delayed and bounded control. The first solution
is delay-dependent in the sense that the delay information is directly used in the feedback design. The second solution is delay-independent as the delay information is not directly used in the feedback. Both of these two kinds of controllers are parameterized by a single scalar \( \gamma \), which in fact ensure a family of solutions. In both designs, we provide explicit ranges of the value of the parameter \( \gamma \) within which the resulting closed-loop system is globally asymptotically stable. Our results complement those in [29], where nonlinear feedback is used to achieve global stabilization of a chain of integrators of arbitrarily length in the presence of actuator saturation and input delay, by showing that linear feedback laws exist when the length of the chain is two. Our results also complement the results in [23], where it is shown that, for a linear system with all its poles located in the closed left-half plane, linear feedback can achieve semi-global stabilization in the presence of input saturation and input time-delay.

This paper is organized as follows. The main results are presented in Section II. For clarity, the proofs of these main results are given in Section III. Section IV presents an example to show the effectiveness of the proposed results. Section V concludes this paper. Finally, some technical results that are used in this paper and the proofs for some results used in Section III are collected in the appendix.

Notation: Throughout this paper, we use \( A^T, \det(A) \) and \( \sigma(A) \) to denote the transpose, the determinant and the set of eigenvalues of matrix \( A \), respectively. The function \( \text{sign} \) is defined as \( \text{sign}(y) = 1 \) if \( y \geq 0 \) and \( \text{sign}(y) = -1 \) if \( y < 0 \). The saturation function \( \text{sat}: \mathbb{R} \rightarrow \mathbb{R} \) is then defined as \( \text{sat}(u) = \text{sign}(u) \min\{1, |u|\} \). For two vectors \( u \) and \( v \), \( \min \{ u, v \} \) denotes their convex hull.

II. MAIN RESULTS

Consider the problem of global stabilization of the following double integrator system subject to both input delay and input saturation:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t - \tau) \\
x(\theta) &= \varphi(\theta), \theta \in [-\tau, 0] \\
|u(t)| &\leq 1, \ t \in \mathbb{R}
\end{align*}
\]

where \( x(t) \in \mathbb{R}^2 \) and \( u(t) \in \mathbb{R} \) are, respectively, the state and input, \( A \) and \( B \) are, respectively, given by

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

the scalar \( \tau < \infty \) is a given positive number and \( \varphi(\theta) \) is a continuous function. We have assumed, without loss of generality, the unity saturation level in the input. Non-unity saturation level can be absorbed by the matrix \( B \) and the feedback gain. Our solutions to this stabilization problem are based on the following parametric Riccati equation:

\[
A^T P + PA - PB B^T P = -\gamma P
\]

which has a unique positive definite solution (for proof and more properties of (3), see [44])

\[
P(\gamma) = \begin{bmatrix} \gamma^3 & \gamma^2 \\ \gamma^2 & \gamma \end{bmatrix}.
\]

We first present a delay-dependent result for global stabilization of system (1). In this solution, the knowledge of the delay \( \tau \) in the input is explicitly used in the feedback gains.

**Theorem 1:** Any member of the family of linear feedback laws

\[
u(t) = -B^T P(\gamma) e^{A t} x(t) \quad \forall \gamma \in \left(0, \frac{4}{25\tau}\right)
\]

where \( P(\gamma) \) is given by (4), globally stabilizes system (1).

**Remark 1:** Clearly, in the absence of input saturation, following a similar procedure used in the proof of Theorem 1, the conclusion in Theorem 1 is still valid. Moreover, as the feedback laws (5) stabilize the double integrator in the absence of actuator saturation and input delay, they globally stabilize system (1) for an arbitrary \( \tau > 0 \) when the double integrator is subject to only actuator saturation (i.e., when \( \tau = 0 \)). See [31] and [37].

**Remark 2:** The control law in (5) requires the exact knowledge of the delay \( \tau \) in the input. If we only have an estimate \( \tau_e \) for the actual delay \( \tau \) in the input, namely, the controller is

\[
u(t) = -B^T P(\gamma) e^{A t} x(t) \quad \forall \gamma \in \left(0, \frac{4}{25\tau_e}\right)
\]
the stability of the closed-loop system (1)–(6) is questionable if \( \tau_0 \neq \tau \). Since it is very difficult to analyze the behavior of the closed-loop system (1) and (6) when \( \tau_0 \neq \tau \) in theory, we will use simulation example to study the behavior of the closed-loop system (see Section IV). We fix \( \tau = 1 \) s in system (1) and use different estimation \( \tau_0 \) in the controller law (6) where \( \gamma \) is chosen as \( \frac{\tau}{4/25\tau_0} \) for simplicity. For a fixed initial condition, the simulation results are plotted in Fig. 2. It follows from Fig. 2 that the closed-loop system is unstable if \( [\tau - \tau_0]/\tau \) (the relative error) is large enough. This fact indicates that an under estimate \( \tau_0 \) of the actual delay \( \tau \) in the actuator can lead to instability, which is reasonable. On the other hand, the stability of the closed-loop system can be guaranteed if \( \tau_0 > \tau \). Even so, the performance of the closed-loop system degenerates as \( \tau_0 \) increases since the feedback gain decreases as \( \tau_0 \) increases. However, whether the stability of the closed-loop system is guaranteed for all \( \tau_0 > \tau \) is impossible to validate by simulation. A further study on this problem is required.

As explained in the above remark, the feedback law given in Theorem 1 requires the exact knowledge of the delay \( \tau \) in the input, which can lead to instability of the closed-loop system if the delay used in the controller is different from the actual delay in the input. To overcome this shortcoming, we provide an alternative solution which does not require the exact information of the delay \( \tau \) in the feedback gains.

**Theorem 2:** Any member of the family of linear feedback laws

\[
    u(t) = -B^T P(\gamma)x(t) \quad \forall \gamma \in \left(0, \frac{1}{\tau_0}\right],
\]

where \( P(\gamma) \) is given by (4), globally stabilizes system (1).

Similar to Remark 1, we can show that, in the absence of input saturation and/or input delay, the family of linear feedback laws (7) still globally stabilize system (1).

**Remark 3:** The family of feedback laws (7) clearly does not require the exact value of \( \tau \). Owning to this advantage, Theorem 2 is applicable to a class of systems in the form of (1) where the input delay is not exactly known but satisfies the bounded condition \( 0 < \tau \leq \tau_{\text{max}} \). In this case, the family of linear feedback laws (7) should be updated as

\[
    u(t) = -B^T P(\gamma)x(t) \quad \forall \gamma \in \left(0, \frac{1}{\tau_{\text{max}}}\right].
\]

**Remark 4:** The advantage of Theorem 1 over Theorem 2 is that the parameter \( \gamma \) in Theorem 1 can be chosen within a larger interval than that in Theorem 2. This is the benefit of the utilization of the information of \( \tau \) in the feedback gains. On the other hand, we see from (4) that the larger the \( \gamma \) the larger the norm of \( P(\gamma) \) will be. Therefore, Theorem 1 allows for higher feedback gains, and thus better transient performances, than Theorem 2. This statement can be validated by our simulation example given in Section IV.

