Detectability and observability of discrete-time stochastic systems and their applications

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\textbf{A B S T R A C T}

This paper studies detectability and observability of discrete-time stochastic linear systems. Based on the standard notions of detectability and observability for time-varying linear systems, corresponding definitions for discrete-time stochastic systems are proposed which unify some recently reported detectability and exact observability concepts for stochastic linear systems. The notion of observability leads to the stochastic version of the well-known rank criterion for observability of deterministic linear systems. By using these two concepts, the discrete-time stochastic Lyapunov equation and Riccati equations are studied. The results not only extend some of the existing results on these two types of equation but also indicate that the notions of detectability and observability studied in this paper take analogous functions as the usual concepts of detectability and observability in deterministic linear systems. It is expected that the results presented may play important roles in many design problems in stochastic linear systems.

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1. Introduction

Since stochastic modeling can describe many practical systems, stochastic systems have received much attention in the past few decades (see Gao, Lam, and Wang (2006), Hu and Mao (2008), Hwang, Balakrishnan, and Tomlin (2003), Kubrusly and Costa (1985), Mao (1994), Vidal, Chiuso, Soatto, and Sastry (2003), Xu and Chen (2002) and the references therein). Many control problems for stochastic systems can be treated quite analogously to the deterministic case. For instance, the solution to linear quadratic control problem for stochastic systems also relies heavily on the stochastic Riccati equations (see, e.g., Ram and Zhou (2000) and Wonham (1968)) and the mean square stability of stochastic linear systems can also be characterized by the spectra locations of certain coefficient matrices of the system (see Feng, Lam, and Xue (2008), Zhang and Chen (2004) and the references given there).

Although some control problems for stochastic systems can be treated quite analogously to that in deterministic systems, there are concepts that are essentially different from each other. Detectability and observability concepts are such cases. It is well-known that in deterministic linear system, the detectability (observability) of a system is equivalent to the stabilizability (controllability) of its dual system. However, this is not the case for stochastic linear systems although many researchers have used this equivalence property in deterministic systems to define detectability for stochastic systems, i.e. the detectability of a system is defined as the stabilizability of its dual system (see, e.g., Freiling and Hochhaus (2003)). Recently, it is shown in Damm (2007) that there exists a natural concept of detectability for a continuous time stochastic linear systems which is weaker than the detectability defined as the stabilizability of its dual system. The most important consequent result is that the new concept of detectability defined in Damm (2007) seems to be served as the usual detectability concept for deterministic linear systems.

The detectability definition given in Damm (2007) is based on the natural idea that any non-observed states corresponds to stable models of the system which leads to the PBH test that is well understood in deterministic linear systems. However, for observability, another kind of concept – exact observability which is firstly introduced in Zhang and Chen (2004) – is adopted. It seems that these two concepts have been defined in different
The discrete-time stochastic linear system (1) with \( u(k) = 0, k \geq 0 \) is said to be mean square stable if \( \lim_{k \to \infty} E \| x(k, x_0) \|^2 = 0, \forall x_0 \in \mathbb{R}^n \). Moreover, we say shortly that \( A \) is mean square stable.

Definition 2. The discrete-time stochastic linear system (1) is said to be mean square stabilizable if there exists a matrix \( F \in \mathbb{R}^{m \times n} \) such that the closed-loop system (1) with \( u(k) = Fx(k) \), i.e.,

\[
x(k + 1) = A\delta x(k) + \sum_{j=1}^N A_j x(k) w_j(k),
\]

is mean square stable for arbitrary \( x_0 \in \mathbb{R}^n \), where

\[
A_i^T F = A_i + B_i F, \quad i \in \mathbb{N}.
\]

Moreover, we will say shortly, \( (A, B) \) is mean square stabilizable.

