Parametric solutions to the generalized discrete Sylvester matrix equation $MXN - X = TY$ and their applications

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In this paper, an explicit, analytical and complete solution to the generalized discrete Sylvester matrix equation $MXN - X = TY$ which is closely related with several types of matrix equations in control theory is obtained. The proposed solution has a neat and elegant form in terms of the Krylov matrix, a block Hankel matrix and an observability matrix. Based on the proposed solution, an explicit solution to the general discrete Lyapunov matrix equation is also derived. As an application, the parametric pole assignment for descriptor linear systems by proportional-plus-derivative state feedback is considered. The results presented here are parallel to our earlier results on the generalized Sylvester matrix equation $AX - XF = BY$.

Keywords: generalized discrete Sylvester matrix equations; parametric solutions; controllability and observability; parametric pole assignment; descriptor linear systems.

1. Introduction

The problem of solving the following generalized discrete Sylvester matrix equation:

$$MXN - X = TY,$$  \hspace{1cm} (1)

with $M \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times r}$ and $N \in \mathbb{R}^{p \times p}$ some known matrices, is motivated by two facts.

Firstly, when $M = N^\top$ and $Y = -T^\top$, the generalized discrete Sylvester equation (1) is reduced to the general discrete Lyapunov matrix equation which has widely applications in the analysis and synthesis problems for conventional discrete linear systems. If a pair of solution $(X, Y)$ to (1) has been established, then set $Y = -T^\top$, generally, under some conditions, $X$ can be immediately obtained. This idea is realized indeed in Section 3.3 of this paper.

Second, it is easy to show that (a detailed discussion is given by Lemma 9) the following generalized Sylvester matrix equation:

$$AX - EXF = BY,$$  \hspace{1cm} (2)

is equivalent to the generalized discrete Sylvester matrix equation (1) with their coefficient matrices satisfying some relations. It is well known that the generalized Sylvester matrix equation (2) is closely related with many synthesis problems in linear systems theory. When $E = I$, (2) is closely related with many problems in conventional linear control systems theory, such as pole/eigenstructure assignment design (Fahmy & O’Reilly, 1983; Kwon & Youn, 1987), Luenberger-type observer design

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(Chen et al., 1996; Luenberger, 1964, 1971; Tsui, 1988), robust fault detection (Park & Rizzoni, 1994; Duan & Patton, 2001) and so on, and has been investigated by many authors (Duan, 1993; Tsui, 1987, 1993; Zhou & Duan, 2005). When dealing with eigenstructure assignment (Duan, 1992, 1996; Zhou & Duan, 2006; Duan & Patton, 1997), observer design (Duan et al., 2002) and model reference control for descriptor linear systems, the generalized Sylvester matrix equation (2) with \( E \) being singular is encountered.

In solving the generalized Sylvester matrix (2), finding the complete parametric solutions, i.e. parametric solutions consisting of the maximum number of free parameters, is of extreme importance since many problems such as robustness in control system design require full use of the design freedom. For (2), with \( F \) being in Jordan form, Duan (1992) has proposed a complete parametric solution. However, this solution is not in a direct, explicit form but in a recursive form. Also, when the matrix \( F \) is assumed in Jordan form and the matrix triple \((E, A, B)\) is assumed to be R-controllable, Duan (1996) has given a complete and explicit solution which uses the right coprime factorization of the input-state transfer function \((sE - A)^{-1}B\).

These existing solutions are directly applicable in problems like eigenstructure assignment since the matrix \( F \) is originally required to be in Jordan form in such problems. However, in some other problems, the matrix \( F \) may be arbitrary. Although we can apply some similar transformation to transform \( F \) into a Jordan form, such a process is generally not numerically reliable (Wilkinson, 1965). Furthermore, such a process may also add additional computation. At the same time, with the method proposed in Duan (1996), the right coprime factorization of \((sE - A)^{-1}B\) is needed. However, right coprime factorization is not generally easy to be obtained.

Lemma 9 in this paper shows that the solutions to the generalized Sylvester matrix equation (2) can be immediately obtained as soon as the solutions to the generalized discrete Sylvester matrix equation (1) are obtained. This is the second fact that motives us to find general solutions to the generalized discrete Sylvester matrix equation (1).

When \( E = I \) and the matrix \( F \) is in companion form or Frobenius form, reference Zhou & Duan (2005) gives an explicit and general solution to this type of (2). This solution is in an extremely neat form represented by a symmetric operator matrix, a Hanker matrix and the controllability matrix of the matrix pair \((A, B)\). The primary advantage of this proposed method is that it does not need right coprime factorization but only the characteristic polynomial of matrix \( A \).

Similar to the results proposed in Zhou & Duan (2005), we present in this paper a parametric solution represented by a block Hankel matrix, the Krylov matrix of matrix pair \((M, T)\) and a controllability matrix with matrix pair \((Z, N)\). We use an arbitrary parameter matrix \( Z \) in the proposed solution to denote the freedom in the solutions of the generalized discrete Sylvester matrix equation (1). Further, we deal with this equation in this paper without assuming matrix \( N \) in any special forms. Such property may bring some conveniences in some design problems. As a demonstration, a simple and effective approach for parametric pole assignment for descriptor linear systems by proportional-plus-derivative (P-D) feedback is proposed.

2. Preliminaries

In this paper, we use \( \det(A) \), \( \text{trace}(A) \), \( \text{adj}(A) \) and \( \sigma(A) \) to denote the determinant, the trace, the adjoint matrix and the set of eigenvalues of some square matrix \( A \), respectively. If \( B \) is a \( p \times q \) matrix, we use \( \text{Im}(B) \), \( B^\top \) and \( \text{rank}(B) \) to denote the image, the transpose and the rank of matrix \( B \), respectively, and \( b_{ij} \) is the \( i \)-th row and \( j \)-th column of matrix \( B \). \( I_p \) is the \( p \times p \) identity matrix, and \( 0 \) will be used as an \( r \times s \) matrix when the dimensions are evident from the context. The symbol \( \otimes \) is to denote the Kronecker
product. The symbol $A \in \mathbb{F}_q^{r \times p}$ means that $A$ is an $r \times p$ matrix in the field $\mathbb{F}$ with its elements all having rank $q$.