**Proofs of Theorems 1 and 2**

**Proof of Theorem 1:** Rewrite the closed-loop system (1) and (5) as follows:

\[
    \dot{x}(t) = A x(t) - B_{\text{sat}} \left(B^T P(\gamma) e^{At}x(t - \tau)\right) + B^T z(t)
\]

where \( \gamma \in (0, 4/25\tau] \). Without loss of generality, we consider the stability of (8) with \( t \geq \tau \). Then

\[
    x(t) = e^{A(t-\tau)}x(\tau) - \int_{\tau}^{t} p(s)ds, \quad t \geq \tau
\]

where

\[
    p(s) = e^{-As}B_{\text{sat}} \left(B^T P(\gamma) e^{A(s+t-\tau)}z(s+t-\tau)\right).
\]

Then it follows from (9) that

\[
    B^T P(\gamma) e^{A(t-\tau)}x(t-\tau) = B^T P(\gamma) x(t) + B^T z(t)
\]

where

\[
    z(t) = P(\gamma) \int_{\tau}^{t} p(s)ds.
\]

Let

\[
    \eta(\gamma, \tau) \triangleq 2\gamma \tau + \frac{\gamma^2}{2} \tau^2.
\]

Note that \( \eta(\gamma, \tau) = 1 \) has a unique positive solution \( \gamma_{0} = [\sqrt{6} - 2]/\tau = 0.4495/\tau \). Therefore we have

\[
    0 < \eta(\gamma, \tau) < 1, \quad \forall \gamma \in \left(0, \frac{4}{25\tau}\right].
\]
Moreover, one can get
\[
\sup_{t \in \mathbb{R}} \left\{ \int_t^0 B^T P(\gamma) p(s) \, ds \right\} = \sup_{s \in \mathbb{R}} \left\{ \int_s^0 B^T P(\gamma) p(s) \, ds \right\}
\]
\[
\leq \int_{-\tau}^0 B^T P(\gamma) e^{-\lambda s} B \, ds
\]
\[
= \int_{-\tau}^0 [2\gamma - \gamma^2 s + \gamma_2^2] \, ds = \int_{-\tau}^0 (2\gamma - \gamma^2 s) \, ds
\]
\[
= 2\gamma \tau + \frac{\gamma^2}{2} \tau^2 = \eta(\gamma, \tau), \quad \forall \gamma \in \left( 0, \frac{4}{25\tau} \right].
\]
(15)

Consider the following state space partition:
\[
\begin{align*}
\text{I} & : B^T P(\gamma) x > 1 + \eta(\gamma, \tau) \\
\text{II} & : B^T P(\gamma) x \leq 1 + \eta(\gamma, \tau) \\
\text{III} & : B^T P(\gamma) x < - (1 + \eta(\gamma, \tau))
\end{align*}
\]

which means that the $x_1 - x_2$ plane is divided into three regions by two lines $B^T P(\gamma) x(t) = \pm (1 + \eta(\gamma, \tau))$. We can state the following claim whose proof is given in Appendix I-B.

**Claim 1:** Consider the nonlinear time-delay system (8).
1. Any bounded initial condition in Region I or III yields a trajectory that enters the boundary of Region II in a finite time.
2. Any state on the boundary of Region II that will enter Region I (or III) will return to the boundary of Region II in a finite time and with a lower energy level with respect to the energy function $V(x(t)) = x^T(t) P(\gamma) x(t)$.

Claim 1 says that the state of system (8) will enter and remain in Region II in a finite time. Therefore, to show the stability of the system, we need only to show that any trajectory that remains in II will enter a contractively invariant level set of $V(x)$ that is within Region II. Let us rewrite Region II as follows:

\[ \Omega_{\gamma, \tau} = \left\{ x : B^T P(\gamma) x > 1 + \eta(\gamma, \tau) \right\}. \]

Then according to Lemma 4 in Appendix I-A, we get

\[ \text{sat} \left( B^T P(\gamma) e^{\lambda s} x(t - \tau) \right) \in \text{co} \left\{ B^T P(\gamma) x(t), B^T P(\gamma) e^{\lambda s} x(t - \tau) \right\}. \]

Therefore, system (8) can be written as

\[ \dot{x}(t) = \alpha(x(t)) f_1 + (1 - \alpha(x(t))) f_2 \]

(16)

where $1 \geq \alpha(x(t)) \geq 0$, and $f_1 = f_1(x(t))$ and $f_2 = f_2(x(t), x(t - \tau))$ are, respectively, given by

\[ f_1 = A x(x(t)) - \frac{BB^T P(\gamma x(t))}{1 + \eta(\gamma, \tau)}, \]
\[ f_2 = A x(t) - BB^T P(\gamma) e^{\lambda s} x(t - \tau) = A(\gamma) x(t) - BB^T z(t) \]

in which $z(t)$ is related with (12) and $A(\gamma) = A - BB^T P(\gamma)$.

We can now state the following lemma, whose proof is provided in Appendix I-B.

**Lemma 1:** We have the following two statements.
1. Consider the linear system
\[ \dot{x}(t) = f_1 = A x(t) - \frac{BB^T P(\gamma) x(t)}{1 + \eta(\gamma, \tau)}, \]
(17)

Assume that $\gamma \in (0, 4/25\tau)$. Then the time derivative of $V(x(t))$ along the trajectories of system (17) satisfies
\[ \dot{V}(x(t)) \leq -\gamma x^T(t) P(\gamma) x(t). \]
(18)

2. Consider the following time-delay system
\[ \dot{x}(t) = f_2 = A(\gamma) x(t) - BB^T z(t) \]
(19)

where $z(t)$ is defined in (12). Then for arbitrary number $\theta > 1$, there holds
\[ V(x(t + \theta)) < \text{sat} V(x(t)) \quad \forall \theta \in [-2\tau, 0] \]
\[ \Rightarrow \dot{V}(x(t)) \leq -\gamma(\tau, \theta) V(x(t)) \]
(20)

where
\[ \gamma(\tau, \theta) = 1 - \frac{2s}{3} \left( 2 + s^2 + s\sqrt{4s^2 + 4s^2} \right) \times \left( 8 + 6s + s^2 - \sqrt{(4s^2 + 4s^2)^3} \right). \]
(22)

Let $E(P(\gamma), r_{\text{max}})$ be the maximal ellipsoid contained in Region II. Then we have
\[ \frac{B^T P(\gamma)}{1 + \eta(\gamma, \tau)} \left( \frac{P(\gamma)}{r_{\text{max}}} \right)^{-1} \left( \frac{P(\gamma) B}{1 + \eta(\gamma, \tau)} \right) = 1 \]

from which we obtain
\[ r_{\text{max}} = \frac{(1 + \eta(\gamma, \tau))^2}{2\gamma} \]

(23)

**Claim 2:** Consider the system (16) with initial condition $x(t_0) = x_0 \in \Omega_{\gamma, \tau}$. If $\gamma \in (0, 4/25\tau)$, then there exists a finite time $t_0 \geq t_0$ such that $x(t) \in M_V(r_{\text{max}}; 2\tau)$, where $M_V(r_{\text{max}}; 2\tau)$ is defined in (44).

Proof for the above claim is given in Appendix I-B. This claim shows that the state of the nonlinear time-delay system (16) within Region II will enter the set $M_V(r_{\text{max}}; 2\tau)$ in a finite time. We will show that $M_V(r_{\text{max}}; 2\tau)$ is a contractively invariant set and thus the system (16) is asymptotically stable.

Invoking of (22), simple calculation shows that
\[ \theta = 1.1, \quad \forall \gamma \in \left( 0, \frac{4}{25\tau} \right) \Rightarrow \gamma(\tau, \theta) > 1/16. \]

Then for $x(t) \in M_V(r_{\text{max}}; 2\tau)$, we conclude from Lemma 1 and $\gamma \in (0, 4/25\tau)$ that the time derivative of $V(x(t))$ along the trajectories of system (19) satisfies, for all $t \in [-2\tau, 0]$

\[ V(x(t + \theta)) < \frac{11}{10} V(x(t)) \Rightarrow \dot{V}(x(t)) \leq -\gamma \frac{V(x(t))}{16}. \]

(23)
Combining (18) and (23), we conclude that the time derivative of $V(x(t))$ along the trajectories of the system (16) within Region $\Pi$ satisfies (23). The proof is completed by invoking the Razumikhin Stability Theorem (Theorem 3 in Appendix I-A).

