A novel nested non-linear feedback law for global stabilisation of linear systems with bounded controls

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A novel nested non-linear feedback law for global stabilisation of a chain of integrators with bounded controls is proposed, which can be directly applied on the system without state transformation as soon as a so-called SP-stable polynomial is constructed. Then the so-called state-dependent saturation function is introduced into this novel nested non-linear feedback law to replace the standard saturation function, which can significantly improve the convergence performance of the closed-loop system. The results are then extended to a wider class of linear system that can be globally stabilised by bounded controls. Two numerical examples show the effectiveness of the proposed approach.

Keywords: bounded controls; non-linear control; nested non-linear functions; state-dependent saturation functions; global stabilisation

1. Introduction

Practical control systems are subject to input constraint. For the special case of time invariant linear systems subject to input saturation, several important control problems have been solved. Among these problems are global stabilisation (see Teel (1992), Sussmann et al. (1994), Suarez et al. (1997) and Marchand and Hably (2005) and the references therein), semi-global stabilisation (Lin and Saberi 1993; Lin et al. 1996), output regulation (Santis and Isidori 2001), servo systems design (Chen et al. 2003), input-output stabilisation (Liu et al. 1996), estimation of attraction domain (Hu and Lin 2002) and robust stabilisation (Saberi et al. 1996). Besides time invariant linear systems with input constraints, non-linear systems, non-autonomous linear systems and uncertain systems with input constraints are also well investigated in the literature (see, for example, Soldatos and Corless (1991), Sun et al. (1998), Kaliora and Astolfi (2004) and Phat and Niamsup (2006)). See Bernstein and Michel (1995) for a summarisation. It is well-known that, for global stabilisation problem, a time invariant linear subject to input saturation is globally stabilisable if and only if the system in the absence of the input saturation is asymptotically null controllable with bounded controls (ANCBC), equivalently, it is stabilisable in the ordinary sense and has all its poles located in the closed left plane. It has been shown in Sussmann and Yang (1991) that a simple linear system of a chain of integrators of length $n \geq 2$ which is ANCBC, can not be globally stabilised by saturated linear feedback. Thus, for general linear systems, non-linear feedback is required. Still for the multiple integrators case, Teel (1992) proposed a non-linear state feedback law of nested saturation type which not only solve the global stabilisation but also can be used to achieve trajectory tracking for a class of bounded trajectories. This technique of using nested saturation functions was latterly successfully applied to achieve global stabilisation of general ANCBC linear systems in Sussmann et al. (1994).

In this paper, we consider the global stabilisation problem of linear systems with bounded controls. Firstly, we propose a novel nested non-linear feedback law for global stabilisation of multiple integrators system with bonded controls. This type of nested non-linear feedback laws is quite different from that
proposed in Teel (1992), Sussmann et al. (1994), Suarez et al. (1997) and Marchand and Hably (2005) for the reason that no state transformation is needed to construct such feedback law. Furthermore, the eigenvalues of the closed-loop system operating in linear form can be complex numbers, while in Teel (1992), Sussmann et al. (1994), Suarez et al. (1997) and Marchand and Hably (2005) only real numbers are allowed, and can be well assigned to improve system performances.

Secondly, we introduce the so-called state-dependent saturation functions firstly proposed in Marchand and Hably (2005) to replace the standard saturation functions appearing in this novel non-linear feedback law. Such state-dependent saturation functions can increase control energy if the states are badly scaled, and thus can significantly improve the performance of the closed-loop system.

Thirdly, we extend our novel nested non-linear feedback law to a wider class of ANCBC linear system by assuming that the imaginary eigenvalues of the closed-loop system operating in linear form can be complex numbers, while in Teel (1992), Sussmann et al. (1994), Suarez et al. (1997) and Marchand and Hably (2005) only real numbers are allowed, and can be well assigned to improve system performances.

The remainder of this paper is organised as follows. The main results of this paper are presented in §2 which contains three subsections. In §2.1, we define and study the so-called \( P \)-stable and \( SP \)-stable polynomials that will be used in the sequel of this paper.

### Definition 1:

Denote a stable polynomial

\[
a(s) = \sum_{j=0}^{n-1} a_{j+1} s^j.
\]

Then we define a series of polynomials \( a_i(s), i \in \mathbb{I}[1,n] \) as follows:

\[
\begin{align*}
    a_0(s) &= s + a_0 \\
    a_{n-1}(s) &= s^2 + a_0 s + a_{n-1} \\
    & \vdots \\
    a_3(s) &= s^{n-1} + a_0 s^{n-2} + \cdots + a_3 s + a_2 \\
    a_1(s) &= s^n + a_0 s^{n-1} + \cdots + a_2 s + a_1 = a(s).
\end{align*}
\]

If \( a_i(s), i \in \mathbb{I}[1,n] \) are stable, then the polynomial \( a(s) \) is called a \( P \)-stable polynomial.

It follows from the definition that if \( a_i(s) \) is \( P \)-stable, then \( a_i(s), i \in \mathbb{I}[2,n] \) are all \( P \)-stable.

Let polynomials \( a_i(s), i \in \mathbb{I}[1,n] \) be defined as (1) and \( A_i \in \mathbb{R}^{m \times n + 1} \), \( i \in \mathbb{I}[1,n] \) be companion matrices associated with polynomials \( a_i(s), i \in \mathbb{I}[1,n], \) i.e.,

\[
A_i = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_i & -a_{i+1} & -a_{i+2} & \cdots & -a_n
\end{bmatrix}.
\]

Since \( A_i, i \in \mathbb{I}[1,n] \) are all stable, \( a_i, i \in \mathbb{I}[1,n] \) are all positive numbers which implies that \( A_i, i \in \mathbb{I}[1,n] \) are diagonal matrix with the diagonal elements being \( A \) and \( B \), and \([p,q]\) to denote the set \( \{p,p+1,\ldots,q\} \), respectively. Moreover, the symbol \( \text{sign}(.) \) denotes the sign function which takes value +1 and −1. With this notation, the standard saturation function can be defined as \( \text{sat}(u) = \text{sign}(u \min(|u|,1)} \) and the saturation function of level \( a > 0 \) can be defined as \( \text{sat}_a(y) = a \text{sat}(y/a) \), respectively. The symbol \( \text{He}(A) \) refers to the Hermite matrix \( A + A^T \).

## 2. Main results

### 2.1 The \( P \)-stable and \( SP \)-stable polynomials

In this subsection we will define and study the so-called \( P \)-stable polynomials and \( SP \)-stable polynomials that will be used in the sequel of this paper.