Definition 3. For the matrix groups \( A \) and \( C \), the linear operators \( \mathcal{L}_A : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) and \( \mathcal{L}_C : \mathbb{R}^{m \times n} \to \mathbb{R}^{p \times p} \) are, respectively, defined as

\[
\mathcal{L}_A (X) = \sum_{i=0}^N A_i X A_i^T, \quad \mathcal{L}_C (X) = \sum_{i=0}^N C_i X C_i^T.
\]

The eigenvalue set of \( \mathcal{L}_A \) is

\[
\sigma (\mathcal{L}_A) = \{ \lambda \in \mathbb{C} \mid \mathcal{L}_A (X) = \lambda X, 0 \neq X \in \mathbb{R}^{m \times n} \}.
\]

In addition, the corresponding nonzero matrix X is said to be an eigenvector associated with the eigenvalue \( \lambda \). Moreover, the spectral radius of the operator \( \mathcal{L}_A \) is defined by \( \rho (\mathcal{L}_A) = \max \{ |\lambda| \mid \lambda \in \sigma (\mathcal{L}_A) \} \).

Let \( \mathcal{L}_A^* (X) : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) be the adjoint operator of \( \mathcal{L}_A \). Since \( X \geq 0 \) implies \( \mathcal{L}_A (X) \geq 0 \), i.e., \( \mathcal{L}_A \) is a positive operator, then according to the well-known Krein–Rutman Theorem (see, e.g., Ziegler (1986)), we have the following result.

Lemma 1. Let \( \rho (\mathcal{L}_A) \) be the spectral radius of the operator \( \mathcal{L}_A \). Then \( \rho (\mathcal{L}_A) \) is an eigenvalue of both \( \mathcal{L}_A \) and \( \mathcal{L}_A^* \) with associated eigenvector \( V \geq 0 \).

With the operator \( \mathcal{L}_A (X) \), the following lemma is easy to prove.

Lemma 2. Consider the discrete-time stochastic linear system (1). Denote

\[
X(k) = E(x(k)x^T(k)), \quad Y(k) = E(y(k)y^T(k)).
\]

If \( u(k) \equiv 0, k \geq 0 \), then

\[
X(k + 1) = \mathcal{L}_A (X(k)), \quad Y(k) = \mathcal{L}_C (X(k)).
\]

Taking vec on both sides of (6) and using the well-known formulation vec \((AXB) = (B^T \otimes A) \text{vec} (X) \) gives

\[
\text{vec}(X(k + 1)) = A \text{vec}(X(k)), \quad \text{vec}(Y(k)) = C \text{vec}(X(k))
\]

where

\[
A = \sum_{j=0}^N A_j \otimes A_j, \quad C = \sum_{j=0}^N C_j \otimes C_j.
\]

By using the method in Feng et al. (2008), the following result can be proven.

Lemma 3. The following statements are equivalent: (a) The discrete-time stochastic linear system (1) with \( u(k) \equiv 0, k \geq 0 \) is mean square stable; (b) \( \sigma (\mathcal{L}_A) \subseteq \mathbb{C}^+ \), (c) \( \rho (\mathcal{L}_A) < 1 \), and (d) \( \sigma (A) \subseteq \mathbb{C}^+ \), where \( \sigma (A) \) denotes the eigenvalue set of the matrix \( A \).
Let $X(k)$ and $Y(k)$ be defined by (5). For $l \geq 1$, denote
\begin{equation}
\varepsilon(l, X_0) = \sum_{k=0}^{l-1} \text{tr}(Y(k)), \quad \|X(k)\|_E = \text{tr}(X(k)).
\end{equation}

**Remark 4.** The function $\varepsilon(l, X_0)$ has the physical interpretation of the accumulated energy of the output process $y(i, x_0)$ on the interval $0 \leq i \leq l - 1$. To see this, we note from (5) that
\begin{equation}
\varepsilon(l, X_0) = \sum_{k=0}^{l-1} \mathbb{E} \left( \| y(k, x_0) \|^2 \right).
\end{equation}
Similarly, $\|X(k)\|_E$ can be viewed as the energy of the state $x(k, x_0)$ at time $k$ since $\|X(k)\|_E = \mathbb{E}(\|x(k, x_0)\|^2)$.

Finally, we define recursively a series of matrices $\phi_k$, $k \geq 0$ associated with the matrix groups $A$ and $C$ as
\begin{equation}
\phi_{k+1} = \sum_{j=0}^{N} A_j^T \phi_j A_j + \sum_{j=0}^{N} C_j^T C_j, \quad k \geq 0, \quad \phi_0 = 0,
\end{equation}
which will play important functions in studying the detectability and observability properties of the discrete-time stochastic linear system (1).