**Proof of Theorem 2:** The closed-loop system composing of (1) and (7) is given by

$$\dot{x}(t) = Ax(t) - B \text{sat}(B^TP(\gamma)x(t - \tau))$$

(24)

where $\gamma \in (0, 1/7\tau]$. Also, we prove the stability of system (24) for $t \geq \tau$. It follows from (24) that

$$x(t) = e^{At}x(t - \tau) - \int_{t-\tau}^{t} e^{A(t-s)}Bk(s)ds, \ t \geq \tau$$

(25)

where

$$k(s) = \text{sat}(B^TP(\gamma)e^{At}x(s + t - \tau)).$$

(26)

Then it follows from (25) that

$$B^TP(\gamma)x(t - \tau) = B^TP(\gamma)(e^{-At}x(t) + q(t))$$

(27)

where

$$q(t) = \int_{-\tau}^{0} e^{-A(s+\tau)}Bk(s)ds.$$  

(28)

Let

$$\rho(\gamma, \tau) = 2\gamma - 1/2(2\gamma)^2.$$  

(29)

Clearly, we have $\rho(\gamma, \tau) > 0$ for any $\gamma \in (0, 1/7\tau]$. Note that

$$\sup_{x \in \mathbb{R}^n} \{B^TP(\gamma)q(t)\} = \sup_{x \in \mathbb{R}^n} \left\{\int_{-\tau}^{0} B^TP(\gamma)e^{-A(s+\tau)}Bk(s)ds\right\} \leq \sup_{x \in \mathbb{R}^n} \left\{\int_{-\tau}^{0} B^TP(\gamma)e^{-A(s+\tau)}Bk(s)ds\right\} \leq \int_{-\tau}^{0} \|B^TP(\gamma)e^{-A(s+\tau)}B\|ds \leq 2\gamma - 1/2(2\gamma)^2 = \rho(\gamma, \tau), \ \forall \gamma \in (0, 1/7\tau].$$  

(30)

Similar to the proof of Theorem 1, we consider the following state space partition:

$$\begin{cases}
\text{I}: B^TP(\gamma)e^{-At}x > 1 + \rho(\gamma, \tau) \\
\text{II}: B^TP(\gamma)e^{-At}x \leq 1 + \rho(\gamma, \tau) \\
\text{III}: B^TP(\gamma)e^{-At}x < -(1 + \rho(\gamma, \tau))
\end{cases}$$

and propose the following claim whose proof is provided in Appendix I-C.

**Claim 3:** Consider the nonlinear time-delay system (24).

1) Any bounded initial condition in Region I or III yields a trajectory that reaches the boundary of Region II in a finite time.

2) Any state on the boundary of Region II that will enter Region I (or III) will return to the boundary of Region II in a finite time and with a lower energy level with respect to the energy function $V(x(t)) = x^T(t)P(\gamma)x(t)$.

Again, it follows from the above claim that the state of the closed loop system (24) will enter and remain in Region II in a finite time. With this, what remains in the proof is to show that any trajectory that remains in Region II will enter a contractively invariant level set of $V(x)$ that is within Region I. Let

$$\overline{\Omega}_{\tau, \gamma} = \left\{ x : \left| B^TP(\gamma)e^{-At}x(t) \right| < 1 \right\}.$$  

(31)

Once again, it follows from Lemma 4 in Appendix I-A that

$$\text{sat}(B^TP(\gamma)x(t - \tau)) \in \text{co}\left\{ \frac{B^TP(\gamma)e^{-At}x(t)}{1 + \rho(\gamma, \tau)}, B^TP(\gamma)x(t - \tau) \right\}$$

and consequently, system (24) can be written as

$$\dot{x}(t) = \lambda(x(t))g_1 + (1 - \lambda(x(t)))g_2$$

(32)

where $1 \geq \lambda(x(t)) > 0$, and $g_1 = g_1(x(t))$ and $g_2 = g_2(x(t), x(t - \tau))$ are, respectively, given by

$$g_1 = Ax(t) - B^TP(\gamma)e^{-At}x(t)$$

(33)

$$g_2 = B^TP(\gamma)e^{-At}x(t) = A_{\alpha}(x(t))x(t) - BB^TP(\gamma)q(t)$$

where $q(t)$ is given in (28) and

$$A_\alpha(\gamma) = A - BB^TP(\gamma)e^{-At}.$$  

(34)

We then have following lemma, whose proof is given in Appendix I-C.

**Lemma 2:** We have the following two statements.

1) Consider the linear system

$$\dot{x}(t) = g_1$$

(35)

Assume that $\gamma \in (0, 1/7\tau]$. Then there exists a scalar $\epsilon(\gamma, \tau) > 0$ such that the time derivative of $V(x(t))$ along the trajectories of system (35) satisfies

$$\dot{V}(x(t)) \leq -\epsilon(\gamma, \tau)\|x(t)\|^2.$$  

(36)

2) Consider the following linear time-delay system:

$$\dot{x}(t) = g_2 = A_{\alpha}(\gamma)x(t) - BB^TP(\gamma)q(t)$$

(37)

where $q(t)$ is related with (28). Then for an arbitrary number $\phi > 1$, there holds

$$V(x(t + \theta)) < \phi V(x(t)) \quad \forall \theta \in [-2\tau, 0]$$

$$\Rightarrow \dot{V}(x(t)) \leq -x^T(t)W(\gamma, \tau, \phi)x(t)$$

(38)

where

$$W(\gamma, \tau, \phi) = \begin{bmatrix}
\gamma^\phi & \gamma^\phi \\
\gamma^\phi & \gamma^\phi \\
\gamma^\phi & \gamma^\phi
\end{bmatrix} (\mu - 2\gamma^\phi)$$

(39)

with $\mu = 1 - \phi \delta(\gamma, \tau)$ and

$$\delta(\phi) = \frac{2s^2}{3(2s^2 + s\sqrt{4 + s^2})(s^2 - 3s + 6)}.$$  

(40)
Let $\mathcal{E}(P(\gamma), \tau_{\text{max}})$ be the maximal ellipsoid contained in Region $\Pi$. Then we have

$$
\tau_{\text{max}} = \frac{(1 + \rho(\gamma, \tau))^2}{\gamma (\gamma^2 \tau^2 - 2\gamma \tau + 2)}.
$$

The following claim is similar to Claim 2. A detailed proof is given in Appendix I-C.

**Claim 4:** Consider the system (31) with initial condition $x(t_0) = x_0 \in \Omega_\infty$. If $\gamma \in (0, 1/7\tau]$, then there exists a finite time $t_0 \geq t_0$ such that $x(t_0) \in M_V(\tau_{\text{max}}, 2\tau)$ which is defined in (44).

It is easy to show that

$$
\phi = 1.5, \quad \gamma \in \left(0, \frac{1}{17}\right] \Rightarrow W(\gamma, \tau, 1.5) > 0.
$$

Then for $x(t) \in M_V(\tau_{\text{max}}, 2\tau)$, we conclude from Item 2 of Lemma 2 and (40) that the time derivative of $V(x(t))$ along the trajectories of system (35) satisfies

$$
V(x(t + \theta)) < 1.5V(x(t)) \quad \forall \theta \in [0, 2\tau, 0] \Rightarrow \dot{V}(x(t)) \leq -\gamma \lambda_{\min} \frac{[W(\gamma, \tau, 1.5)]}{2} ||x(t)||^2.
$$

Combining (41) and (34), we conclude from Item 1 of Lemma 2 that the time derivative of $V(x(t))$ along the trajectories of the system (31) within Region $\Pi$ satisfies

$$
V(x(t + \theta)) < 1.5V(x(t)) \quad \forall \theta \in [0, 2\tau, 0] \Rightarrow \dot{V}(x(t)) \leq -\kappa(\gamma, \tau, \phi) ||x(t)||^2
$$

with

$$
\kappa(\gamma, \tau, \phi) = \min \left\{ \frac{\lambda_{\min} [W(\gamma, \tau, \phi)]}{2}, \epsilon(\gamma, \tau) \right\} > 0.
$$

The proof is completed by invoking the Razumikhin Stability Theorem (Theorem 3 in Appendix I-A).