### Notations:

Throughout this paper, the state vectors are denoted by bold symbols. We use \( y_i \) to denote the \( i \)th row of the state vector \( y \), \( A^T \) to denote the transpose of matrix \( A \), \( \text{diag}(A,B) \) to denote
all non-singular. In fact
\[
A_i^{-1} = \begin{bmatrix}
\frac{a_{i+1}}{a_i} & \frac{a_{i+2}}{a_i} & \cdots & \frac{a_n}{a_i} & \frac{1}{a_i} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
0 & 1 & \cdots & 1 & 0
\end{bmatrix}.
\] (3)

Furthermore, we define the following matrices associated with \(a_i(s), i \in I[1, n]::
\[
h_i = [0 \cdots 0 1]^T \in \mathbb{R}^{(n-i+1) \times 1}
g_i = [1 0 \cdots 0] \in \mathbb{R}^{1 \times (n-i+1)}
c_i = \begin{bmatrix}
a_{i+1} \\
a_i \\
a_{i+2} \\
\vdots \\
\vdots \\
a_I \\
\end{bmatrix} \in \mathbb{R}^{1 \times (n-i+1)}.
\] (4)

For a series of scalars \(\varepsilon_n \geq \varepsilon_{n-1} \geq \cdots \geq \varepsilon_0\), we define
\[
\varepsilon = [\varepsilon_0 \varepsilon_1 \cdots \varepsilon_n]^T \in \mathbb{R}^{n+1}.\] (5)

We are now ready to give the following result.

**Proposition 1:** Let \(\varepsilon \in \mathbb{R}^{n+1}\) be a vector defined as (5). If \(a_i(s), i \in I[2, n]\) is a stable polynomial, then there exists a positive definite matrix \(P_i \in \mathbb{R}^{(n-i+1) \times (n-i+1)}\) and a positive scalar \(\gamma_i, i \in I[2, n]\) such that
\[
\begin{bmatrix}
\text{He}(A_i^T P_i + \gamma_i g_i^T c_i) & P_i h_i \\
\gamma_i g_i P_i & -1
\end{bmatrix} < 0.
\] (6)

Furthermore, if \(a_{i-1}, i \in I[2, n]\) is chosen such that
\[
\gamma_i > \frac{\varepsilon_i}{2 \varepsilon_{i-2}} a_{i-1}, \quad i \in I[2, n],\]
or equivalently
\[
0 < a_{i-1} < \frac{2\varepsilon_i - 2\varepsilon_i^2}{\varepsilon_{i-2}} a_{i-1}, \quad i \in I[2, n],\] (7)

then the polynomial \(a_i(s) = sa_i(s) + a_{i-1}, i \in I[2, n]\) is also a stable polynomial.

**Proof:** Note that (6) is equivalent to
\[
\text{He}(A_i^T P_i + \gamma_i g_i^T c_i) + P_i b_i^T P_i < 0
\]
and which is also equivalent to
\[
\text{He}(K_i A_i^T) + \gamma_i K_i \text{He}(g_i^T c_i) K_i + b_i b_i^T < 0
\]
by denoting \(P_i^{-1} = K_i\). Since \(A_i\) is stable, there exists matrix \(K_i\) such that
\[
K_i A_i^T + A_i K_i + b_i b_i^T < 0
\]
\[
K_i > 0.
\]

Obviously, (10) is true if \(\gamma_i\) is sufficiently small.

We now prove the second conclusion. Note that the companion matrix \(A_{i-1}\) can be rewritten as
\[
A_{i-1} = \begin{bmatrix}
0 & g_i \\
-a_{i-1} b_i & A_i
\end{bmatrix}.\] (11)

Denote a symmetric matrix as follows:
\[
Q_{i-1} = \begin{bmatrix}
\gamma_i & y_i \\
\gamma_i^T & P_i
\end{bmatrix},\] (12)

where \(y_i\) will be determined later and \(\gamma_i, P_i\) are as that in (6) and (8). Using (11) and (12), we have
\[
Q_{i-1} A_{i-1} + A_{i-1}^T Q_{i-1} = \begin{bmatrix}
-2a_{i-1} y_i b_i & y_i g_i + y_i A_i - a_{i-1} b_i^T P_i \\
y_i g_i^T + A_i^T y_i - a_{i-1} P_i b_i & \text{He}(P_i A_i + y_i g_i c_i)
\end{bmatrix}
\] (13)

Set \(y_i g_i^T + y_i A_i = 0\) which is equivalent to \(y_i = -y_i g_i A_i^{-1}\).

Substituting this relation into (13) and using (3), gives
\[
Q_{i-1} A_{i-1} + A_{i-1}^T Q_{i-1} = T \begin{bmatrix}
-2\gamma_i & P_i b_i \\
\gamma_i^T & P_i b_i
\end{bmatrix} T,
\]

where \(T = \text{diag}(-a_{i-1}, I_{n-i+1})\). Using the first inequality in (9) and the inequality (7), we get
\[
\text{He}(P_i A_i + y_i g_i c_i) + \frac{a_{i-1} a_{i-1}^T}{2\gamma_i} P_i b_i b_i^T P_i < 0
\]
\[
< \text{He}(P_i A_i + y_i g_i c_i) + \frac{a_{i-1} a_{i-1}^T}{2\gamma_i} P_i b_i b_i^T P_i
\]
\[
\leq \text{He}(P_i A_i + y_i g_i c_i) + P_i b_i b_i^T P_i < 0.
\]

By using the Schur complement, the matrix in the right hand side of (14) is negative definite. So we have
\[
Q_{i-1} A_{i-1} + A_{i-1}^T Q_{i-1} < 0.
\]

Further by using \(y_i = -y_i g_i A_i^{-1}\) and (6) we obtain
\[
P_i - \frac{1}{\gamma_i} y_i^T y_i = P_i - y_i^T c_i,
\]

which implies that \(Q_{i-1}\) defined as (12) is positive definite. Thus we can conclude that \(A_{i-1}\) is stable. \(\square\)
Remark 1: It follows from (7) and the definition of $\varepsilon$ that
\[
\gamma_i > \frac{\varepsilon_i}{2\varepsilon_i - 2} a_i a_{i-1} \geq \frac{\varepsilon_i - 2}{2\varepsilon_i - 2} a_i a_{i-1} = \frac{a_i a_{i-1}}{2}.
\] (15)

Proposition 1 in fact implies a method to construct a $\mathcal{P}$-stable polynomial $a(s)$ with degree $n$.

Corollary 1: Let $\varepsilon \in \mathbb{R}^{n+1}$ be a vector defined as (5). Suppose that the polynomial $a_i(s)$ as in (1) is stable. Solve the optimization problem
\[
\max_{p_i, \gamma_i > 0} \gamma_i \quad \text{s.t.} \quad (6),
\] and get a maximal value $\gamma_i^{\text{max}} > 0$. Then set $0 < a_{i-1} < (2\varepsilon_i - 2\gamma_i^{\text{max}})/\varepsilon_i a_i$. This procedure is initialised with $i = n$ and $a_0(s) = s + a_0$ where $a_0 > 0$ can be arbitrarily chosen. Then $a_i(s)$ is $\mathcal{P}$-stable with degree $n$.