Further, let $M \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{q \times r}$, we can define the following so-called Krylov matrix with matrix pair $(M, T)$:

$$Q_c(M, T, k) = [T \quad MT \quad M^2T \quad \cdots \quad M^{k-1}T].$$

(3)

The dual form of the Krylov matrix with matrix pair $(C, N)$, where $C \in \mathbb{R}^{p \times q}$ and $N \in \mathbb{R}^{p \times p}$, is defined as

$$Q_o(C, N, k) = [C^\top \quad (CN)^\top \quad \cdots \quad (CN^{k-1})^\top]^\top,$$

(4)

which can also be considered as a generalized observability matrix (since the index $k$ may be lower or higher than $p$).

We define a polynomial in $s$ with degree $\omega$ as

$$d(s, \omega) = d_0 + d_1s + \cdots + d_{\omega-1}s^{\omega-1} + d_\omega s^\omega,$$

(5)

where $d_0$ is not necessary different from zero. The conjugate polynomial of $d(s, \omega)$ is denoted by $\tilde{d}(s, \omega)$ with the following form:

$$\tilde{d}(s, \omega) = d_0s^\omega + d_1s^{\omega-1} + \cdots + d_{\omega-1}s + d_\omega,$$

(6)

where $d_0$ is not necessary different from zero too. For an arbitrary polynomial matrix in the form of

$$D(s, \omega) = D_0 + D_1s + \cdots + D_{\omega-1}s^{\omega-1} + D_\omega s^\omega,$$

we can define the following so-called block Hankel matrix:

$$S(D, \omega) = \begin{bmatrix}
D_0 & D_1 & \cdots & D_{\omega-2} & D_{\omega-1} \\
0 & D_0 & \ddots & \ddots & D_{\omega-2} \\
\vdots & 0 & \ddots & \ddots & \vdots \\
& 0 & \ddots & D_0 & D_1 \\
& & 0 & \cdots & 0 & D_0 \\
\end{bmatrix}. (7)$$

Finally, we define, for an $m \times n$ matrix $R = [r_{ij}]$, the so-called stretching function $\text{vec}(R)$ as

$$\text{vec}(R) = [r_{11} \quad r_{21} \quad \cdots \quad r_{m1} \quad r_{1n} \quad r_{2n} \quad \cdots \quad r_{mn}]^\top.$$  

(8)

The following result can be regarded as an extension of normal Cayley–Hamilton theorem.

**Lemma 1** Let $M \in \mathbb{R}^{n \times n}$ be an arbitrary square matrix and define the so-called inverse characteristic polynomial of $M$ as

$$\beta(s, \psi) = \det(sM - I) = \sum_{i=0}^{\psi} \beta_i s^i, \quad \beta_\psi \neq 0,$$

(9)

then the following identity holds:

$$\sum_{i=0}^{\psi} \beta_i M^{\psi-i} = 0, \quad \forall \phi \geq n.$$  

(10)
Proof. Assume that the adjoint matrix of the polynomial matrix \((sM - I)\) is \(R(s, n - 1)\), then
\[
(sM - I)^{-1} = \frac{\text{adj}(sM - I)}{\det(sM - I)} = \frac{1}{\det(sM - I)} \sum_{i=0}^{n-1} R_i s^i,
\]
which can be equivalently rewritten as
\[
\sum_{i=0}^{\psi} (\beta_i I) s^i = (sM - I) \sum_{i=0}^{n-1} R_i s^i.
\]
Now both sides of the above equation are polynomial in \(s\), by equating the power of \(s\), we obtain the following series of equations:
\[
\begin{align*}
-R_0 &= \beta_0 I, & MR = R_{\psi} - R_{\psi+1} &= 0, \\
MR_0 - R_1 &= \beta_1 I, & \vdots, \\
& \vdots & \vdots, \\
MR_{\psi-1} - R_{\psi} &= \beta_{\psi} I, & MR_{n-1} &= 0.
\end{align*}
\]
Premultiplying the \(i\)th equation by \(M^{n+1-i}\) and summing together produce
\[
0 = M^n \beta_0 + M^{n-1} \beta_1 + \cdots + M^{n-\psi} \beta_{\psi-1} + M^{n-\psi} \beta_{\psi}.
\]
The proof is then completed.

Let \(M \in \mathbb{R}^{n \times n}\) be an arbitrary matrix and \(B_i, i = 0, 1, \ldots, n - 1\), and \(\alpha_i, i = 0, 1, \ldots, n\), satisfy the following recursive relations:
\[
\begin{align*}
\alpha_n &= 1, & B_{n-1} &= I, \\
\alpha_{n-1} &= \frac{\text{trace}(B_{n-1}M)}{n-1}, & B_{n-2} &= B_{n-1}M + \alpha_{n-1} I, \\
& \vdots & \vdots, \\
\alpha_1 &= \frac{\text{trace}(B_1M)}{n}, & B_1 &= B_2M + \alpha_2 I, \\
\alpha_0 &= \frac{\text{trace}(B_0M)}{n}, & B_0 &= B_1M + \alpha_1 I,
\end{align*}
\]
then according to the well-known algorithm of Faddeev and Leverrier, we have
\[
(sI - M)^{-1} = \frac{\text{adj}(sI - M)}{\det(sI - M)} = \frac{B(s, n - 1)}{\alpha(s, n)}.
\]
Using the above algorithm, we have the following result concerning the calculation of the inverse characteristic polynomial \(\beta(s, n)\) of matrix \(M\) and the adjoint matrix \(R(s, n - 1)\) of \((sM - I)\).

**Lemma 2** Let \(M \in \mathbb{R}^{n \times n}\) be an arbitrary matrix and \(B_i, i = 0, 1, \ldots, n - 1\), and \(\alpha_i, i = 0, 1, \ldots, n\), satisfy the recursive relation (12), then
\[
(sM - I)^{-1} = \frac{\text{adj}(sM - I)}{\det(sM - I)} = \frac{R(s, n - 1)}{\beta(s, n)}
\]
with $R_i, i = 0, 1, \ldots, n - 1$, and $\beta_i, i = 0, 1, \ldots, n$, satisfying
\[
\begin{cases}
\beta_{n-i} = -\alpha_i, & i = 0, 1, \ldots, n-1, n, \\
R_{n-1-i} = B_i, & i = 0, 1, \ldots, n-2, n-1.
\end{cases}
\tag{15}
\]

**Proof.** Note that
\[
(sM - I)^{-1} = \left(-s \left(\frac{1}{s} I - M\right)\right)^{-1} = -\frac{1}{s} \left(\frac{1}{s} I - M\right)^{-1}.
\]
Using (13), we have
\[
(sM - I)^{-1} = -\frac{1}{s} \frac{B_i(i, \frac{1}{s}, n-1)}{\alpha_i(i, \frac{1}{s}, n)}
= -\frac{1}{s} \sum_{i=0}^{n-1} B_i(i, \frac{1}{s}, n-1) \alpha_i(i, \frac{1}{s}, n)
= -\frac{1}{s} \sum_{i=0}^{n-1} B_i s^{n-1-i} \alpha_i s^{n-i}.
\tag{16}
\]
Comparing (14) with (16), we get the relation (15). \hfill \Box

The following lemma is to be used in Section 3.