**Lemma 5.** Let $x_0$ be an initial condition and $\phi_k$, $k \geq 0$ be defined as (10). Then
\begin{equation}
\varepsilon(m, X_0) = \text{tr}(X_0 \phi_m), \quad m \geq 1.
\end{equation}

**Proof.** We show (11) by induction. If $m = 1$, then by definition and using (6), we can show $\varepsilon(1, X_0) = \text{tr}(X_0 \phi_1)$. Assume that (11) is true with $m = k$, i.e.,
\begin{equation}
\varepsilon(k, X_0) = \phi_k (X_0),
\end{equation}
we show (11) is true with $m = k + 1$. By definition
\begin{equation}
\varepsilon(k + 1, X_0) = \sum_{l=0}^{k+1} \text{tr}(Y(l)) = \sum_{l=0}^{k} \text{tr}(Y(l)) + \text{tr}(Y(k+1)) = \phi_k (X_0) + \text{tr}(Y(k+1)).
\end{equation}
Note that if (12) is true with initial condition $X(0)$, then it is true for arbitrary initial condition $X(0)$, i.e.,
\begin{equation}
\varepsilon(k, X(1)) = \sum_{l=0}^{k} \text{tr}(Y(l+1)) = \phi_k (X(1)).
\end{equation}
Substituting (14) into (13) and using (6) and (10) gives
\begin{equation}
\varepsilon(k + 1, X_0) = \phi_k (X_0) + \text{tr}(X(1) \phi_k) = \phi_{k+1} (X_0),
\end{equation}
which ends the proof. 

### 3. Detectability and observability of stochastic linear systems

Throughout this section, we will assume that $u(k) \equiv 0$, $k \geq 0$ in system (1).

#### 3.1. Detectability and the criteria

We give the following definition of detectability of the discrete-time stochastic linear system (1).

**Definition 4.** Consider the stochastic linear system (1). We say that $(A, C)$ is detectable if there exist integers $N_d, k_t > 0$ and scalars $0 \leq \delta < 1, \gamma > 0$ such that
\begin{equation}
\|X(k_2)\|_E \geq \delta \|X_0\|_E \Rightarrow \varepsilon(N_d, X_0) \geq \gamma \|X_0\|_E.
\end{equation}

The above definition of detectability adopts from the standard concepts of detectability of linear time-varying systems (see, e.g., Anderson and Moore (1981) and Peters and Iglesias (1997)). It is worth of pointing out that such definition is also used to study detectability and observability of Markovian jump linear systems (Costa & do Val, 2001). In view of Remark 4, the concept of detectability realizes the basic idea that any unstable models should be reflected by the output process.

The following lemma is essential in studying the detectability of the stochastic system (1).

**Lemma 6.** Let $x_0$ be any initial condition and $X(k)$ and $Y(k)$ be defined as (5). Then $\varepsilon(n^2, X_0) = 0 \Rightarrow Y(k) = 0, k \geq 0 \Rightarrow \varepsilon(l + 1, X(s)) = 0$, where $l, s \geq 0$.

**Proof.** By definition (8) and the time invariance of system (6), we have $\varepsilon(l + 1, X(s)) = \sum_{j=0}^{l} \text{tr}(Y(s + k)), \forall l, s \geq 0$. Therefore, $Y(k) = 0, k \geq 0 \Rightarrow \varepsilon(l + 1, X(s)) = 0, \forall l, s \geq 0$ is obvious since $Y(k)$ is nonnegative for arbitrary $k \geq 0$. In the following, we need only to show $\varepsilon(n^2, X_0) = 0 \Rightarrow Y(k) = 0, k \geq 0$.