Such optimisation problem (16) is a standard LMIs based optimisation problem and can be well solved by some software such as Matlab (Boyd et al. 1994). For a particular purpose, we give the following definition.

Definition 2: Let $\varepsilon \in \mathbb{R}^{n+1}$ be a vector defined as (5). A $\mathcal{P}$-stable polynomial $a_i(s)$ produced by Corollary 1 is called an $\mathcal{SP}$-stable polynomial with respect to $\varepsilon_i$ and is denoted by $a_i(s)$.

Remark 2: It follows from Corollary 1 that all the polynomials $a_i(s), i \in \{1, n\}$ are $\mathcal{SP}$-stable w.r.t. $\varepsilon_i$ according to Corollary 1 as follows:
\[
a_i(s) = s^3 + 2.0000s^2 + 0.9496s + 0.2185,
\] which has eigenvalue set $\{-1.4486, -0.2757 \pm 0.2736i\}$.

2.2 Multiple integrators system

In this subsection we consider the following $n$th order integrators system
\[
\Sigma_n: \dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \ldots, \dot{x}_{n-1} = x_n, \quad \dot{x}_n = u,
\] (19)
with bounded input
\[
-u_{\text{max}} \leq u \leq u_{\text{max}},
\] (20)
where $u_{\text{max}}$ is a priori known strictly positive real number and denotes the control amplitude limitation.

Such kind of system is well studied in the literature (Lin 1995, Zhu and Krishnan 1997, Rao and Bernstein 2001, Bozano and Dimogianopoulos 2003, Kannan and Johnsony 2003). The problem considered in this paper is stated as follows.

Problem 1: Find a function $u = u(x)$ satisfying the amplitude constraint (20) such that the system (19) is globally asymptotically stable, i.e., for arbitrarily given initial condition $x(0) \in \mathbb{R}^n$ and a positive scalar $\varepsilon$, the closed-loop system (19) with $u = u(x)$ has a unique and bounded solution $x(x(0), t)$, and there exists a number $T$ such that for arbitrary $t > T$, there holds $\|x(x(0), t)\| < \varepsilon$.

For simplicity, we give some notations associated with the system (19). Firstly, for $i \in \{1, n-1\}$, we define
\[
\xi_{i+1} = [x_{i+1} \cdots x_n]^T, \quad y_i = [x_i \xi_i^T]^T
\]
and two series of matrices $A_i^w, A_i^d \in \mathbb{R}^{(n-1)\times(n-i+1)}$ for $i \in \{1, n-1\}$ as follows:
\[
\begin{align*}
A_i^w &= \begin{bmatrix}
0 & 1 & \cdots & 0 \\
& 0 & \ddots & \vdots \\
& & \ddots & 1 \\
& & & 0 \end{bmatrix}, & A_i^d &= \begin{bmatrix}
0 & 0 & \cdots & 0 \\
& 0 & \ddots & \vdots \\
& & \ddots & 1 \\
& & & 0 \end{bmatrix},
\end{align*}
\]
With the notations $g_{i+1}, A_{i+1}$ and $b_{i+1}$ defined as in (2) and (4), respectively, we redefine
\[
\begin{align*}
A_i^g &= \begin{bmatrix} 0 & g_{i+1} \\ 0 & A_{i+1} \end{bmatrix} \in \mathbb{R}^{(n-i+1)\times(n-i+1)}, & b_i^g &= \begin{bmatrix} 0 \\ b_{i+1} \end{bmatrix},
\end{align*}
\]
for $i \in \{1, n-1\}$. Then we can define the following series of systems
\[
\Sigma_i: \begin{bmatrix} w_i \\ y_i \end{bmatrix} = \begin{bmatrix} A_i^w & A_i^d \end{bmatrix} \begin{bmatrix} w_i \\ y_i \end{bmatrix} + \begin{bmatrix} 0 \\ b_i^g \end{bmatrix} u_i,
\] (21)
for $i \in \{2, n-1\}$. When $i = 1$, we define
\[
\Sigma_1: \begin{bmatrix} w_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} A_1^w & A_1^d \end{bmatrix} \begin{bmatrix} w_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1^g \end{bmatrix} u_1,
\]
and when $i = 0$ we define
\[
\Sigma_0: \begin{bmatrix} w_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} A_0^w & A_0^d \end{bmatrix} \begin{bmatrix} w_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0^g \end{bmatrix} u_1,
\]
with $u_0 = 0$.

We then have the following result regarding solution to Problem 1.

Theorem 1: Consider the multiple integrators system (19). Let $\varepsilon_i, i \in \{0, n\}$ satisfy
\[
\varepsilon_i > 2\varepsilon_{i-1}, i \in \{1, n\}, \quad \varepsilon_0 = \varepsilon_1, \varepsilon_n < u_{\text{max}},
\] (22)
and $a_i(s) = \sum_{j=0}^{n-1} a_{i+j} s^j$ be an SP-stable polynomial w.r.t. $s$. Then the control law given by $u(x) = u_n$ with

$$u_i = -\varepsilon_i \text{sat} \left( \frac{a_i x_i}{\varepsilon_i} - \frac{1}{\varepsilon_i} u_{i-1} \right), \quad i \in \{2, n\}, \quad u_1 = -\varepsilon_1 \text{sat} \left( \frac{a_1 x_1}{\varepsilon_1} \right),$$

(23)

solves Problem 1. Furthermore, there exists a time $T > 0$ such that for arbitrary $t > T$, the closed-loop system (19) and $u_n$ will operate in linear form with characteristic polynomial $a_i(s)$.

**Proof:** Since $\varepsilon_i > 2\varepsilon_{i-1}, i \in \{2, n\}$, there exists a series of scalars $L_i, i \in \{2, n\}$ such that

$$\varepsilon_{i-1} < L_i a_i < \frac{\varepsilon_i}{2}, \quad i \in \{2, n\}.$$  

(24)

We prove this theorem by adopting the idea in Kaliora and Astolfi (2004).

**Step n.** We assume that $|x_0| > L_n$. Consider the Lyapunov function $V(x_n) = \frac{1}{2} x_n^2$. The time derivative of $V(x_n)$ along the trajectory of the closed-loop system is given by $\dot{V}(x_n) = x_n u_n$. In view of (24), we have

$$L_n a_n > \varepsilon_{n-1} \iff \frac{a_n}{\varepsilon_n} L_n > \frac{\varepsilon_{n-1}}{\varepsilon_n} \iff \text{sign}(u_n) = \text{sign} \left( -\varepsilon_n \text{sat} \left( \frac{a_n x_n}{\varepsilon_n} - \frac{1}{\varepsilon_n} u_n - 1 \right) \right) = -\text{sign}(x_n).$$

So $\dot{V}(x_n)$ is negative. According to the Lyapunov stability theory $x_n$ necessarily joints $[-L_n, L_n]$ at finite time. On the other hand, by using (22) and (24), we obtain

$$L_m a_n < \frac{\varepsilon_n}{2} \iff \frac{a_n}{\varepsilon_n} L_n > \varepsilon_{n-1} \iff \frac{\varepsilon_{n-1}}{\varepsilon_n} < 1.$$

So the control law (23) can be simplified as $u_n = -a_n x_n + u_{n-1}$. It is easy to verify that with this simplified feedback law, the system $\Sigma_n$ under the input (23) reduces to the system $\Sigma_{n-1}$.