**Lemma 3** Let $D_i \in \mathbb{R}^{r \times r}, i = 0, 1, \ldots, \omega$, be some square matrices, then matrix $\sum_{i=0}^{\omega} (A^i \otimes D_i)$ is nonsingular if and only if the matrices
\[
D(\lambda, \omega) = \sum_{i=0}^{\omega} D_i \lambda^i, \quad \forall \lambda \in \sigma(A),
\tag{17}
\]
are all nonsingular.

The proof of the above lemma is trivial and thus is omitted here. The following result is standard.

**Lemma 4** Let $R, X$ and $S$ be some matrices with appropriate dimensions, then
\[
\text{vec}(RXS) = (S^T \otimes R)\text{vec}(X).
\tag{18}
\]

3. Main results

3.1 Basic results

Firstly, we give the following lemma about the degree of freedom in the solution $(X, Y)$ to the generalized discrete Sylvester matrix equation (1).

**Lemma 5** The degree of freedom in the solution $(X, Y)$ to the generalized discrete Sylvester matrix equation (1) is $rp$ if and only if
\[
\text{rank}[\lambda M - I]^T = n, \quad \forall \lambda \in \sigma(N).
\tag{19}
\]
Proof. Putting vec(\cdot) on both sides of the (1) and applying (18) yield

$$[N^\top \otimes M - I_p \otimes I_n \ - I_p \otimes T] \begin{bmatrix} \text{vec}(X) \\ \text{vec}(Y) \end{bmatrix} = 0,$$

(20)

which is equivalent to (1). Let $P$ and $J$ be the eigenvector matrix and the Jordan form of the matrix $N^\top$, respectively, then by definition

$$N^\top = P J P^{-1}.$$  \hspace{1cm} (21)

By substituting (21) into (20), we obtain

$$[N^\top \otimes M - I_p \otimes I_n \ - I_p \otimes T] = P \otimes I_n [J \otimes M - I_{np} \ - I_p \otimes T] \text{diag}(P^{-1} \otimes I_n, -P^{-1} \otimes I_r).$$

Noting that $P \otimes I_n$, $P^{-1} \otimes I_n$ and $P^{-1} \otimes I_r$ are all nonsingular, we have

$$\text{rank}[N^\top \otimes M - I_p \otimes I_n \ - I_p \otimes T] = \text{rank}[J \otimes M - I_{np} \ - I_p \otimes T]$$

\hspace{1cm} (22)

where the terms denoted by $\ast$ may be zero and $s_i, i = 1, 2, \ldots, p$, are the eigenvalues of the matrix $N^\top$, which are not necessarily distinct. Therefore,

$$\text{rank}[N^\top \otimes M - I_p \otimes I_n \ - I_p \otimes T] = np$$

(23)

holds if and only if (19) holds. According to linear equation theory, when (23) holds, the number of free parameters of the solution $(X, Y)$ is

$$\nu = np + rp - \text{rank}[N^\top \otimes M - I_p \otimes I_n \ - I_p \otimes T] = rp.$$

Finally, recalling that $N^\top$ and $N$ have the same Jordan form, we complete the proof. \hfill \square

We now give our main results in this paper.

**Theorem 1** Let $M \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times r}$ and $N \in \mathbb{R}^{p \times p}$ be some known matrices. If there exists a scalar $\omega$ and a series of matrices $D_i \in \mathbb{R}^{r \times r}, i = 0, 1, \ldots, \omega$, such that

$$MTD_\omega + M^2TD_{\omega-1} + \cdots + M^{\omega}TD_1 + M^{\omega+1}TD_0 = 0,$$

(24)

then

1. matrices pair characterized as

$$\begin{align*}
X & = Q_c(M, T, \omega + 1)S(D, \omega + 1)Q_\omega(Z, N, \omega + 1), \\
Y & = -[D_0 \ D_1 \ \cdots \ \ D_{\omega-1} \ D_\omega]Q_\omega(Z, N, \omega + 1),
\end{align*}$$

(25)
with \( Z \in \mathbb{R}^{r \times p} \) an arbitrary parameter matrix, satisfies the generalized discrete Sylvester matrix equation (1);

2. further, if \( M \in \mathbb{R}^{n \times n}, T \in \mathbb{R}^{n \times r} \) and \( N \in \mathbb{R}^{p \times p} \) satisfy (19) and

\[
\det(D(\lambda)) \neq 0, \quad \forall \lambda \in \sigma(N),
\]

(26) holds with \( D(\lambda) = \sum_{i=0}^{\omega} D_i \lambda^i \), then all the matrices \( X \in \mathbb{R}^{n \times p}, Y \in \mathbb{R}^{r \times p} \) satisfying the generalized discrete Sylvester matrix equation (1) can be parameterized as (25).

**Proof.** With the \( X \) expression and \( S(D, \omega + 1) \) defined as (7), we have

\[
MXN = MQ_c(M, T, \omega + 1)S(D, \omega + 1)Q_o(Z, N, \omega + 1)N
\]

\[
= [MT \quad M^2T \quad M^3T \quad \cdots \quad M^{\omega+1}T]S(D, \omega + 1)Q_o(Z, N, \omega + 1)N
\]

\[
= \left[ MT D_0 \quad MT D_1 + M^2T D_0 \quad \cdots \quad \sum_{i=1}^{\omega+1} M^iT D_{\omega+1-i} \right] Q_o(Z, N, \omega + 1)N.
\]

Further noting (24), we obtain that

\[
MXN = \left[ MT D_0 \quad MT D_1 + M^2T D_0 \quad \cdots \quad \sum_{i=1}^{\omega} M^iT D_{\omega-i} \right] Q_o(Z, N, \omega + 1)N
\]

\[
= \Psi Q_o(Z, N, \omega + 1),
\]

where

\[
\Psi = \left[ 0 \quad MT D_0 \quad MT D_1 + M^2T D_0 \quad \cdots \quad \sum_{i=1}^{\omega} M^iT D_{\omega-i} \right].
\]

By expanding the \( X \) expression in (25), we obtain that

\[
X = \Pi Q_o(Z, N, \omega + 1),
\]

where

\[
\Pi = \left[ TD_0 \quad TD_1 + MT D_0 \quad \cdots \quad TD_\omega + \sum_{i=1}^{\omega} M^iT D_{\omega-i} \right].
\]

Therefore,

\[
MXN - X = (\Psi - \Pi)Q_o(Z, N, \omega + 1)
\]

\[
= -[TD_0 \quad TD_1 \quad \cdots \quad TD_\omega]Q_o(Z, N, \omega + 1)
\]

\[
= -[D_0 \quad D_1 \quad \cdots \quad D_\omega]Q_o(Z, N, \omega + 1)
\]

\[
= TY.
\]

This states that \( X \) and \( Y \) given in (25) satisfy the generalized discrete Sylvester matrix equation (1).