Clearly, $Y(k) = 0, k \geq 0$ implies $\varepsilon(n^2, X_0) = 0$. Therefore, the remaining problem is to show
\begin{equation}
\varepsilon(n^2, X_0) = 0 \Rightarrow Y(k) = 0, \quad k \geq 0.
\end{equation}
Again, as $Y(k)$ is nonnegative for arbitrary $k \geq 0$, it follows from $\varepsilon(n^2, X_0) = 0$ that
\begin{equation}
Y(k) = 0, \quad \forall 0 \leq k \leq n^2 - 1.
\end{equation}
We next show that $Y(k) = 0$ is true for arbitrary $k \geq n^2$. By definition of $\varepsilon(l, X_0)$ and using (7), we have
\begin{equation}
Y(k) = \text{vec}^{-1}(\text{vec}(X_0) A^k), \quad k \geq 0.
\end{equation}
According to the Cayley–Hamilton theory, we have $A^{n^2} = \sum_{i=0}^{n^2-1} a_i A^i$ where $a_i, i \in \{0, n^2 - 1\}$ are some scalars. Consequently, we can write
\begin{equation}
A^{n^2+1} = \sum_{i=0}^{n^2-1} a_i (l+1) A^i, \quad l \geq 0,
\end{equation}
in which $a_i(l), i \in \{0, n^2 - 1\}, l \geq 0$ are some scalars dependent on $l$. Then by substituting (18) into (17) and in view of (16), for $l \geq 0$, we have $Y(n^2 + l) = \sum_{i=0}^{n^2-1} a_i (l) Y(i) = 0$, which, together with (16), is equivalent to (15). The proof is done. 

We introduce the following lemma whose proof is quite similar to the proof of Lemma 6 in Costa and do Val (2001) and thus omitted.

**Lemma 7.** The system in (1) is detectable in the sense of Definition 4 if and only if for $\forall x_0 \in \mathbb{R}^n$
\begin{equation}
\varepsilon(n^2, X_0) = 0 \Rightarrow \lim_{k \to \infty} \mathbb{E}(\|x(k, x_0)\|^2) = 0.
\end{equation}
The stochastic linear system (1) is said to be detectable if for all $x_0 \in \mathbb{R}^n$, $E \{ y^T(k, x_0) y(k, x_0) \} = 0$, $\forall k \geq 0$ implies $\lim_{k \to \infty} E \{ x^T(k, x_0)x(k, x_0) \} = 0$.

The main result of this subsection is presented as follows.

**Theorem 8.** The following statements are equivalent.

1. The discrete-time stochastic system (1) is detectable in the sense of Definition 4.
2. The discrete-time stochastic system (1) is detectable in the sense of Definition 5.
3. $\ell^2 (n^2, x_0) = \lim_{n \to \infty} E \{ x^T(k, x_0)x(k, x_0) \} = 0$, $\forall x_0 \in \mathbb{R}^n$.
4. The following inequality holds true for every eigenvector $0 \neq V \geq 0$ of $Z$ corresponding to the eigenvalue $|\lambda| \geq 1$.

**Proof.** Clearly, (1)$\iff$(3) follows from Lemma 7, (2)$\iff$(4) is the discrete-time version of Theorem 3 in Damm (2007) and the proof is quite similar and thus omitted. Since $\text{tr} \{ Y(k) \} = E \{ y^T(k, x_0) y(k, x_0) \}$, it follows from Lemma 6 that $E \{ y^T(k, x_0) y(k, x_0) \} = 0$, $\forall k \geq 0 \iff \ell^2 (n^2, x_0) = 0$. Then comparing (19) with Definition 5, we clearly have (1)$\iff$(2). $\blacksquare$

**Remark 9.** The stochastic linear system (1) is said to be MS-detectable if $(A^T, C^T)$ is mean square stabilizable. Similarly to the continuous-time case considered in Damm (2007), it is not difficult to show that the detectability (both in the sense of Definitions 4 and 5) can be implied by the MS-detectability but the converse is not true.

### 3.2. Observability and the criteria

By setting $\delta = 0$ in Definition 4, we get the following definition of observability.

**Definition 6.** The discrete-time stochastic linear system (1) is said to be observable if there exist an integer $N_0 > 0$ and a scalar $\gamma > 0$ such that

$$
\ell^2 (N_0, x_0) \geq \gamma \|x_0\|_E. 
$$

holds for each initial condition $x_0 \in \mathbb{R}^n$.

Different from the detectability concept, observability requires that any models (stable and unstable) should be reflected by the output.

**Lemma 10.** The discrete-time stochastic linear system (1) is observable in the sense of Definition 6 if and only if $\ell^2 > 0$.

**Proof.** (If) If $\ell^2 > 0$, then it follows from Lemma 5 that $\ell^2 (n^2, x_0) = \text{tr} \{ X_0 \ell^2 \} \geq \lambda_{\min} (\ell^2) \|X_0\|_E$, which indicates that system (1) is observable by definition.