**Step i, i = n-1, n-2, \ldots, 2.** We now consider the system $\Sigma_i$. Note that $\Sigma_i$ has a block form. Thus for system $\Sigma_i$, we focus on the subsystem with state $y_i$. Take the Lyapunov function

$$V(y_i) = y_i^T \begin{bmatrix} \gamma_i + 1 & I_{i+1} \\ I_{i+1}^T & P_{i+1} \end{bmatrix} y_i,$$

(25)

where $I_{i+1}$ is to be determined later, into consideration. The time derivative of $V(y_i)$ along the trajectory of the low subsystem of system (21) is given by

$$\dot{V}(y_i) = y_i^T \begin{bmatrix} 0 & v_{i+1} \\ v_{i+1}^T & -\Pi_{i+1} \end{bmatrix} y_i + 2y_i^T \begin{bmatrix} I_{i+1} b_{i+1}^T \\ P_{i+1} b_{i+1}^T \end{bmatrix} u_i,$$

(26)

where $v_{i+1} = \gamma_{i+1} g_{i+1} + t_{i+1} A_{i+1}$ and

$$\Pi_{i+1} = -\text{He} \left[ A_{i+1}^T P_{i+1} + t_{i+1}^T g_{i+1} \right].$$

(27)

Set $v_{i+1} = 0$, then (26) becomes

$$\dot{V}(y_i) = -\xi_i \Theta_i y_i^T + 2x_i t_{i+1} b_{i+1} u_i + u_i^2,$$

where $\xi_i = [u_i \quad \Theta_i]$, and

$$\Theta_i = \begin{bmatrix} 1 & P_{i+1} b_{i+1} \\ b_{i+1}^T P_{i+1} & \Pi_{i+1} \end{bmatrix}.$$  

Since $v_{i+1} = 0$ and $A_{i+1}$ is invertible, $t_{i+1}$ can be expressed as

$$t_{i+1} = -\gamma_{i+1} g_{i+1} A_{i+1}^{-1},$$

(28)

With this expression, the centre matrix in (25) is positive definite since

$$P_{i+1} = \frac{1}{\gamma_{i+1}} t_{i+1}^T t_{i+1},$$

$$= P_{i+1} - \gamma_{i+1} A_{i+1}^T g_{i+1} A_{i+1}^{-1},$$

$$= P_{i+1} - \gamma_{i+1} c_{i+1}^T c_{i+1} > 0.$$  

Substituting (28) into (27) gives

$$-\Pi_{i+1} = \text{He} \left[ A_{i+1}^T P_{i+1} + \gamma_{i+1} A_{i+1}^T g_{i+1} A_{i+1}^{-1} \right] = \text{He} \left( A_{i+1}^T P_{i+1} + \gamma_{i+1} c_{i+1}^T c_{i+1} \right),$$

where $g_{i+1}$ and $c_{i+1}$ are defined in (4). Since the polynomial $a_i(s)$ is SP-stable w.r.t. $s$, the matrix $\Theta_i$ is positive definite. Hence by using (28) again, we obtain

$$\dot{V}(y_i) < 2x_i t_{i+1} b_{i+1} u_i + u_i^2 = u_i (2t_{i+1} b_{i+1} x_i + u_i) = u_i \left( \frac{2\gamma_{i+1}}{a_{i+1}} x_i + u_i \right).$$

(29)

Noting that the polynomial $a_i(s)$ is SP-stable w.r.t. $s$, we have

$$y_{i+1} > \frac{\varepsilon_{i+1}}{2\varepsilon_{i+1}} a_{i+1} a_i \iff \frac{\varepsilon_{i+1} a_{i+1}}{2\gamma_{i+1}} < \frac{\varepsilon_{i+1}}{a_i}.$$

In view of the inequality (24), we further have

$$\frac{\varepsilon_{i+1} a_{i+1}}{2\gamma_{i+1}} \cdot \frac{\varepsilon_{i-1}}{a_i} \cdot \frac{\varepsilon_{i+1}}{a_i} \cdot \frac{\varepsilon_{i+1}}{a_i} \cdot \frac{\varepsilon_{i+1}}{a_i} \cdot \frac{\varepsilon_{i+1}}{a_i} \cdot \frac{\varepsilon_{i+1}}{a_i} < \frac{\varepsilon_{i+1}}{a_i}.$$
Assume that $|x_i| > L_i$. Then we can obtain
\[
\begin{align*}
\text{sign} \left( \frac{2y_{i+1}}{a_{i+1}} x_i + u_i \right) \\
= \text{sign} \left( \frac{2y_{i+1}}{a_{i+1}} x_i - \varepsilon_i \text{sat} \left( \frac{a_i x_i}{\varepsilon_i} + \cdots \right) \right) \\
= \text{sign}(x_i).
\end{align*}
\]
(30)

Also, by using the inequality (24), we have $(a_i/\varepsilon_i) L_i > (\varepsilon_{i-1})/\varepsilon_i$ which implies that $\text{sign}(u_i) = -\text{sign}(x_i)$. This together with equation (30) imply
\[
V(y_i) < u_i \left( \frac{2y_{i+1}}{a_{i+1}} x_i + u_i \right) < 0.
\]

Then according to the Lyapunov stability theory, the state $x_i$ necessarily joints $[-L_i, L_i]$ at finite time. When $x_i \in [-L_i, L_i]$ by noting (24), we can get
\[
\frac{|a_i x_i - u_{i-1}|}{\varepsilon_i} \leq \frac{a_i}{\varepsilon_i} |x_i| + \frac{|u_{i-1}|}{\varepsilon_i} \\
= \frac{a_i}{\varepsilon_i} L_i + \frac{|u_{i-1}|}{\varepsilon_i} \\
= \frac{a_i L_i + |u_{i-1}|}{\varepsilon_i} < 1,
\]

which implies that the control law $u_i$ can be simplified as $u_i = -a_i x_i + u_{i-1}, i \geq 2$. With this simplification, the system $\Sigma_i$ is then reduced to $\Sigma_{i-1}$.