We now prove the second conclusion. Putting \( \text{vec}(\cdot) \) on both sides of the expressions \( X \) and \( Y \) in (25) and using (18) produce

\[
\begin{bmatrix}
\text{vec}(X) \\
\text{vec}(Y)
\end{bmatrix}
= \Omega \text{vec}(Z),
\]

(27)
where
\[
\Omega = \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix}, \quad \Pi_2 = \sum_{i=0}^{\omega} (N^T)^i \otimes D_i, \\
\Pi_1 = \sum_{i=0}^{\omega} (N^T)^i \otimes \left( \sum_{j=1}^{i+1} M^i T D_{i+1-j} \right).
\]

According to Lemma 5, the number of free parameters of the solution \((X, Y)\) to the generalized discrete Sylvester matrix equation (1) is \(rp\). Recalling the fact that \(Z \in \mathbb{R}^{r \times p}\) is an arbitrary parameter matrix, we only need to show that the elements in \(Z\) are independent under the condition (26). By observing (27), we know that the elements in \(Z\) are independent if and only if \(\text{rank}(\Omega) = rp\). Obviously, \(\text{rank}(\Omega) = rp\) holds if \(\Pi_2\) is nonsingular. According to Lemma 3, \(\Pi_2\) is nonsingular if and only if (26) holds. The proof is completed. □

The following theorem is similar to Theorem 1.

**Theorem 2** Let \(M \in \mathbb{R}^{n \times n}, T \in \mathbb{R}^{n \times r}\) and \(N \in \mathbb{R}^{p \times p}\) be some known matrices. If there exists a scalar \(\omega\) and a series of matrices \(D_i \in \mathbb{R}^{r \times r}, i = 0, 1, \ldots, \omega\), such that
\[
TD_\omega + MTD_{\omega-1} + \cdots + M^{\omega-1}TD_1 + M^\omega TD_0 = 0,
\]
then
1. matrices pair in the form of
\[
\begin{cases} 
X = Q_c(M, T, \omega)S(D, \omega)Q_o(Z, N, \omega), \\
Y = -[D_0 \quad D_1 \quad \cdots \quad D_\omega]Q_o(Z, N, \omega + 1),
\end{cases}
\]
with \(Z \in \mathbb{R}^{r \times p}\) an arbitrary parameter matrix, satisfies the generalized discrete Sylvester matrix equation (1);
2. further, if \(M \in \mathbb{R}^{n \times n}, T \in \mathbb{R}^{n \times r}\) and \(N \in \mathbb{R}^{p \times p}\) satisfy (19) and the condition (26) holds, then all the matrices \(X \in \mathbb{R}^{n \times p}, Y \in \mathbb{R}^{r \times p}\) satisfying the generalized discrete Sylvester matrix equation (1) can be parameterized as (29).

**Proof.** The proof is similar to the proof of Theorem 1. □

**Remark 1** Since the restriction (28) implies (24), the scalar \(\omega\) existing in Theorem 2 may be higher than in Theorem 1. Obviously, if \(M\) is nonsingular, the restrictions (28) and (24) are equivalent.

According to Theorems 1 and 2, in order to get a complete parametric solution to the generalized discrete Sylvester matrix equation (1), we should guarantee that the matrices series \(D_i \in \mathbb{R}^{r \times r}, i = 0, 1, \ldots, \omega\), determined by (24) or (28) satisfy (26). Regarding the existence and the computation of \(D_i \in \mathbb{R}^{r \times r}, i = 0, 1, \ldots, \omega\), we have the following simple result.

**Lemma 6** Let \(D_0\) be chosen to be a nonsingular matrix and \(\omega \geq \phi\), where
\[
\phi = \min(t | \text{rank}(MQ_c(M, T, t))) = \text{rank}(MQ_c(M, T, t + 1)).
\]
Then, there exists a group of matrices \(D_i, i = 1, 2, \ldots, \omega\), satisfying (24).
Proof. When \( D_0 \) is selected to be nonsingular, (24) is equivalent to

\[
M Q_c(M, T, \omega)[D_\omega^T \ D_{\omega-1}^T \ \cdots \ D_1^T \ D_0^T]^T = -M^{\omega+1}T D_0,
\]

which is a nonhomogeneous linear equation when \( D_0 \) is prescribed. Noting that

\[
\text{rank}[M Q_c(M, T, \omega) - M^{\omega+1}T D_0] = \text{rank}\left[M Q_c(M, T, \omega) \ M^{\omega+1}T\right] = \text{rank}(M Q_c(M, T, \omega + 1))
\]

and, when \( \omega \geq \phi \), \( \text{rank}(M Q_c(M, T, \omega + 1)) = \text{rank}(M Q_c(M, T, \omega)) \), we have

\[
\text{rank}[M Q_c(M, T, \omega) - M^{\omega+1}T D_0] = \text{rank}[M Q_c(M, T, \omega)],
\]

which implies that the nonhomogeneous linear equation (30) has a solution. \( \square \)

**Remark 2** Let \( D_i, i = 1, 2, \ldots, \omega \), be the solution of (24) under the condition of the lemma, then we can declare that the condition (26) will hold for `almost all' matrix \( N \in \mathbb{R}^{p \times p} \). In fact, since \( D_0 \) is nonsingular, we have \( \det(D(\lambda, \omega))|_{\lambda = 0} = \det(D_0) \neq 0 \), which implies that \( D(\lambda, \omega) \) is not identical zero. Further, \( \deg \det(D(\lambda, \omega)) \leq r \omega \), the set \( \Omega = \{ \lambda | \det(D(\lambda, \omega)) = 0 \} \) is a finite one with its elements no more than \( r \times \omega \). Then, we can declare that for almost all matrix \( N \), the condition (26) holds.