**III. EXAMPLE**

In this section, we give a simulation result to illustrate the theory developed in this paper. Consider a double integrator system with input saturation and time delay $\tau = 2s$. According to Theorems 1 and 2, we choose $\gamma = 4/25\tau$ and $\gamma = 1/7\tau$ in the feedback laws (5) and (7), respectively. Note that $\tau = 2s$ is a “large” delay such that the result in [29] cannot be applied directly before using a time rescaling procedure. Following the development in [29], the resulting controller is given by

$$
u = -\text{sat} \left( \frac{\text{sat} \left( \frac{x_2\tau^2}{\tau} + 8 \text{sat} \left( \frac{8\gamma^2}{\tau^2} \left( x_1 + \frac{\tau x_2}{\tau_0} \right) \right) \right)}{\tau} \right)
$$

(42)

where $\tau \geq \tau_0$ with the constant $\tau_0$ determined by $\tau_0 = \min \left\{ \frac{1}{6}, 1/9, 7/(8 \times 2 + 4.8^2 - 3.2) \right\} = 1/9$. For a given initial condition $\varphi(\theta) = [-10, 10]^T$, $\theta \in [-2, 0]$, the state responses of the closed-loop system with different controllers are plotted in Fig. 1. It is clear that the controller in Theorem 1 leads the best transient performance, which is not surprising because it not only uses the delay information but also requires fewer saturation elements in the feedback law. Moreover, the controller given in Theorem 2 also performs better than (42) because it also requires fewer saturation elements in the feedback law than that in (42).

We next examine the behavior of the closed-loop system with the controller in Theorem 1 when the delay used in the controller is an estimate of the actual delay $\tau$ in the actuator, namely, the controller (6) where $\tau_0$ is the estimate of $\tau$ is used. The initial conditions are chosen as before and $\tau = 1s$ is fixed. For different estimates $\tau_0$ and with $\gamma = 4/25\tau$, the behaviors of the closed-loop system are shown in Fig. 2, from which we clearly validate the statements in Remark 2: the closed-loop system is unstable if $|\tau - \tau_0|/\tau$ (the relative error) is large enough, and the performance of the closed-loop system degenerates as $\tau_0$ increases if $\tau_0 > \tau$. However, as commented in Remark 2, whether the stability of the closed-loop system is guaranteed for all $\tau_0 > \tau$ is impossible to validate by simulation and a further study on this problem is required in the future. No matter how, these results show that Theorem 2 is more convenient to use than Theorem 1 if the delay in the input is not precisely known.

To the best of our knowledge, no stability-guaranteed controller except for the one in [29] has been reported in the literature for global stabilization of the double integrator system subject to both input saturation and time-delay. On the contrary, in the absence of actuator delay, many approaches have been developed to deal with such a system. These approaches include minimum time controller, minimum energy controller, discontinuous and continuous sliding mode controller, direct adaptive controller, LQR controller, trap door controller, saturated linear controller, and homogeneous controller [31]. Reference [31] provides a complete comparison of these controllers. The results in [31] indicate that the LQR (LQG instead if the state is not available) controller leads to the best system performances in the absence of delay. For this reason, we compare our methods with the LQR controller. This class of controller is in the form of $u(t) = -R^{-1}B^TSx(t)$, where $S > 0$ is the solution to the algebraic Riccati equation

$$
A^TS + SA - SBR^{-1}B^TS = -Q
$$

with $R > 0$ and $Q > 0$. Here we choose $R = 1$ and $Q = I_2$. Consequently, the controller is given by

$$
u(t - \tau) = -\text{sat} \left( [1, \sqrt{3}, x(t - \tau)] \right).
$$

Again, the initial conditions are chosen as before. For a couple of delays $\tau$, the evolutions of the states of the closed-loop system under different controllers are shown in Fig. 3. Here, we choose $\gamma = 4/25\tau$ and $\gamma = 1/7\tau$ in the feedback laws (5) and (7), respectively. From Fig. 3 we clearly see that the controllers in Theorems 1 and 2 lead to better transient performances than the nonlinear controller in (42). Although the closed-loop system with the LQR controller has a very good transient performance if $\tau$ is small, it becomes unstable for $\tau > 0.65s$. These simulation results validate the effectiveness of the proposed results.
IV. CONCLUSION

This paper revisited the problem of global stabilization of linear systems with actuator saturation and delayed input. For a double integrator system, we proposed two families of linear feedback solutions to the problem, both of which are parameterized in a single parameter $\gamma$. The first solution is delay-dependent in the sense that the delay information is explicitly used in the controller design, while the second solution is delay-independent as the delay information is not directly used in the feedback. Explicit ranges of the value of $\gamma$ are provided to guarantee the global stability of the closed-loop system. Generally speaking, for an arbitrary large but bounded delay in the input, the resulting closed-loop system is globally stable provided the parameter $\gamma$ is sufficiently small, which means that our solutions fall into the methodology of low gain feedback.

APPENDIX

Some Technical Lemmas: In this subsection, we provide some technical results that are used in the paper. The first result is the following well-known Razumikhin Stability Theorem.

Theorem 3 ([3] and [12]): Consider the functional differential equation

$$\begin{cases} \dot{x}(t) = f(t, x_t), \quad x(t) \in \mathbb{R}^n, \quad t \geq t_0 \\ x_{t_0}(\theta) = \varphi(\theta), \quad \forall \theta \in [-\tau, 0] \end{cases}$$

where $x_t = x(t + \theta)$, $\forall \theta \in [-\tau, 0]$. Let $C_{n, \tau} = C([-\tau, 0], \mathbb{R}^n)$ denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The function $f : \mathbb{R} \times C_{n, \tau} \rightarrow \mathbb{R}^n$ is such that the image of $f$ is a bounded subset of $C_{n, \tau}$ and the functions $u, v, w, p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous, nondecreasing and positive for all $s > 0, u(0) = v(0) = 0$, $v$ is strictly increasing and $p(s) > s$. If there is a continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and a positive number $\rho$, such that for all $x_t \in M_V(\rho)$ where

$$M_V(\rho, \tau) \triangleq \{\psi \in C_{n, \tau} | V(\psi(\theta)) \leq \rho, \quad \forall \theta \in [-\tau, 0]\}$$

1) $u(||x||) \leq V(t, x), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n$,
2) $\dot{V}(t, x) \leq -w(||x||)$ if $V(t + \theta, x(t + \theta)) < p(V(t, x(t)))$, $\forall \theta \in [-\tau, 0]$.

Then, the trivial solution $x(t) \equiv 0$ of the differential (43) is asymptotically stable. Moreover, the set $M_V(\rho, \tau)$ is an invariant set inside the domain of attraction. If Items 1 and 2 hold for all $x \in \mathbb{R}^n$ and $\lim_{\theta \rightarrow -\infty} u(s) = \infty$, then the solution $x(t) \equiv 0$ is globally asymptotically stable.

The second result is the following well-known Jensen integral inequality, whose proof can be found in [10].

Lemma 3: For any positive definite matrix $M > 0$, scalars $\gamma_2$ and $\gamma_1$ with $\gamma_2 \geq \gamma_1$, vector function $\omega : [\gamma_1, \gamma_2] \rightarrow \mathbb{R}^n$ such that the integrations in $x = \int_{\gamma_1}^{\gamma_2} \omega(\beta) d\beta$ and the following are well-defined, then:

$$x^T M x \leq (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^T(\beta) M \omega(\beta) d\beta.$$
The third result is adopted from [17], where this result was established for vector valued saturation functions. See also [8].

**Lemma 4:** Let \( u, v \in \mathbb{R} \) with \( |v| \leq 1 \). Then \( \text{sat}(u) \in \mathcal{C}^0(u, v) \).

The following simple result is also recalled.

**Lemma 5:** Let \( x, y \in \mathbb{R}^n \) be two arbitrary vectors and \( P \in \mathbb{R}^{n \times n} \) be a positive definite matrix. Then

\[
x^T y + y^T x \leq x^T P x + y^T P^{-1} y.
\]

The following results are concerned with the solutions to the parametric algebraic Riccati equation (3).