**Step 1.** With $i = 1$, we begin with the inequality (29), i.e.,
\[
\dot{V}(y_1) < u_1 \left( \frac{2y_2}{a_2} x_1 + u_1 \right),
\]
where $u_1 = -\varepsilon_1 \text{sat}(a_1 x_1/\varepsilon_1)$. Suppose $L_1$ is a positive scalar and $|x_1| > L_1$. If $2y_2/a_2 |x_1| + u_1 = \text{sign}(x_1)$ and $\dot{V}(y_1) < 0$. With the same arguments, the state $x_1$ necessarily joints $[-L_1, L_1]$ at finite time. Moreover, if $(a_1/\varepsilon_1) L_1 < 1$, we have $u_1 = -a_1 x_1$. Note that
\[
2 \frac{y_2^2}{a_2^2} L_1 > \varepsilon_1, \quad \frac{a_1}{\varepsilon_1} L_1 < 1 \\
\iff \frac{a_1^2}{2y_2^2} < 1 \\
\iff y_2 > \frac{\sqrt{2} a_1}{2}.
\]

The last inequality is guaranteed by inequality (15). Now, with these procedures, the closed-loop system will be simplified as $\Sigma_0$ at finite time. The system $\Sigma_0$ is stable since it is linear and its characteristic polynomial $a_0(s)$ is stable. With this, we complete the proof. 

**Remark 3:** According to the above theorem, we can construct the non-linear feedback law by using directly the system state, while a state transformation is required firstly in Teel (1992), Sussmann et al. (1994), Suarez et al. (1997) and Marchand and Hably (2005). Such advantage makes the implementation of the controller easier. Also, the eigenvalues of the closed-loop system operating in linear form can be chosen as complex numbers, while in Teel (1992), Sussmann et al. (1994), Suarez et al. (1997) and Marchand and Hably (2005) only real eigenvalues are allowed.

Similar to the improvement for the saturation function proposed in Marchand and Hably (2005), we introduce the following theorem which uses the so-called state-dependent saturation functions to replace the standard saturation functions used in Theorem 1.

**Theorem 2:** Consider the multiple integrators system (19). Let $\varepsilon_i, i \in [0, n]$ and $L_i, i \in [2, n]$ satisfy
\[
L_i > \frac{\varepsilon_{i-1}}{\varepsilon_i}, \quad 1 > L_i + \frac{\varepsilon_{i-1}}{\varepsilon_i}, \quad i \in [2, n],
\]
and $a_i(s) = \sum_{j=0}^{i-1} a_i s^j$ be an $\mathcal{SP}$-stable polynomial w.r.t. $s$. Furthermore, let $\phi_0 = 1$ and
\[
\phi_i = 1 + \frac{\varepsilon_{i-1}}{\varepsilon_i} \left[ L_{i+1} - \left| \text{sat}_{L_{i+1}} \left( \frac{a_i x_i}{\varepsilon_i} \right) \right| \right],
\]
(32)

for $i \in [1, n-1]$. Then the control law $u(x) = u_n$ with
\[
\begin{align*}
\dot{u}_i &= -\varepsilon_i \text{sat}_{\phi_i} \left( \frac{a_i x_i}{\varepsilon_i} - \frac{1}{\varepsilon_i} u_{i-1} \right) \quad i \in [2, n], \\
\dot{u}_1 &= -\varepsilon_1 \text{sat}_{\phi_1} \left( \frac{a_1 x_1}{\varepsilon_1} \right),
\end{align*}
\]
(33)
solves Problem 1. Moreover, there exists a time $T > 0$ such that for arbitrary $t > T$, the closed-loop system (19) and $u_n$ will operate in linear form with characteristic polynomial $a_0(s)$.

**Remark 4:** It is easy to verify that the inequalities (31) and (22) are equivalent. Therefore, the $\mathcal{SP}$-stable polynomial w.r.t. $\varepsilon$ in Theorem 1 and Theorem 2 can be chosen as the same.

**Proof of Theorem 2:** The process is quite similar to that of Theorem 1.

**Step n.** We assume that $|x_n| > (\varepsilon_n/a_n)L_n$ which implies $\phi_{n-1} = 1$. Consider the Lyapunov function $V(x_n) = (1/2) x_n^2$. The time derivative of $V(x_n)$ along the trajectory of the closed-loop system is given by $\dot{V}(x_n) = x_n \dot{u}_n$. By using (31), we have
\[
\dot{V}(x_n) > \frac{\varepsilon_{n-1}}{\varepsilon_n} \iff \frac{\varepsilon_{n-1}}{\varepsilon_n} L_n \frac{\varepsilon_n}{a_n} > \frac{\varepsilon_{n-1}}{\varepsilon_n} \phi_{n-1}.
\]

So, we have $\text{sign}(u) = -\text{sign}(x_n)$ and consequently, $\dot{V}(x_n)$ is negative. For the same reason given in the
proof of Theorem 1, $x_n$ necessarily joints $[-(e_n/a_n)L_{an}, (e_n/a_n)L_{an}]$ at finite time. Then in view of (31) and (32), we get

$$\frac{a_n}{e_n} x_n - u_{n-1} - \frac{a_n}{e_n} x_n = \frac{\dot{e}_{n-1}}{e_n} \phi_{n-1}$$

$$= \frac{\dot{e}_{n-1}}{e_n} \left( L_n - \left| \text{sat}_L \left( \frac{a_n x_n}{e_n} \right) \right| \right) + \frac{a_n}{e_n} x_n$$

$$= \frac{a_n}{e_n} x_n - \left| \text{sat}_L \left( \frac{a_n x_n}{e_n} \right) \right| + \frac{\dot{e}_{n-1}}{e_n} L_n$$

$$< 1 = \phi_n.$$ 

So the control law $u_n$ can be simplified as $u_n = -a_n x_n + u_{n-1}$ and the system $\Sigma_n$ under the input (23) can be simplified as $\Sigma_{n-1}.$

Step $i, i = n - 1, n - 2, \ldots, 2.$ The front part of this step is similar to Step $i$ in the Proof of Theorem 1. We begin our proof with the equation (29). That is, to show negativity of the right hand side of the following inequality

$$\dot{V}(y_{i+1}) < u_i \left( 2\frac{\dot{y}_{i+1}}{a_{i+1}} x_i + u_i \right).$$

Assume that $|x_i| > (e_i/a_i)L_i$ which implies $\phi_{i-1} = 1.$ Since $a_i(s)$ is an $S^{1}$-stable polynomial w.r.t. $e_i$, we have

$$y_{i+1} > e_{i+1} a_{i+1} \dot{a}_{i+1}$$

$$\Rightarrow \frac{a_{i+1} \dot{e}_{i+1} a_{i+1}}{2\dot{y}_{i+1} e_{i+1}} < \frac{\dot{e}_{i+1}}{e_{i+1}} < L_i$$

$$\Rightarrow \frac{2y_{i+1} \dot{e}_{i+1} L_i}{a_{i+1} a_{i+1}} > e_{i+1} > e_{i+1} L_{i+1} + e_i$$

$$\Rightarrow \frac{2y_{i+1} \dot{e}_{i+1} L_i}{a_{i+1} a_{i+1}} > e_{i+1} L_{i+1} + e_i - e_{i+1} \left| \text{sat}_L \left( \frac{a_{i+1} x_{i+1}}{e_{i+1}} \right) \right|$$

$$\Rightarrow \frac{2y_{i+1}}{a_{i+1}} x_i > e_i \phi_i$$

$$\Rightarrow \text{sign} \left( \frac{2y_{i+1}}{a_{i+1}} x_i + u_i \right) = \text{sign}(x_i). \quad (34)$$