### 3.2 Special solutions

Though Lemma 6 gives us a method to establish the matrices series \( D_i, i = 0, 1, \ldots, \omega \), such that solutions characterized as (25) are complete for almost all matrix \( N \), the choice of matrix \( D_0 \) is still in technical difficulty, and especially, solving \( D_i, i = 1, 2, \ldots, \omega \), from the nonhomogeneous linear equation (30) may lead to bad numerical condition. So we introduce in the following a more applicable method.

Firstly, we assume that the series of matrices \( D_i, i = 0, 1, \ldots, \omega \), are in the form of \( D_i = d_i I_r \), \( i = 0, 1, \ldots, \omega \). Then, condition (24) is then reduced to

\[
\tilde{d}(M, \omega)MT = 0,
\]

where \( \tilde{d}(s, \omega) \) is the conjugate polynomial of \( d(s, \omega) = \sum_{i=0}^{\omega} d_i s^i \).

With this simplification, we have the following corollary.

**Corollary 1** Let \( M \in \mathbb{R}^{n \times n} \), \( T \in \mathbb{R}^{n \times r} \) and \( N \in \mathbb{R}^{p \times p} \) be some known matrices. If there exists a scalar \( \omega \) and a polynomial \( d(s, \omega) \) with its conjugate polynomial \( \tilde{d}(s, \omega) \) satisfying (31), then the matrix pair \( (X, Y) \) characterized as

\[
\begin{align*}
X &= Q_c(M, T, \omega + 1)[S(d, \omega + 1) \otimes I_r]Q_0(Z, N, \omega + 1), \\
Y &= -(d_0 Z + d_1 Z N + \cdots + d_{\omega-1} Z N^{\omega-1} + d_\omega Z N^\omega),
\end{align*}
\]

(32)
where $Z \in \mathbb{R}^{r \times p}$ is an arbitrary parameter matrix, satisfies the generalized discrete Sylvester matrix equation (1). Further, if condition (19) holds, such solutions (32) are complete if

$$\det \tilde{d}(\lambda, \omega) \neq 0, \quad \forall \lambda \in \sigma(N). \quad (33)$$

According to (31), a possible choice of $\tilde{d}(s, \omega)$ is the characteristic polynomial or the minimal polynomial of matrix $M$. However, the degrees of such two special polynomials may be not the lowest.

**Definition 1** For a matrix pair $(A, B)$ with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, a matrix $A_{c} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ is called the restriction of $A$ to the controllable subspace of $(A, B)$ if there exists a nonsingular matrix $P$ such that

$$P^{-1}AP = \begin{bmatrix} A_{c} & A_{1} \\ 0 & A_{2} \end{bmatrix}, \quad P^{-1}B = \begin{bmatrix} B_{c} \\ 0 \end{bmatrix}$$

and $(A_{c}, B_{c})$ is controllable.

The following lemma gives us a method to reduce the degree of $\tilde{d}(s, \omega)$.

**Lemma 7** (Souza & Bhattacharyya, 1981, Lemma 3) The polynomial $\tilde{d}(s, \omega)$ of the lowest degree for (31) holds is $\tilde{d}^{*}(s, \omega_{*})$, the minimal polynomial of $M_{c}$, which denotes the restriction of $M$ to the controllable subspace of $(M, MT)$.

**Remark 3** Corresponding corollary to Theorem 2 can also be obtained via the similar simplification and thus is omitted.

By using the extended Cayley–Hamilton theorem established in Lemma 1, we have the following corollary.

**Corollary 2** Let $\beta(s, \psi)$ be the inverse characteristic polynomial of $M$ defined in (9). If

$$\beta(\lambda, \psi) = \det(\lambda M - I) \neq 0, \quad \forall \lambda \in \sigma(N), \quad (34)$$

then the complete parametric solutions to the generalized discrete Sylvester matrix equation (1) can be expressed as

$$\begin{align*}
X &= Q_{c}(M, T, n)[S(d, n) \otimes I_{r}]Q_{o}(Z, N, n), \\
Y &= -(d_{0}Z + d_{1}ZN + \cdots + d_{\psi}ZN^{\psi}),
\end{align*} \quad (35)$$

where $Z \in \mathbb{R}^{r \times p}$ is an arbitrary parameter matrix and $d(s, n)$ is a polynomial in $s$ with

$$\begin{align*}
d_{n} = d_{n-1} = \cdots = d_{\psi+1} = 0, \\
d_{i} = \beta_{i}, \quad i = \psi, \psi - 1, \ldots, 1, 0. \quad (36)
\end{align*}$$

**Proof.** According to Lemma 1, we can see that

$$\begin{align*}
D_{i} &= d_{i}I = \beta_{i}I, \quad i = 0, 1, \ldots, \psi, \\
D_{i} &= d_{i}I = 0, \quad i = \psi + 1, \psi + 2, \ldots, n,
\end{align*}$$

satisfy (24) with $\omega = n$. Further, if (34) holds, then (19) and (26) hold automatically. By using the second conclusion of Theorem 2, we complete the proof. \qed
Based on the extended algorithm of Faddeev and Leverrier proposed in Lemma 2 and the above corollary, we have the following result which is similar to Theorem 2 proposed in Zhou & Duan (2005).

**Corollary 3** If (34) holds, then the complete parametric solution to the generalized discrete Sylvester matrix equation (1) can be expressed as

\[
X = \sum_{i=0}^{n-1} R_i T Z N^i, \quad Y = \sum_{i=0}^{\nu} \beta_i Z N^i,
\]

(37)

where \(Z \in \mathbb{R}^{r \times p}\) is an arbitrary parameter matrix, \(\beta_i, i = 0, 1, \ldots, n\), and \(R_i, i = 0, 1, \ldots, n-1\), satisfy (15) and (12).