**Fact 1:** Let \((A, B)\) be given by (2) and \(P(\gamma)\) be given by (4). Then

\[
P(\gamma) BB^T P(\gamma) - 2\gamma P(\gamma) = \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 0 \end{bmatrix} \leq 0,
\]

**Fact 2:** Let \((A, B)\) be given by (2) and \(P(\gamma)\) be the unique positive definite solution to (3). Then for \(\forall s \geq 0\)

\[
e^{-AT} s P(\gamma) e^{As} - \beta(s) P(\gamma) = -R_1(s, \gamma) \leq 0
\]

\[
e^{-AT} s P(\gamma) e^{-As} - \beta(s) P(\gamma) = -R_2(s, \gamma) \leq 0
\]

where

\[
\beta(t) = 1 + \frac{1}{2} t^2 + \frac{1}{2} \sqrt{4t^2 + t^4}.
\]

**Proof:** Direct calculation gives that

\[
R_1(s, \gamma) = \begin{bmatrix} \frac{1}{4} \gamma^4 s (r_1 + \gamma s) & \frac{3}{2} \gamma^2 (r_1 + \gamma s - 2) \\ \frac{3}{2} \gamma^2 (r_1 + \gamma s - 2) & \gamma^2 s (r_1 - 2) \end{bmatrix}
\]

\[
R_2(s, \gamma) = \begin{bmatrix} \frac{1}{2} \gamma^4 s (\gamma s + r_1) & \frac{3}{2} \gamma^2 (2 + \gamma s + r_1) \\ \frac{3}{2} \gamma^2 (2 + \gamma s + r_1) & \gamma^2 s (2 + r_1) \end{bmatrix}
\]

where \( r_1 = \sqrt{4 + \gamma^2 s^2} \) and it follows that

\[
\sigma(R_1(s, \gamma)) = \left\{ 0, \frac{1}{2} \gamma^2 s (\gamma^2 s + 2r_1 + r_1 \gamma^2 - 4) \right\}
\]

\[
\sigma(R_2(s, \gamma)) = \left\{ 0, \frac{1}{2} \gamma^2 s (\gamma^3 s + 2r_1 + r_1 \gamma^2 + 4) \right\}
\]

This proves the result.

Finally, we provide the following result.

**Lemma 6:** Consider the nonlinear system \( \dot{x}(t) = Ax(t) + B \text{sat}(u(t)) \), where \(A\) and \(B\) are given in (2). Let \( V(x(t)) = x^T(t) P(\gamma) x(t) \) with \( P(\gamma) \) being given by (4). Then for any \( \omega \geq 0 \) and \( \rho > 0 \),

\[
V(x(t)) > \rho \Rightarrow V(x(t - \omega)) < \hat{\rho} V(x(t))
\]

\[
V(x(t)) \leq \rho \Rightarrow V(x(t - \omega)) \leq \hat{\rho} \rho
\]

where \( \hat{\rho} = \hat{\rho}(\omega, \gamma, r, \rho) \) and

\[
\hat{\rho} = (1 + r) \beta(\omega \gamma) + \left( 1 + \frac{1}{r} \right) \frac{\gamma \omega^2 (\gamma^2 \omega^2 - 3 \gamma \omega + 6)}{3 \rho}.
\]

Moreover, let \( \theta(\gamma, 2r, \rho) = \min_{r > 0} \max_{\omega \in [0, 2\pi]} \left\{ \theta(\omega, \gamma, r, \rho) \right\} \). Then

\[
\sqrt{\theta(\gamma, 2r, \rho)} = \sqrt{\beta(2\gamma \rho)} + \sqrt{\frac{8 \gamma^2 r^2 (2 \gamma^2 r^2 - 3 \gamma r + 3)}{3 \rho}}.
\]

**Proof:** Note that for an arbitrary \( \omega \geq 0 \), we have

\[
x(t) = e^{L(\omega)(t - \omega)} + \int_{-\omega}^{0} e^{-L(\omega)s} B \text{sat}(u(t + s)) \, ds
\]

Consequently, by using Lemmas 3 and 5, we obtain, for arbitrary \( r > 0 \)

\[
x^T(t - \omega) P(\gamma) x(t - \omega)
\]

\[
\leq (1 + r) x(t) e^{-AT} P(\gamma) e^{A(\omega - 1)} (t + \left( 1 + \frac{1}{r} \right) \omega)
\]

\[
\times \int_{-\omega}^{0} e^{-A(s + \omega)} B \text{sat}(u(t + s)) \, ds
\]

\[
\leq (1 + r) x(t) e^{-AT} P(\gamma) e^{A(\omega - 1)} (t + \left( 1 + \frac{1}{r} \right) \omega)
\]

\[
\times \int_{-\omega}^{0} \text{sat}(u(t + s)) B e^{-A(s + \omega)} P(\gamma) e^{-A(\omega + 1)B}
\]

\[
\times \text{sat}(u(t + s)) \, ds
\]

\[
\leq (1 + r) x(t) e^{-AT} P(\gamma) e^{A(\omega - 1)} (t + \left( 1 + \frac{1}{r} \right) \omega)
\]

\[
\times \int_{-\omega}^{0} B e^{-A(s + \omega)} P(\gamma) e^{A(\omega + 1)B} \, ds.
\]

On the other hand, direct calculation gives

\[
\int_{-\omega}^{0} v_i^T P(\gamma) v_i \, ds = \frac{\gamma \omega^2 (\gamma^2 \omega^2 - 3 \gamma \omega + 6)}{3}
\]

where \( \beta(\omega \gamma) = e^{-A(\omega + 1)B} \). If \( V(x) > \rho \), then in view of Fact 2, inequality (49) can be continued as follows:

\[
x^T(t - \omega) P(\gamma) x(t - \omega)
\]

\[
\leq (1 + r) \beta(\omega \gamma) x^T(t) P(\gamma) x(t)
\]

\[
+ \frac{1}{3} \left( 1 + \frac{1}{r} \right) \gamma \omega^2 (\gamma^2 \omega^2 - 3 \gamma \omega + 6)
\]

\[
\leq (1 + r) \beta(\omega \gamma) x^T(t) P(\gamma) x(t) + \frac{1}{3 \rho} \left( 1 + \frac{1}{r} \right)
\]

\[
\times \gamma \omega^2 (\gamma^2 \omega^2 - 3 \gamma \omega + 6) x^T(t) P(\gamma) x(t),
\]

which is (46). Inequality (47) can be proven similarly.

Note that \((\beta(\omega \gamma))\) is strictly increasing with respect to \( \omega \) and \( \frac{d \omega^2 (\gamma^2 \omega^2 - 3 \gamma \omega + 6)}{d \omega} = (4/3 \omega^2 (\gamma^2 \omega^2 - 3 \gamma \omega + 4) > 0 \)

which implies that \( \beta(\omega \gamma, r, \rho) \) is strictly increasing with respect to \( \omega \). Relation (48) then follows as \( \hat{\theta}(2r, \gamma, r, \rho) \) is minimized when

\[
r = r_{\text{opt}} = \sqrt{\frac{8 \gamma^2 r^2 (2 \gamma^2 r^2 - 3 \gamma r + 3)}{3 \rho \beta(2\gamma r)}}.
\]

This ends the proof.

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Proofs of the Technical Results in Section III-A:

Proof of Claim 1: The proof is quite similar to that given in [8, 36] and [45].

Proof of Item 1: Assume that \( x_0 = [x_{10}^T, x_{20}^T]^T \) is an initial condition in Region I, i.e.,
\[
B^T P(\gamma) x_0 > 1 + \eta(\gamma, \tau),
\]
(50)
It follows from (11) and (15) that the closed-loop system (8) in Region I reduces to
\[
\dot{x}(t) = Ax(t) - B
\]
whose closed form solution can be expressed as
\[
x(t) = e^{At} x_0 - \int_0^t e^{As} B ds.
\]
(52)
If we set \( B^T P(\gamma) x(t) - (1 + \eta(\gamma, \tau)) = 0 \), we get the following equation:
\[
B^T P(\gamma) e^{At} x_0 - B^T P(\gamma) \int_0^t e^{As} B ds = 1 + \eta(\gamma, \tau)
\]
from which it follows that
\[
1 + \eta(\gamma, \tau) - B^T P(\gamma) x_0 = -\frac{1}{2} \gamma^2 t^2 + (x_{20} \gamma^2 - 2\gamma) t.
\]
In view of (50), the above equation clearly has a unique positive solution \( t > 0 \). That is, the state \( x(t) \) will arrive at the boundary of Region II in a finite time. The same argument holds for Region III by symmetry.