Furthermore, using (31) again, we obtain

$$L_i > \frac{e_{i-1}}{e_i} \Leftrightarrow \frac{a_i}{e_i} \left| \frac{x_i}{e_i} \right| > \frac{\dot{e}_{i-1}}{e_i} \phi_{i-1}$$

$$\Rightarrow \frac{a_i}{e_i} |x_i| > \frac{\dot{e}_{i-1}}{e_i} \phi_{i-1}$$

$$\Rightarrow \text{sign}(u_i) = -\text{sign}(x_i).$$

With this, we can conclude that $\dot{V}(y_i) < 0.$ So the state $x_i$ necessarily joints $[-L_i(e_i/a_i), L_i(e_i/a_i)]$ at finite time. Note that

$$|a_i| \frac{x_i}{e_i} - \frac{u_{i-1}}{e_i} - \frac{a_i}{e_i} |x_i| = \frac{\dot{e}_{i-1}}{e_i} \phi_{i-1}$$

$$= \frac{\dot{e}_{i-1}}{e_i} \left( 1 + \frac{e_i}{\dot{e}_{i-1}} \left( L_i - \left| \text{sat}_L \left( \frac{a_i x_i}{e_i} \right) \right| \right) \right) + \frac{a_i}{e_i} |x_i|$$

$$= \frac{a_i}{e_i} |x_i| - \left| \text{sat}_L \left( \frac{a_i x_i}{e_i} \right) \right| + \frac{\dot{e}_{i-1}}{e_i} L_i$$

$$|a_i| |x_i| < 1 \leq \phi_i.$$ 

Then the control law $u_i$ can be simplified as $u_i = -a_i x_i + u_{i-1}, i \geq 2.$ The rest of the proof is similar to that of the proof for Theorem 1.

Step 1. With the property of $a_i(s),$ we have

$$\gamma_1 > \frac{\dot{e}_2}{e_1} = \frac{\dot{e}_2}{e_1}$$

$$\Rightarrow \frac{\dot{e}_2}{e_1} < 1$$

$$\Rightarrow \exists L_1 < 1, \frac{\dot{e}_2}{e_1} < L_1.$$ 

With the same arguments as in (34), we have $\text{sign}(2(\gamma_2/a_2)x_1 + u_1) = \text{sign}(x_1)$ if $|x_i| > (e_i/a_i)L_i.$ Obviously, we have $\text{sign}(u_i) = -\text{sign}(x_i).$ So the state $x_i$ necessarily joins $[-(e_i/a_i)L_i, (e_i/a_i)L_i]$ at finite time and $u_i$ then becomes $-a_i x_i.$ The closed-loop system is reduced to $\Sigma_0$ which is stable since it is a linear system with a stable characteristic polynomial $a_i(s).$ With this, we complete the proof. \hfill $\Box$

Remark 5: It follows from (32) that

$$\phi_i > 1 \text{ if } |x_i| < |\frac{\dot{e}_i}{a_{i+1}} L_{i+1}, i \in I[1, n - 1].$$

This modified control law (33) allows us to increase the saturation level in the feedback if some of the states are small. Such property can significantly improve the convergence performances of the closed-loop system.

2.3 Extension to a class of linear systems

In this subsection, we extend the results given in the precious subsection to a wider class of linear system

$$\dot{x} = Ax + bu, \quad x \in \mathbb{R}^n,$$  

(35)

which is ANCBC, i.e., (Sussmann et al. 1994) (i) the matrix $A$ has no eigenvalue with positive real part and (ii) the matrix pair $(A, b)$ is stabilisable in the ordinary
sense (i.e., all the uncontrollable modes of the system have negative real parts). The input also has the constraint (20). In this paper, the system (35) should further satisfy the following additional condition.

**Condition 1:** All the pure imaginary eigenvalues of $A$ are different from each other.

Under the ANCBC assumption, there exists a linear change of coordinates of the state space that transforms the system (35) into the block form

$$
\begin{align*}
\dot{x}_1 &= A_1 x_1 + A_{12} x_2 + b_1 u, \quad x_1 \in \mathbb{R}^{n_1}, \\
\dot{x}_2 &= A_2 x_2 + b_2 u, \quad x_2 \in \mathbb{R}^{n_2},
\end{align*}
$$

where (i) $n_1 + n_2 = n$, (ii) all the eigenvalues of $A_1$ are stable, (iii) all the eigenvalues of $A_2$ are critical, and (iv) $(A_2, b_2)$ is controllable. So we only need to design the control law $u = u(x_2)$ to stabilise the second subsystem. Hence without loss of generality, in this paper, we will suppose that the system (35) is already in this "critical" form. Similarly to Problem 1, the global stabilisation problem for system (35) can be stated as follows.

**Problem 2:** Find a function $u = u(x)$ satisfying the amplitude constraint (20) such that the system (35) is globally asymptotically stable, i.e., for arbitrarily given initial condition $x(0) \in \mathbb{R}^n$ and a positive scalar $\varepsilon$, the closed-loop system (35) with $u = u(x)$ has a unique and bounded solution $x(x(0), t)$, and there exists a number $T$ such that for arbitrary $t > T$, there holds $\|x(x(0), t)\| < \varepsilon$.

To solve this problem, we need a special state space representation of the linear system (35).

**Lemma 1:** Let the matrix $A$ in (35) satisfy Condition 1 and the number of pure imaginary eigenvalues and the number of zero eigenvalues of $A$ be $2q$ and $p$, respectively, so $p + 2q = n$. Furthermore, let

$$
a_i(s) = s^p + \sum_{j=0}^{p-1} a_{i+j}s^j,
$$

be an $\mathcal{SP}$-stable polynomial w.r.t. $s \in \mathbb{R}^{p+1}$. Then there exists a unique nonsingular matrix $T$ such that the linear transformation $[x^T\xi^T]^T = T x$ transforms the system (35) into the following form

$$
\begin{align*}
\dot{z} &= A_z z + M_z\xi + b_z u, \\
\dot{\xi} &= A_\xi \xi + b_\xi u,
\end{align*}
$$

where $z \in \mathbb{R}^{2q}$, $\xi \in \mathbb{R}^p$, $A_z$ is a skew symmetric matrix with eigenvalues $\pm \omega_j$, $j \in \{1, \ldots, q\}$, $u_p = u$, $M_z$ has the form

$$
M_z = b_z [a_1 \cdots a_{p-1} a_p],
$$

with appropriate dimensions, $(A_z, b_z)$ is controllable and

$$
A_\xi^+ = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad b_\xi = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

The proof of the above lemma is simple and omitted here. With the help of this lemma, the system (35) is globally stabilised if the system (37) is globally stabilised. Firstly we give the following technical lemma about stabilising a neutral stable system by saturated linear feedback.