**Proof.** Comparing (35) with (37), we need only to show that

\[
\sum_{i=0}^{n-1} R_i T Z N^i = -Q_c(M, T, n)[S(d, n) \otimes I_r]Q_o(Z, N, n).
\]

(38)

Using (15) and (12) produces

\[
\sum_{i=0}^{n-1} R_i T Z N^i = [R_0 T \quad R_1 T \quad \cdots \quad R_{n-1} T]Q_o(Z, N, n)
\]

\[
= [B_{n-1} T \quad B_{n-2} T \quad \cdots \quad B_0 T]Q_o(Z, N, n)
\]

\[
= [T \quad (M + \alpha_{n-1} I) T \quad \cdots \quad \sum_{i=1}^{n} \alpha_i M^{i-1} T]Q_o(Z, N, n)
\]

\[
= [T \quad MT \quad \cdots \quad M^{n-1} T][S(\bar{a}, n) \otimes I_r]Q_o(Z, N, n),
\]

(39)

where \(\alpha(s, n) = \sum_{i=0}^{n} \alpha_i s^i\) with \(\alpha_i, i = 0, 1, \ldots, n\), satisfying (12). Further, it follows from (15) and (36) that

\[
d(s, n) = \beta(s, n) = -\bar{\alpha}(s, n).
\]

Therefore, (39) reduces to (38). With this, we complete the proof. \(\square\)

**Remark 4** Parametric solutions given by Corollaries 1–3 are significantly simple than the results given in Theorems 1 and 2, however, it is worthy of pointing out that they are still not numerical reliable due to the fact that the algorithm of Faddeev and Leverrier is even badly conditioned. In addition, all these proposed solutions need to calculate some matrices in the form of \(Q_c(M, T, t)\), which involves the power of matrix \(M\) and will certainly accumulate the error in matrix \(M\). Therefore, due to their inherent shortcomings, we have not attempted to discuss any numerical and implementation issues in this paper.

### 3.3 The discrete Lyapunov matrix equations

With Corollary 2 proposed in Section 3.2, we can give the following result regarding the unique solution to the discrete Lyapunov matrix equation

\[
M X M^\top - X = -T T^\top.
\]

(40)
Corollary 4 Let $M \in \mathbb{R}^{n \times n}$ and $\lambda_i, i = 1, 2, \ldots, \psi$, be the roots of polynomial equation $\beta(s, \psi) = \det(sM - I) = 0$. Denote a polynomial $d(s, n)$ in the form of (5) with coefficient $d_i, i = 0, 1, \ldots, n$, satisfying (36). If

$$\lambda_i \lambda_j \neq 1, \quad i, j = 1, 2, \ldots, \psi,$$

then the unique solution to the discrete Lyapunov matrix equation (40) is given by

$$X = Q_c(M, T, n)[S(d, n) \otimes I_r]Q_o(Z^\top, M^\top, n), \quad (42)$$

where

$$Z = \left[\sum_{i=0}^{\psi} \beta_i M^i \right]^{-1} \, T. \quad (43)$$

Proof. Since (41) holds, the condition (34) is true. Such condition also implies that $\sum_{i=0}^{\psi} \beta_i M^i$ is nonsingular. By applying Corollary 2, we can obtain (42) and (43) immediately. □

Remark 5 It is easy to show that the condition (41) is equivalent to the classical result that the discrete Lyapunov matrix equation (40) has an unique solution if and only if

$$\mu_i \mu_j \neq 1, \quad \forall \mu_i, \mu_j \in \sigma(M), \quad i, j = 1, 2, \ldots, n.$$

The explicit solution (42) has a very nice form. But this representation is useless for the computation of $X$ (especially for large $n$), since it involves computing the matrices $M^j$ for $j = 1, 2, \ldots, n$, and the coefficients $\beta_i$ of the polynomial $\det(sM - I)$. For a numerical reliable computation of $X$, the quadratically convergent squared Smith method is more appropriate.

Remark 6 The unique solution to the following Sylvester matrix equation:

$$MXN - X = TS,$$

corresponding to Corollary 4, can also be readily obtained.

4. Parametric pole assignment for descriptor linear systems by P-D feedback

As an application of the proposed solution to the generalized discrete Sylvester matrix equation (1), in this section we consider the parametric pole assignment in the following descriptor linear system:

$$E \dot{x} = Ax + Bu, \quad (44)$$

where $A, E \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times r}$ are known coefficient matrices. When the following P-D state feedback:

$$u = -K_px - K_d \dot{x}, \quad (45)$$

is applied to the system (44), the following closed-loop system is resulted in:

$$(E + BK_d) \dot{x} = (A - BK_p)x. \quad (46)$$
It follows from Duan & Patton (1997) that the closed-loop system (46) is regular if and only if
\[
\det(E + BK_d) \neq 0. \tag{47}
\]
Therefore, when the closed-loop system is assumed to be regular, the closed-loop system (46) can be rewritten as
\[
\dot{x} = A_c x, \quad A_c = (E + BK_d)^{-1}(A - BK_p). \tag{48}
\]
We now state the parametric pole assignment by P-D feedback as follows.

**Problem 1** Given the descriptor linear system (44) and a set of desired closed-loop poles \( s_i, i = 1, 2, \ldots, n \), find a parameterization of all the matrices \( K_p \) and \( K_d \) satisfying the regular constraint (47) and
\[
\sigma(A_c) = \sigma((E + BK_d)^{-1}(A - BK_p)) \quad = \{s_i, i = 1, 2, \ldots, n\}. \tag{49}
\]

Regarding the solution to the above parametric pole assignment by P-D feedback, we first give the following lemma.

**Lemma 8** Problem 1 has a solution if and only if there exists a matrix \( K_d \) satisfying (47) and a nonsingular matrix \( X \) satisfying the generalized Sylvester matrix equation (2) with
\[
\sigma(F) = \sigma(A_c) = \{s_i, i = 1, 2, \ldots, n\}. \tag{50}
\]
When (47) and (50) are satisfied, all the feedback gain matrix \( K_p \) can be parameterized as
\[
K_p = (Y - K_d XF)X^{-1}. \tag{51}
\]

**Proof.** Note that the regularity constraint (47) is necessary for the existence of \( n \) finite poles of closed-loop system (46). Further, (49) is satisfied if and only if there exists a nonsingular matrix \( X \) and a matrix \( F \) satisfying (50) such that
\[
X^{-1}[(E + BK_d)^{-1}(A - BK_p)]X = F,
\]
which is equivalent to the generalized Sylvester matrix equation (2) with
\[
Y = K_p X + K_d XF.
\]
Since \( X \) is nonsingular, the above equation is equivalent to (51). \( \square \)

It follows from the above lemma that the pivotal step of solving Problem 1 is to solve the generalized Sylvester matrix equation (2). The following lemma shows us that such generalized Sylvester matrix equation (2) is equivalent to the generalized discrete Sylvester matrix equation (1) discussed in the past sections.