Proof of Item 2: Let \( x(0) = x_0 \) be an initial condition on the boundary of Region II, that is
\[
B^T P(\gamma) x_0 = 1 + \eta(\gamma, \tau).
\]
(53)
Note that system (8) can also be written as (51) in this case. To enter Region I, we must have \( B^T P(\gamma) \dot{x}(0)|_{t=0} > 0 \), that is
\[
B^T P(\gamma) (Ax_0 - B) > 0 \iff x_{20} > \frac{2}{\gamma}.
\]
(54)
Assume that the state \( x \) has been in Region I. We have shown in Item 1 that \( x \) will return to the boundary of Region II in a finite time \( t_b > 0 \), that is, \( B^T P x_{20} = B^T P x_b(t_b) \), or equivalently,
\[
B^T P(\gamma) (e^{At_b} - I) x_0 + B^T P(\gamma) \int_0^{t_b} e^{As} B ds = 0.
\]
Solving the above equation and using (54) gives \( t_b = 2/(x_{20} \gamma^2 - 2) > 0 \). Then by using (52)–(54), we have
\[
V(x(t)) = V(x_0) = x^T(x_0) = x^T(t_b) P(\gamma) x(t_b) - x_0^T P(\gamma) x_0
\]
\[
= \left( e^{At_b} x_0 - \int_0^{t_b} e^{As} B ds \right)^T P(\gamma)
\times \left( e^{At_b} x_0 - \int_0^{t_b} e^{As} B ds \right) - x_0^T P x_0
\]
\[
= \frac{4}{\gamma} (x_{20} \gamma^2 - 2)(x_{10} \gamma^2 + 2x_{20} \gamma - 4)
\]
\[
= \frac{4}{\gamma} (x_{20} \gamma^2 - 2)(\eta(\gamma, \tau) - 3)
\]
\[
= 0.
\]
Similarly, if the initial \( x_0 \) is on the boundary of Region II and is going to enter Region III, namely, \( B^T P(\gamma) x_0 = -1 + \eta(\gamma, \tau) \), then the same argument can be proven by symmetry.

Proof of Lemma 1:

Proof of Item 1: Let \( A_{\lambda_1}(\gamma) = A - 1/[1 + \eta(\gamma, \tau)] B B^T P(\gamma) \). Then direct manipulation shows that \( \dot{V}(x(t)) = x^T(t) Q(\gamma, \tau) x(t) \) with
\[
Q(\gamma, \tau) = A_{\lambda_1}^T(\gamma) P(\gamma) + P(\gamma) A_{\lambda_1}(\gamma)
\]
\[
= A^T P(\gamma) + P(\gamma) A - 2 P(\gamma) (1 + \eta(\gamma, \tau))^{-1}
\]
\[
= -\gamma P(\gamma) \frac{1}{1 + \eta(\gamma, \tau)} P(\gamma) B B^T P(\gamma)
\]
\[
< -\gamma P(\gamma), \forall \gamma \in \left( 0, \frac{4}{\gamma(0, 1)} \right]
\]
where we have used the parametric algebraic Riccati (3) and inequality (14).

Proof of Item 2: With the help of Lemma 5 and (3), one gets
\[
\dot{V}(x(t)) = -\gamma x^T(t) P(\gamma) x(t) - x^T(t) P(\gamma) B B^T P(\gamma) x(t) + z^T(t) B B^T P(\gamma) x(t) + x^T(t) P(\gamma) B B^T z(t)
\]
\[
\leq -\gamma x^T(t) P(\gamma) x(t) + z^T(t) B B^T z(t)
\]
(55)
where \( z(t) \) is defined in (12). In view of Lemma 3 and Fact 1 in Appendix I-A, we have
\[
z^T(t) B B^T z(t)
\]
\[
= \left( \int_{-\tau}^0 p(s) ds \right)^T P(\gamma) B B^T P(\gamma) \int_{-\tau}^0 p(s) ds
\]
\[
\leq 2\gamma \left( \int_{-\tau}^0 p(s) ds \right)^T P(\gamma) \int_{-\tau}^0 p(s) ds
\]
\[
\leq 2\gamma \int_{-\tau}^0 p(s) P(\gamma) p(s) ds
\]
(56)
where \( p(s) \) is given by (10). Let \( y(s) = B^T P(\gamma) e^{As} \), then in view of Fact 2, inequality (56) can be continued as follows:
\[
z^T(t) B B^T z(t)
\]
\[
\leq 2\gamma \int_{-\tau}^0 \beta(s) y(s) y(s) P(\gamma) B B^T p(s) y(s) ds
\]
\[
= 4\gamma^2 \int_{-\tau}^0 \beta(s) y(s) y(s) P(\gamma) B B^T p(s) y(s) ds
\]
\[
\leq 4\gamma^2 \int_{-\tau}^0 \beta(s) y(s) y(s) P(\gamma) B B^T p(s) y(s) ds
\]
\[
= 4\gamma^2 \int_{-\tau}^0 \beta(s) e^{At} P(\gamma) e^{At} \dot{z}(s) P(\gamma) B B^T e^{At} \dot{z}(s) ds
\]
\[
\leq 4\gamma^3 \int_{-\tau}^0 \beta(s) e^{At} P(\gamma) e^{At} \dot{z}(s) P(\gamma) e^{At} \dot{z}(s) ds
\]
\[
\leq 4\gamma^3 \int_{-\tau}^0 \beta(s) e^{At} P(\gamma) e^{At} \dot{z}(s) P(\gamma) e^{At} \dot{z}(s) ds
\]
where $\dot{x}(s) = x^T (s + t - \tau)$. Thus, under condition (20), we can obtain
\[
z^T (t) BB^T z(t) \leq \left( 8\gamma^2 \beta(\gamma) \int_{-\tau}^0 \beta(s) \, ds \right) V(x(t)) .
\]
Inserting (57) into (55) gives (21), where $\zeta(\tau, \gamma, \vartheta) = 1 - 8\gamma^2 2\beta(\gamma) \int_{-\tau}^0 \beta(s) \, ds$ which is exactly in the form of (22). This ends our proof.

**Proof of Claim 2:** Let $r_{\text{min}}$ be defined as follows:
\[
r_{\text{min}} = \left( 1 + \eta(\gamma, \tau) - 4\gamma \sqrt{\frac{1}{3} (2\gamma^2 \tau^2 - 3\gamma \tau + 3)} \right) \frac{2\gamma (2\gamma \tau)}{
\frac{1}{2} (2\gamma^2 \tau^2 - 3\gamma \tau + 3)}.
\]
where $\eta(\gamma, \tau)$ and $\beta(s)$ are, respectively, defined by (13) and (45). It is easy to verify that $r_{\text{min}} > 0$ for arbitrary $\gamma \in (0, 4/2\tau)$ and
\[
\varrho(\gamma, 2\tau, r_{\text{min}}) r_{\text{min}} = r_{\text{max}} .
\]
We consider two cases.