**Lemma 2:** Let $A_z \in \mathbb{R}^{2q \times 2q}$ be a skew symmetric matrix and $(A_z, b_z)$ be controllable. Then the system

$$
\dot{z} = A_z z + b_z u,
$$

can be globally stabilised by

$$
u = -\varepsilon_0 \text{sat}\left(\frac{a_0 b^T_z z}{\varepsilon_0}\right),
$$

where $\varepsilon_0$ and $a_0$ are some arbitrary positive scalars. Furthermore, there exists a $T_\zeta > 0$ such that for all $t > T_\zeta$, the closed-loop system (39) and (40) operates in linear region with characteristic polynomial $a_z(s) = \text{det}(sI - A_z + a_0 b_z b_z^T)$.

**Proof:** Consider the Lyapunov function $V(z) = z^T z$. The time derivative of $V(z)$ along the trajectory of the closed-loop system (39) and (40) is given by

$$
\dot{V}(z) = 2z^T z = z^T (A_z + A_z^+ z) + 2z^T b_z u
\leq -2\varepsilon_0 \text{sat}\left(\frac{a_0}{\varepsilon_0}\right) \leq 0,
$$

where $= b_z^T z$. By LaSalle’s invariant principle, the trajectories of the closed-loop system (39) and (40) asymptotically converge to the set

$$
\Omega = \{z \in \mathbb{R}^{2q} : \text{sat} \neq 0\} = \{z \in \mathbb{R}^{2q} : b_z^T z = 0\}.
$$

Furthermore, the closed-loop system (39) and (40) in the set $\Omega$ becomes $\dot{z} = A_z z$. So we have

$$
b_z^T \dot{z} = b_z^T A_z z = 0, \ldots, b_z^T A_z^{2q-1} z = 0.
$$

That is

$$
z^T [b_z A_z b_z \cdots b_z A_z^{2q-1}] = 0.
$$

Since $(A_z, b_z)$ is controllable, equation (41) in turn implies that $z = 0$ and $\Omega = \{0\}$. Hence, the closed-loop system (39) and (40) is asymptotically stable. Furthermore, global stability of the closed-loop
system (39) with input (40) implies that there exists a $T_z$ such that for $\forall t > T_z$, there holds

$$\frac{\|a_0 b_T^T z\|}{\|e_0\|} \leq \|a_0 b_T^T \| \|z\| \leq 1.$$ 

This implies that $u_0$ can be simplified as $-a_0 b_T^T z$ for $\forall t > T_z$ and from that time on, the closed-loop system (39) and (40) will operate in linear region with system matrix $A - a_0 b_T b_T^T$.

Now applying Theorem 1 and Lemma 2 on the system (37), we have the following result regarding solution to Problem 2.

**Theorem 3:** Let $\epsilon_i, i \in \{0, p\}$ be some positive scalars satisfying

$$\epsilon_i > 2\epsilon_{i-1}, i \in \{1, p\}, \quad \epsilon_p \leq u_{\max}, \quad (42)$$

and $a_i(s)$ defined as (36) be an $SP$-stable polynomial w.r.t. $s$. Then the control law $u(\xi, z) = u_p$ with

$$u_i = -\epsilon_i \text{sat} \left( \frac{a_i \xi_i}{\epsilon_i} - \frac{1}{\epsilon_i} u_{i-1} \right), \\
i \in \{1, p\}, \quad u_0 = -\epsilon_0 \text{sat} \left( \frac{a_0 b_T^T \xi}{\epsilon_0} \right), \quad (43)$$

where $a_0 > 0$ is an arbitrary scalar, solves Problem 2. Furthermore, the closed-loop system (35) and $u_p$ will operate in linear region at finite time and has characteristc polynomial

$$\alpha(s) = a_i(s) \det(sI - A_z + a_0 b_T b_T^T). \quad (44)$$

**Proof:** Similar to the proof of Theorem 1, the low subsystem of (37) given by

$$\Sigma^0_z: \dot{\xi} = A^0_{\xi} \xi + b_T u_p,$$

will reduce to the system

$$\Sigma^0_z: \dot{\xi} = A^0_{\xi} \xi + b_T u_0,$$

after some time $T$ with $u_0$ being in the form of (43) and $A^0_{\xi}$ being a companion matrix in the form of

$$A^0_{\xi} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & \cdots & -a_p \end{bmatrix}.$$ 

Note that the system (37) can then be written as

$$\begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_z & 0 \\ 0 & A^0_{\xi} \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix} + \begin{bmatrix} b_z \\ b_\xi \end{bmatrix} u_0. \quad (45)$$

Now take the high subsystem of (45), i.e., (39), into consideration. According to Lemma 2, this subsystem is globally stable if $\epsilon_0 > 0$ and $a_0 > 0$ and the closed-loop system (45) can be further simplified as

$$\begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_z - a_0 b_T b_T^T & 0 \\ 0 & A^0_{\xi} \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix},$$

which is obviously stable and has characteristic polynomial (44). The proof is then completed.

Now combining Theorem 2 and Lemma 2, we can obtain the following.

**Theorem 4:** Let $\epsilon_i, L_i, i \in \{1, p\}$ satisfy

$$L_i > \frac{\epsilon_{i-1}}{\epsilon_i}, \quad 1 > L_i + \frac{\epsilon_{i-1}}{\epsilon_i}, \quad i \in \{1, p\}, \quad \epsilon_p \leq u_{\max}, \quad (46)$$

and $a_i(s) = \sum_{j=0}^{p-1} a_{i,j}s^j$ be an $SP$-stable polynomial w.r.t. $s$. Furthermore, let $M_z$ be given by (38), $\phi_p = 1$ and

$$\phi_i = 1 + \frac{\epsilon_{i+1}}{\epsilon_i} \left( L_{i+1} + \text{sat}_{L_{i+1}, \phi_i \left( \frac{a_{i+1} \xi_{i+1}}{\epsilon_{i+1}} \right)} \right), \quad i \in \{0, p-1\}.$$ 

Then the control law $u(\xi, z) = u_p$ with

$$u_i = -\epsilon_i \text{sat}_{\phi_i} \left( \frac{a_i \xi_i}{\epsilon_i} - \frac{1}{\epsilon_i} u_{i-1} \right), \quad i \in \{1, p\},$$

$$u_0 = -\epsilon_0 \text{sat}_{\phi_0} \left( \frac{a_0 b_T^T \xi}{\epsilon_0} \right),$$

where $a_0 > 0$ is an arbitrary scalar, solves Problem 2. Moreover, the closed-loop system (35) and $u_p$ will operate in linear region at finite time and has characteristic polynomial (44).