**Lemma 9** The generalized Sylvester matrix equation (2), where \( A, E \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r} \) and \( F \in \mathbb{R}^{p \times p} \) are known and the matrix pair \((E, A)\) is regular, is equivalent to the generalized discrete Sylvester matrix equation (1) with
\[
M = (\gamma E - A)^{-1}E, \quad N = \gamma I - F, \quad T = (\gamma E - A)^{-1}B, \tag{52}
\]
where \( \gamma \) is an arbitrary scalar such that \((\gamma E - A)\) is nonsingular.
Proof. Since the matrix pair \((E, A)\) is regular, there exists a scalar \(\gamma\) such that \((\gamma E - A)\) is nonsingular. Premultiplying (2) by \((\gamma E - A)^{-1}\) produces

\[
(\gamma E - A)^{-1}AX - (\gamma E - A)^{-1}EXF = (\gamma E - A)^{-1}BY. \tag{53}
\]

Let \(M = (\gamma E - A)^{-1}E\) and note that

\[
\gamma M - (\gamma E - A)^{-1}A = \gamma (\gamma E - A)^{-1}E - (\gamma E - A)^{-1}A = (\gamma E - A)^{-1}(\gamma E - A) = I,
\]

we have \((\gamma E - A)^{-1}A = \gamma M - I\). So (53) is equivalent to

\[
(\gamma M - I)X - MXF = TY
\]
or

\[
MX(\gamma I - F) - X = TY.
\]

Let \(N = \gamma I - F\), then the above equation is reduced to (1). \(\square\)

Remark 7 Regarding general solutions to the generalized Sylvester matrix equation (2), the existing results are only given under the assumption that \(F\) is in Jordan form (Tsui, 1987; Volker & Xu, 1997). It follows from the above lemma that the generalized Sylvester matrix equation (2) can be solved for arbitrary square matrix \(F\) since it is equivalent to the generalized discrete Sylvester matrix equation (1) which can be solved for arbitrary matrix \(N = \gamma I - F\) in this paper. Also, in these references mentioned above, the eigenvalues of \(F\) need to be known priori but in this paper the eigenvalues of matrix \(N\) do not need to.

With the complete solution to the generalized discrete Sylvester matrix equation (1) by Corollary 2, we have the following result about solutions to Problem 1.

Theorem 3 Let the closed-loop eigenvalues be different from the open-loop finite eigenvalues, i.e.

\[
\sigma(E, A) \cap \sigma(A_c) = \sigma(E, A) \cap \sigma(F) = \emptyset, \tag{54}
\]

then

1. Problem 1 has a solution if and only if there exists a matrix \(Z \in \mathbb{R}^{r \times n}\) such that

\[
\det(Q_c(M, T, n)[S(\beta, n) \otimes I_r]Q_o(Z, N, n)) \neq 0, \tag{55}
\]

where \(M, T\) and \(N\) are defined in (52) and \(\beta(s, \psi)\) defined in (9) is the inverse characteristic polynomial of matrix \(M\);

2. when the above condition is met, all the matrix \(K_p\) to Problem 1 can be parameterized as (51) with \(K_d\) satisfying (47) and \(X, Y\) in the form of (35).

Proof. We only need to show that when (54) holds, the solution given in (35) is complete. According to Corollary 2, (35) is the complete solutions to matrix equation (1) if

\[
\det(\lambda M - I) \neq 0, \quad \forall \lambda \in \sigma(N),
\]
which is equivalent to
\[
\det[\lambda(\gamma E - A)^{-1}E - I] \neq 0, \quad \forall \lambda \in \sigma(\gamma I - F),
\]
or
\[
\det[(\gamma - \lambda)E - A] \neq 0, \quad \forall (\gamma - \lambda) \in \sigma(F).
\]
This is obviously equivalent to (54). □

It follows from the above theorem that the nonsingularity of matrix \(X\) is crucial to the P-D feedback pole assignment problem. To give a necessary condition, we firstly give the following two lemmas.

**Lemma 10** The matrix pair \((M, T)\) with \(M\) and \(T\) defined in (52) is controllable, i.e. (Kailath, 1980)
\[
\text{rank}[sI - M T] = n, \quad \forall s \in \mathbb{C},
\]
if and only if the matrix triple \((E, A, B)\) is C-controllable, i.e.
\[
\text{rank}[sE - A B] = n, \quad \forall s \in \mathbb{C},
\]
and
\[
\text{rank}[E B] = n.
\]

**Lemma 11** Let \(Z \in \mathbb{R}^{r \times p}\) and \(N\) and \(F\) satisfy the second equation in (52), then \((Z, N)\) is observable if and only if \((Z, F)\) is observable.

**Proof.** It is well known that the matrix pair \((Z, N)\) is observable if and only if
\[
\text{rank}\begin{bmatrix} sI - N \\ Z \end{bmatrix} = p, \quad \forall s \in \mathbb{C}.
\]
Substituting \(N = \gamma I - F\) into the above equation yields
\[
\text{rank}\begin{bmatrix} sI - (\gamma I - F) \\ Z \end{bmatrix} = \text{rank}\begin{bmatrix} (\gamma - s)I - F \\ Z \end{bmatrix} = p, \quad \forall s \in \mathbb{C},
\]
which is equivalent to
\[
\text{rank}\begin{bmatrix} \lambda I - F \\ Z \end{bmatrix} = p, \quad \forall \lambda \in \mathbb{C}.
\]
Obviously, (58) is equivalent to the observability of matrix pair \((Z, F)\). □

**Proposition 1** Let the closed-loop eigenvalues be different from the open-loop eigenvalues, then Problem 1 has a solution only if \((E, A, B)\) is C-controllable and there exists a matrix \(Z\) such that \((Z, F)\) is observable where \(F\) satisfies (50).

**Proof.** Note that \(X\) is nonsingular only if there exists a scalar \(\omega\) such that \(Q_c(M, T, \omega + 1)\) has full row rank and \(Q_o(Z, N, \omega + 1)\) has column rank. Obviously, \(Q_c(M, T, \omega + 1)\) has full row rank if and only if matrix pair \((M, T)\) is controllable, and \(Q_o(Z, N, \omega + 1)\) has column rank if and only if the
matrix pair \((Z, N)\) is observable, respectively. According to Lemmas 10 and 11, the matrix pair \((M, T)\) is controllable if and only if the matrix triple \((E, A, B)\) is C-controllable, and the matrix pair \((Z, N)\) is observable if and only if the matrix pair \((Z, F)\) is observable, respectively. □

So in order to assign the closed-loop poles arbitrary, the C-controllability of the matrix triple \((E, A, B)\) must be guaranteed firstly. However, a sufficient and necessary condition for the nonsingularity of matrix \(X\) is difficult to seek. One can refer to Zhou & Duan (2005).

For the solutions to matrix \(K_d\), we give in the following proposition all the parametric solution \(K_d\) satisfying (47). Without loss of generality, we assume that the matrix \(B\) has full column rank.