*Case 1:* $x(t_0) \in \Pi, \mathcal{E}(P, r_{\text{min}})$, namely, $V(x(t_0)) > r_{\text{min}}$. According to Lemma 6, the time derivative of $V(x(t))$ along the trajectories of system (19) within Region $\Pi$ satisfies
\[
V(x(t - \omega)) \leq \min_{\vartheta > 0} \max_{\omega \in [0, 2\tau]} \left\{ \varrho(\omega, \gamma, r, r_{\text{min}}) \right\} V(x(t)) \leq \varrho(\gamma, 2\tau, r_{\text{min}}) V(x(t)), \forall \omega \in [-2\tau, 0]
\]
in which $\varrho(\omega, \gamma, r, r_{\text{min}})$ and $\varrho(\gamma, 2\tau, r_{\text{min}})$ are defined in Lemma 6. Then according to Item 2 of Lemma 1, inequality (59) implies
\[
\hat{V}(x(t)) \leq -\gamma \zeta(\tau, \gamma, \varrho(\gamma, 2\tau, r_{\text{min}})) V(x(t)) .
\]
By substituting (48) into (22), we get $t(\tau) \equiv \zeta(\tau, \gamma, \varrho(\gamma, 2\tau, r_{\text{min}}))$ in which
\[
t(\tau) = 1 - \frac{t_1(v)}{6 + 12v + 3v^2 - 8v\sqrt{6v^2 - 9v + 9} + \frac{3}{2}}
\]
with
\[
t_1(v) = 6v \left( \sqrt{1 + v^2 + v} \right)^2 \left( 2 + 4v + v^2 \right)^2 \times \left( 2 + v^2 + v \sqrt{4 + v^2} \right) \times \left( 8 + 6v + v^2 - 4v^3/2 \right).
\]
Direct calculation shows that $t(\tau) > 1/128$ if $v \in (0, 4/25]$. Therefore, it follows from (60) that the time derivative of $V(x(t))$ along the trajectories of system (19) within Region $\Pi$ satisfies
\[
\hat{V}(x(t)) = 2x^T P(\gamma) f_2 \leq -\gamma \frac{1}{128} V(x(t)).
\]
Therefore, invoking of Item 1 of Lemma 1, we conclude that the time derivative of $V(x(t))$ along the trajectories of system (16) within Region $\Pi$ satisfies
\[
\hat{V}(x(t)) \leq \max \{ 2x^T P(\gamma) f_1, 2x^T P(\gamma) f_2 \} = -\frac{\gamma}{128} V(x(t))
\]
which, by using the Lyapunov stability theorem, indicates that there exists a finite time $t_f \geq t_0$ such that $x^T (t_f) P(\gamma) x(t_f) = r_{\text{min}}$.

*Case 2:* $x(t_0) \in \mathcal{E}(P, r_{\text{min}}) \subset \mathcal{E}(P, r_{\text{max}}) \subset \Pi$. In this case, we clearly have $x^T (t_0) P(\gamma) x(t_0) \leq r_{\text{min}}$.

In both cases, we conclude that there exists a finite time $t_f \geq t_0$ such that $x^T (t_f) \begin{bmatrix} P(\gamma) x(t_f) \leq r_{\text{min}}. Then by using Lemma 6 and (58), we know that
\[
V(x(t_f - \omega)) \leq \varrho(\gamma, 2\tau, r_{\text{min}}) r_{\text{min}} = r_{\text{max}}, \forall \omega \in [0, 2\tau].
\]
The result then follows.

**Proofs of the Technical Results in Section III-B:**

**Proof of Claim 3:**

**Proof of Item 1:** Let $x_0 = [x_{10}, x_{20}]^T$ be an initial condition in Region $\bar{I}$, namely
\[
B^T P(\gamma) e^{-\beta_t x_0} > 1 + \rho(\gamma, \tau).
\]
In view of (27) and (30), the closed-loop system (24) in Region $\bar{I}$ can also be simplified as (51). Similarly, setting $B^T P(\gamma) e^{-\beta_t x(t)} = (1 + \rho(\gamma, \tau)) / 2$ gives the following equation:
\[
1 + \rho(\gamma, \tau) = B^T P(\gamma) e^{-\beta_t x(t)} x_0 = x_0 \gamma^2 t - \frac{1}{2} \gamma t(\gamma t - 2\gamma^2 + 4)
\]
which, in view of (62), clearly has a unique positive solution $t > 0$. That is to say, the state $x$ will arrive at the boundary of Region $\bar{I}$ in a finite time. The same argument holds for Region $\bar{I}^c$ by symmetry.

**Proof of Item 2:** Let $x(t_0) = x_0$ be an initial condition on the boundary of Region $\bar{I}$, that is, $B^T P(\gamma) e^{-\beta_t x_0} = 1 + \rho(\gamma, \tau)$. The state $x$ will enter Region $\bar{I}$ only if $B^T P(\gamma) e^{-\beta_t x(t)} \mid_{t=0} > 0$, that is
\[
B^T P(\gamma) e^{-\beta_t (A_0 x_0 - B) > 0} \iff x_0 > \frac{2}{\gamma - \tau}.
\]
Assume that the state $x$ has been in Region $\bar{I}$. We have shown in Item 1 of this claim that $x$ will return to the boundary of Region $\bar{I}$ in a finite time $t_b > 0$, i.e., $B^T Pe^{-\beta_t x_0} = B^T Pe^{-\beta_t x(t)}$, from which we solve $t_b = 2(x_0 + \tau - 2\gamma^2)$. Then the difference of the Lyapunov function $V(x(t)) = x^T (t) P(\gamma) x(t)$ between the two states $x(t_b)$ and $x_0$ can be given by
\[
V(x(t_b)) - V(x_0) = x^T (t_b) P(\gamma) x(t_b) - x_0^T P(\gamma) x_0
\]
\[
= x_0^T \left( e^{A_0 t_b} P(\gamma) e^{A_0 t_b} - P(\gamma) \right) x_0
\]
\[
-2 \left( \int_0^{t_b} e^{As} Bds \right)^T P(\gamma) e^{A t_b} x_0
\]
\[
\begin{align*}
&= \frac{6}{\gamma^2} \left( x_{0} + \tau - \frac{2}{\gamma} \right) (\gamma \tau - 3) \\
&\times \left( \left( \gamma - \frac{2}{3} \right)^2 + \frac{2}{9} \right) < 0, \quad \forall \gamma \in \left( 0, \frac{1}{\tau} \right]
\end{align*}
\]

where we have used (63).

Similarly, for the initial condition \( x_{0} \) on the boundary of Region \( \Pi \) that will enter region \( \Pi' \), namely, \( B^{T}P(\gamma)e^{-A\tau}x_{0} = -1 - \rho(\gamma, \tau) \), the result can be proven by symmetry. This proves the second item of the claim.

**Proof of Lemma 2:**

**Proof of Item 1:** Let

\[
A_{c2}(\gamma) = A - \frac{1}{1 + \rho(\gamma, \tau)}BB^{T}P(\gamma)e^{-A\tau}.
\]

Then straightforward manipulation shows that

\[
\dot{V}(x(t)) = -x^{T}(t)M(\gamma, \tau) x(t)
\]

in which

\[
M(\gamma, \tau) = -A_{c2}^{T}(\gamma)P(\gamma) - P(\gamma)A_{c2}(\gamma)
\]

\[
= \begin{bmatrix}
\frac{1}{2}4^{4} & -\frac{1}{2}4^{3}(\gamma^{2} - 6) \\
-\frac{1}{2}4^{3}(\gamma^{2} - 6) & \frac{1}{2}4^{2}(\gamma^{2} - 6(6-\gamma^{2}))
\end{bmatrix}.
\]

Direct calculation gives

\[
\det(M(\gamma, \tau)) = \frac{\gamma^{6}(12 + 8s - 40s^{2} + 12s^{3} - s^{4})}{(2 + 4s - s^{2})^{3}}.
\]

where \( s = \gamma^{2} \). It follows that \( M(\gamma, \tau) > 0 \) if \( 0 < \gamma^{2} \leq p^{*} = 0.757 \) where \( p^{*} \) is the unique positive real solution to the equation

\[
12 + 8p - 40p^{2} + 12p^{3} - p^{4} = 0.
\]

Therefore, we conclude from \( \gamma \in (0, 1/\gamma^{2}) \) that there exists a scalar \( \epsilon(\gamma, \tau) > 0 \) such that \( M(\gamma, \tau) \geq \epsilon(\gamma, \tau) I > 0 \).