**Remark 6:** Similar to Remark 4, the inequalities (42) and (46) are equivalent and therefore the $SP$-stable polynomial in Theorem 3 and Theorem 4 can be chosen as the same.

### 3. Illustrative examples

In this section, we consider two examples to illustrate the efficiency of the proposed approach.

**Example 1:** A chain of three integrators system with $u$ satisfying $|u| \leq u_{\max} = 1$ (Marchand and Hably 2005):

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u. \quad (47)$$

The following feedback laws are applied on the system (47).
Controller 1: This control law is proposed in (Sussmann et al. 1994) where the authors have extended the results given in (Marchand and Hably 2005), and has the form
\[ u = -\frac{u_{\text{max}}}{\sum_{i=1}^{3} e^{3-i+1} \text{sat}_M(y_i)}, \] (48)
where \( y \) is a linear combination of \( x \) (the exact expression is omitted to save space) and \( M_i = 1, i = 1, 2, 3 \). As analysed in Marchand and Hably (2005), the parameter \( \varepsilon \) should be chosen as \( \varepsilon \leq 0.618 \) to ensure stability.

Controller 2: This control law is proposed in Marchand and Hably (2005) and is also in the form of (48) with \( M_3 = 1 \) and
\[ M_j = 1 + \frac{1}{\varepsilon} \left[ M_{j+1} - \left( \text{sat}_{M_i}(y_{j+1}) \right) \right], \]
for \( j = 2, 1 \).

Controller 3: This control law is proposed in Teel (1992) and has the form
\[ u = -\text{sat}_M(y_3 + \text{sat}_M(y_2 + \text{sat}_M(y_1))), \]
where \( y_{3-i} = \sum_{i=0}^{m} \beta(i-i)\varepsilon x_{j-i}, j \in [0, 2] \). Taking \( M_{j-1} = (1/2.00001)M_j \) for \( i \in [2, 3] \) and \( M_3 = u_{\text{max}} = 1 \) to satisfy the stability condition and bounded condition.

Our control laws given in Theorems 1 and 2 take the following parameters: \( a_0, a_1, a_2, a_3 \) in the expression (18) and \( L_1 = 1.001(a_2/a_3) \) \( L_2 = 1.001(a_1/a_2) \). The initial condition is set to \( x_0 = [-3, -3, -3]^T \). The simulation result is given in Figure 1.

It follows from Figure 1 that the convergence performance of the closed-loop system under the control laws given in Theorems 1 and 2 appears to be better than that under the Controllers 1 and 2. Especially the feedback control using state-dependent saturation functions given in Theorem 2 is significantly better than Controllers 1 and 2 and also superior to the control law given in Teel (1992).

We now show how the parameters \( a_i(s) \) in our presented control laws influence the convergence speed of the corresponding closed-loop system. Using Corollary 1 by letting \( a_{3-i} = (2e^{2-sy_{\text{max}}})/1.01e\varepsilon a_3 \) and different initial condition \( a_3 \), we can yield a series of different \( SP \)-stable polynomials \( a_is(s) \). Note that an \( SP \)-stable polynomial \( a_is(s) \) w.r.t. \( s \) is uniquely determined by \( a_3 \). In Figure 2, the legend “\( a_3 = 1 \)” means that an \( SP \)-stable polynomial \( a_is(s) \) is obtained associated with \( a_3 = 1 \).

It follows from Figure 2 that when \( a_3 \) reduces from 8 to 2, the peak value and the regulation time are also reduced. But, reduce \( a_3 \) more, the peak value is increasing! This observation indicates that there exists an optimal \( a_3^* \) such that the peak value and regulation time are minimised. Furthermore, the optimal value of \( a_3^* \) is around 2. Simulation results indicate that \( a_3^* \) is dependent on \( x_0 \) i.e., \( a_3^* = a_3^*(x_0) \). However, it is difficult to determine the optimal value in theory.

Example 2: Consider a linear system in the form of (35) with the following parameter matrices
\[ A = \begin{bmatrix} 0 & I_6 \\ 0 & \sigma \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \] (49)
where
\[ \sigma = \begin{bmatrix} 0 & -\omega_1^2 \omega_2^2 & 0 & -\left( \omega_1^2 + \omega_2^2 \right) & 0 \end{bmatrix}. \]
Such a system is an ANCBC system since the eigenvalues of matrix $A$ are $\pm \omega_1, \pm \omega_2$, 0, 0, 0 and $(A, b)$ is controllable. The input satisfies (20) with $u_{\text{max}} = 1$. We here choose $\omega_1 = 1$ and $\omega_2 = 2$ to satisfy Condition 1.

There are a lot of existing control laws in the literature that globally stabilise an ANCBC linear system by bounded input. Two typical of them are non-linear low gain feedback given in Sussmann et al. (1994) and linear low gain feedback given in Lin and Saberi (1993), which are compared with our proposed approaches given by Theorems 3 and 4 in Figure 3. The parameter $\varepsilon$ is chosen as $[0.11 \ 0.24 \ 0.49 \ 1]$. Then the $SP^\text{-}$stable polynomials can be constructed as

$$a_i(s) = s^3 + 2.00s^2 + 0.958s^2 + 0.1029,$$

by using Corollary 1 and letting $a_{i-1} = (2\varepsilon (\text{max})^3)/1.01\varepsilon a_i$. Shown in Figure 3 are the convergence performance of the closed-loop system with different feedback laws and the initial condition

$$x_0 = [2 \ -1 \ 2 \ -1 \ -2 \ 2]^T.$$

It is clearly to see from Figure 3 that the convergence speed of the states under the feedback laws given by Theorems 3 and 4 is significantly higher than that under the nonlinear feedback low proposed in Sussmann et al. (1994). Though the closed-loop system under the control law given by Theorem 3 has a longer regulation time than that under the control law of Lin and Saberi (1993), the former one has lower peak value than the last one. Nevertheless, convergence performance of the closed-loop system under the feedback law consisting of state-dependent saturation functions proposed in Theorem 4 is significantly better than those under all the other three kind of feedback laws.

4. Conclusion

In this paper, the global stabilisation problem of a chain of integrators system is reconsidered. A novel nested non-linear feedback law is proposed by introducing the so-called $SP^\text{-}$stable polynomial which gives a transparent and tractable procedure to construct such type of non-linear feedback laws. Then the so-called state-dependent saturation functions are used to replace the standard saturation functions which will significantly improve the performances of the closed-loop systems. At last, these two non-linear feedback laws are extended to a class of ANCBC linear systems in which all the open-loop pure imaginary eigenvalues are different from each other. Simulation examples show that this class of control laws using state-dependent saturation functions significantly improve the convergence performance of the closed-loop system, and is also superior to some other global stabilisation controllers existing in the literature.

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