**Proposition 2** There exists a matrix \(K_d\) such that (47) holds if and only if (57) holds. Further, let (57) hold and \(U \in \mathbb{R}^{n \times n}\) be a nonsingular matrix such that

\[
UB = \begin{bmatrix} \Pi_b \\ 0 \end{bmatrix}, \quad UE = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},
\]

where \(\Pi_b \in \mathbb{R}^{r \times r}\), then all the matrix \(K_d\) satisfying (47) can be characterized as

\[
K_d = \Pi_b^{-1} (P - E_1)
\]

with \(P \in \mathbb{R}^{r \times n}\) satisfying

\[
\text{Im}(E_2^\top) \oplus \text{Im}(P^\top) = \mathbb{R}^n.
\]

**Proof.** It follows from (47) that the derivative feedback gain matrix \(K_d\) must satisfy

\[
\det[\begin{bmatrix} E & B \\ I & K_d^\top \end{bmatrix}] \neq 0.
\]

Premultiplying the matrix in \(\det(\cdot)\) by \(U\) and applying (59) yield

\[
\det \begin{bmatrix} E_1 & \Pi_b \\ E_2 & 0 \end{bmatrix} \begin{bmatrix} I \\ K_d \end{bmatrix} \neq 0.
\]

The above inequality holds if and only if

\[
E_2 = Q \quad \text{and} \quad E_1 + \Pi_b K_d = P,
\]

where

\[
Q \in \mathbb{R}^{(n-r) \times n}, \quad P \in \mathbb{R}^{r \times n}, \quad \text{Im}(Q^\top) \oplus \text{Im}(P^\top) = \mathbb{R}^n.
\]

Obviously, \(Q \in \mathbb{R}^{(n-r) \times n}\) and \(\Pi_b \in \mathbb{R}^{r \times r}\) imply (57). Further, if (57) holds, there always exists a matrix \(P\) such that (63) holds. With this, (60) and (61) can be deduced from (62) and (63) immediately. □

5. Illustrative example

To illustrate our approaches of solving the generalized discrete Sylvester matrix equation \(MXN - X = TY\) with their applications on parametric pole assignment for descriptor linear systems by P-D state
feedback, we consider a regular descriptor linear system in the form of (44) with system matrices in the following (borrowed from Duan & Patton, 1997, and Chen & Chang, 1993):

\[
E = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 1 \\
0 & 0
\end{bmatrix}.
\]

It is easy to verify that such a system is C-controllable. Choose \( \gamma = 0 \), then according to (52), we get \( M \) and \( T \) as

\[
M = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad T = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
-1 & 0 \\
0 & 0 \\
-1 & -1 \\
0 & 0
\end{bmatrix}.
\]

The inverse characteristic polynomial of \( M \) is

\[
\beta(s, \psi) = \beta(s, 3) = \det(sM - I) = s^3 + 1.
\]

So according to (36), we have

\[
d(s, n) = d(s, 6) = 6s^6 + 0s^5 + 0s^4 + s^3 + 1
\]

and

\[
[S(d, 6) \otimes I_2] = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \otimes I_2.
\]

We now consider a P-D feedback law in the form of (45). For simplicity, we select the closed-loop eigenvalue set as

\[
\sigma(A_c) = \sigma(F) = \{-1, -1, -2, -2, -1 \pm i\}.
\]
So matrix \( F \) can be chosen as

\[
F = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -2 & -2 \\
\end{bmatrix}, \quad N = \gamma I_6 - F = -F.
\]

5.1 Parametric solutions to matrices \( X \) and \( Y \)

It is easy to verify that the condition (34) holds. So according to Corollary 2, a complete parametric solution to the generalized discrete Sylvester matrix equation (1) can be expressed as

\[
\begin{align*}
X &= Q_c(M, T, 6)[S(d, 6) \otimes I_2]Q_o(Z, N, 6), \\
Y &= -Z - ZN^3,
\end{align*}
\]

(64)

where \( Z \in \mathbb{R}^{2 \times 6} \) is an arbitrary parameter matrix. In order to guarantee that \( X \) is nonsingular, we must have

\[
(z_{25}^2 - 2z_{26}z_{25} + 2z_{26}^2)(z_{22}z_{11} - z_{21}z_{12})(-z_{14}z_{23} + z_{13}z_{24}) \neq 0.
\]

(65)

5.2 Parametric solutions to derivative gain matrix \( K_d \)

The matrices \( U, E_1, E_2 \) and \( \Pi_b \) defined in (59) can be obtained as

\[
U = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad [E_1 E_2] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad \Pi_b = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix}.
\]

Then, according to Proposition 2, all the derivative gain matrix \( K_d \) can be parameterized as

\[
K_d = \Pi_b^{-1}(P - E_1)
\]

(66)

\[
= \begin{bmatrix}
p_{11} - 1 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\
p_{21} + 1 - p_{11} & p_{22} - p_{12} & p_{23} - p_{13} & p_{24} - p_{14} & p_{25} - p_{15} & p_{26} - p_{16}
\end{bmatrix},
\]

where \( P \in \mathbb{R}^{2 \times 6} \) is a parameter matrix satisfying

\[
p_{26}p_{14} - p_{16}p_{24} \neq 0.
\]

(67)
5.3 Parametric solutions to proportional gain matrix $K_p$

According to Lemma 8, all the parametric solutions of the proportional matrix $K_p$ can be obtained as

$$K_p = (Y - K_dXF)X^{-1},$$

where $K_d$ is given in (66) with the constraint (67) and $X, Y$ given in (64) with the constraint (65). By specially choosing

$$Z = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we have

$$K_d = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix},$$

$$K_p = \begin{bmatrix} -11 & -13 & -5 & 3 & 8 & -2 \\ 27 & 23 & 9 & -3 & -11 & 8 \end{bmatrix}.$$

6. Conclusions

An explicit parametric solution to the generalized discrete Sylvester matrix equation

$$MXN - X = TY,$$

with $M \in \mathbb{R}^{n \times n}, T \in \mathbb{R}^{n \times r}$ and $N \in \mathbb{R}^{p \times p}$ some known matrices, is established, which possesses the following two features:

1. it is in a very neat form and can be immediately obtained as soon as a series of matrices $D_i, i = 0, 1, \ldots, \omega$, satisfying (24) is established;

2. it gives all the degrees of freedom to the equation, which are represented by the parameter matrix $Z$.

As a demonstration of the proposed explicit solutions, parametric pole assignment for descriptor linear systems by P-D feedback is considered. A numerical example shows the effectiveness of the proposed method.

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