**Proof of Item 2:** By using (3), Lemma 5 and Fact 1, the time derivative of \( V(x(t)) \) along the trajectories of system (35) can be evaluated as follows:

\[
\dot{V}(x(t)) = x^{T}(t)(A_{c3}(\gamma)P(\gamma) + P(\gamma)A_{c3}(\gamma)) x(t)
\]

\[
= -2x^{T}(t)P(\gamma)BB^{T}P(\gamma) x(t)
\]

\[
= -x^{T}(t)(-\gamma P(\gamma) + P(\gamma)BB^{T}P(\gamma) - P_{1}(\gamma))
\]

\[
\times x(t) - 2x^{T}(t) P(\gamma)BB^{T}P(\gamma) x(t)
\]

\[
\leq -\gamma P(\gamma) + (2P(\gamma)BB^{T}P(\gamma) - P_{1}(\gamma))
\]

\[
\times x(t) + q^{T}(t)P(\gamma)BB^{T}P(\gamma) x(t)
\]

\[
\leq -x^{T}(t)(-\gamma P(\gamma) + (2P(\gamma)BB^{T}P(\gamma) - P_{1}(\gamma))
\]

\[
\times x(t) + q^{T}(t)P(\gamma)BB^{T}P(\gamma) x(t)
\]

where \( A_{c3}(\gamma) \) is given by (32) and

\[
P_{1}(\gamma) = P(\gamma)BB^{T}P(\gamma)e^{-A\tau} + e^{-A\tau}P(\gamma)BB^{T}P(\gamma).
\]

On the other hand, invoking Lemma 3 and Fact 2, we obtain

\[
q^{T}(t)P(\gamma)q(t) = \left( \int_{-\tau}^{0} e^{-A(s+\tau)}Bk(s)ds \right)^{T}
\]

\[
\times P(\gamma) e^{-A(s+\tau)}Bk(s)ds
\]

\[
\leq \tau \int_{-\tau}^{0} e^{-A(\gamma s+\tau)}Bk(s)ds
\]

\[
\times P(\gamma)e^{-A(\gamma s+\tau)}Bk(s)ds
\]

\[
= \tau \int_{-\tau}^{0} k^{T}(s)B^{T}e^{-AT(s+\tau)}P(\gamma)e^{-A(\gamma s+\tau)}Bk(s)ds
\]

\[
\leq \tau \int_{-\tau}^{0} k^{T}(s)B^{T}e^{-AT(s+\tau)}P(\gamma) e^{-A(\gamma s+\tau)}Bk(s)ds
\]

where \( k(s) \) is related with (26). Denote \( \dot{x}(s) = x(s + t - \tau) \).

Then it follows the above inequality that

\[
q^{T}(t)P(\gamma)q(t) \leq 2\gamma \tau \int_{-\tau}^{0} B^{T}e^{-AT(s+\tau)}P(\gamma) e^{-A(\gamma s+\tau)}Bk(s)ds
\]

\[
\leq 2\gamma \tau \int_{-\tau}^{0} B^{T}e^{-AT(s+\tau)}P(\gamma) e^{-A(\gamma s+\tau)}Bk(s)ds
\]

\[
< 2\gamma \tau \int_{-\tau}^{0} \beta(\gamma s)B^{T}e^{-AT(s+\tau)}P(\gamma) e^{-A(\gamma s+\tau)}Bk(s)ds
\]

Therefore, under the condition (36), the above inequality can be continued as follows:

\[
q^{T}(t)P(\gamma)q(t) \leq 2\gamma \tau \phi(\gamma s)B^{T}e^{-AT(s+\tau)}P(\gamma) e^{-A(\gamma s+\tau)}Bk(s)ds
\]

\[
< 2\gamma \tau \phi(\gamma s)B^{T}e^{-AT(s+\tau)}P(\gamma) e^{-A(\gamma s+\tau)}Bk(s)ds
\]

\[
= 2\gamma \tau \phi(\gamma s) \left( \frac{1}{3} \gamma s^{3} - (\gamma s)^{2} + 2\gamma s \right)
\]

\[
\times \dot{x}(s),
\]

where

\[
W(\gamma, s, \phi) = \left( \gamma - 4\gamma s^{2}\phi(\gamma s) \right)
\]

\[
\times \left( \frac{1}{3I} (\gamma s)^{3} - (\gamma s)^{2} + 2\gamma s \right)
\]

\[
\times P(\gamma) - 2P(\gamma)BB^{T}P(\gamma) + P_{1}(\gamma)
\]

which is exactly in the form of (38). This completes the proof.
Proof of Claim 4: Define
\[
\bar{T}_{\min} = \frac{1 + \rho(\gamma, \tau)}{\sqrt{\gamma^2 + 2\gamma T + 1}} = \frac{\gamma T \sqrt{\gamma^2 + 2\gamma T + 3}}{\gamma T}.
\]
where \(\beta(2\gamma T)\) and \(\rho(\gamma, \tau)\) are, respectively, given in (48) and (29). Then direct manipulation shows that
\[
\theta(\gamma, 2\tau, \bar{T}_{\min}) \bar{T}_{\min} = \tau_{\max}.
\]
(66)

Similar to the proof of Claim 2, we consider two cases.

Case 1: \(x(t_0) \in \mathcal{E}(P, \tau_{\min})\), namely, \(V(x(t_0)) > \tau_{\min}\). According to Lemma 6, the time derivative of \(V(x(t))\) along the trajectories of system (35) within Region \(\mathcal{P}\) satisfies
\[
V(x(t - \omega)) < \theta(\gamma, 2\tau, \tau_{\min}) V(x(t)), \omega \in [-2\tau, 0].
\]
(67)

Then according to Item 2 of Lemma 2, inequality (67) implies
\[
\dot{V}(x(t)) \leq -\alpha(t) \bar{W}(\tau, \gamma, \tau_{\min}) x(t)
\]
where
\[
\bar{W}(\tau, \gamma, \tau_{\min}) = \begin{bmatrix}
\gamma^4 \mu_1 \\
\gamma^3 (\mu_1 - \gamma T)
\end{bmatrix}
\]
and
\[
\gamma^2 (\mu_1 - \gamma T) 2\gamma T + 1.
\]
(69)
in which \(\mu_1 = 1 - \tau_{\max}/\tau_{\min}\). We have used the relationship in (66). By using the definition of \(\rho(\gamma, 2\tau, r)\) and (66) again, we get
\[
\bar{T}_{\max} = \frac{\beta(2\gamma T) (1 + \rho(\gamma, \tau))}{d(\gamma, \tau)}
\]
(70)
in which \(\tau_{\gamma} = \begin{cases} 0, & 1 \end{cases} \Rightarrow \bar{W}(\gamma, \gamma, \tau_{\min}) > 0\).

Hence, invoking Item 1 of Lemma 2, we conclude that the time derivative of \(V(x(t))\) along the trajectories of system (31) within Region \(\mathcal{P}\) satisfies
\[
\dot{V}(x(t)) \leq \min \left\{ 2x^T(t) P(\gamma) g_1(t), 2x^T(t) P(\gamma) g_2(t, \tau) \right\}
\]
\[
= \min \left\{ \lambda_{\min} \frac{\bar{W}(\tau, \gamma, \tau_{\min})}{8}, \epsilon(\gamma, \tau) \right\} \| x(t) \|^2
\]
which, by using the Lyapunov stability theorem, indicates that there exists a finite time \(t_{\bar{T}} \geq t_0\) such that \(x^T(t_{\bar{T}}) P(\gamma) x(t_{\bar{T}}) = \tau_{\min}\).

Case 2: \(x(t_0) \in \mathcal{E}(P, \tau_{\min}) \subset \mathcal{E}(P, \tau_{\max}) \subset \mathcal{P}\). In this case, we clearly have \(x^T(t_0) P(\gamma) x(t_0) \leq \tau_{\min}\).

In both cases, we know that there exists a finite time \(t_{\bar{T}} \geq t_0\) such that \(x^T(t_{\bar{T}}) P(\gamma) x(t_{\bar{T}}) = \tau_{\min}\). Then by using Lemma 6 again and in view of (66), we have
\[
V(x(t_{\bar{T}} - \omega)) \leq \theta(\gamma, 2\tau, \tau_{\min}) \tau_{\min} = \tau_{\max}, \forall \omega \in [0, 2\tau].
\]
This proves the result.

REFERENCES

ZHOU et al.: GLOBAL STABILIZATION OF THE DOUBLE INTEGRATOR SYSTEM


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