(Only If) We need to prove that if the system is observable, then $\ell^2 > 0$. We show this by contradiction. Assume that there exists a nonzero vector $\eta$ such that $\eta^T \ell^2 \eta = 0$. By Lemma 5, we have

$$
\ell^2 (n^2, \eta \eta^T) = \text{tr} \{ \eta \eta^T \ell^2 \} = \eta^T \ell^2 \eta = 0.
$$

Now by using Lemma 6, we know that $\ell (I, \eta \eta^T) = 0$, $\forall t \geq 0$, which contradicts with the observability of the system.

(II) The following notion of observability is firstly considered in Zhang and Chen (2004) (see also Huang, Zhang, and Zhang (2006) and Liu (1999)).

**Definition 7.** The discrete-time stochastic linear system (1) is said to be exactly observable if $y(k, x_0) = 0$, a.s. $\forall k \geq 0 \Rightarrow x_0 = 0$, $\forall x_0 \in \mathbb{R}^n$.

We will show that this observability concept is in fact equivalent to that given in Definition 6.

**Lemma 11.** The discrete-time stochastic linear system (1) is observable in the sense of Definition 6 if and only if it is observable in the sense of Definition 7.

**Proof.** (Only if) Assume that $y(k, x_0) = 0$, a.s. $\forall k \geq 0$. Then we have $\ell (k, x_0) = 0$, $\forall k \geq 0$. Therefore, we conclude from Definition 6 that $0 \geq \gamma \|x_0\|_E$, namely, $x_0 = 0$ which is further equivalent to $x_0 = 0$.

(If) Since $y(k, x_0) = 0$, a.s. $\forall k \geq 0 \Rightarrow x_0 = 0$, $\forall x_0 \in \mathbb{R}^n$ is equivalent to $Y(k) \not\rightarrow 0 \Rightarrow \exists k_0 \geq 0$, $y(k, x_0) = 0$, a.s. Then we will show that there exist $\gamma > 0$ and $N_0 > 0$ such that (21) is satisfied for $Y(k) \not\rightarrow 0$. Indeed, let $N_0 \geq k_0 + 1$, we have $\ell (N_0, x_0) = \sum_{k=0}^{N_0-1} \text{tr} \{ Y(k) \} \geq \sum_{k=0}^{k_0} \text{tr} \{ Y(k) \} \geq \text{tr} \{ Y(k_0) \} > 0$, which clearly implies that there exists a $\gamma > 0$ such that (21) is satisfied.

**Lemma 12.** Let $\ell^2 > 0$, $\ell > 0$, be recursively determined by (10). Then $\ell^2 > 0 \iff \exists \ell > 0$ such that $\ell > 0$.

**Proof.** Note that $\ell > 0$ clearly implies that there exists an integer $l$ such that $\ell > 0$. In the following, we will prove the inverse, i.e., $\exists \ell > 0$ such that $\ell > 0 \Rightarrow \ell > 0$. We show this by contradiction. Assume that there exists a nonzero vector $\eta$ such that $\eta^T \ell \eta = 0$. Then we have (22), which, in view of Lemma 6, indicates $0 = \text{tr} \{ \ell \eta \eta^T \} = \ell (I, \eta \eta^T) = \eta^T \ell \eta$, $\forall t \geq 0$. That is to say, $\ell > 0$ is not strict positive definite for arbitrary $l \geq 0$ which is a contradiction.

**Remark 13.** When $C = 0$, $i \in T \subset N$, it has been shown in Dragan and Morozan (2006) that the stochastic linear system (1) is stochastic observable if and only if $\exists \ell > 0$ such that $\ell > 0$. The above lemma and Lemma 10 clearly generalize this result as we have considered general stochastic linear systems and provided a simpler condition $\ell^2 > 0$.

Now, by combining Lemmas 10–12, we have the following result.

**Theorem 14.** The following statements are equivalent.

1. The discrete-time stochastic linear system (1) is observable in the sense of Definition 6.
2. The discrete-time stochastic linear system (1) is observable in the sense of Definition 7.
3. The matrix $\ell^2$ is positive definite.
4. There exists an $l \geq 0$ such that $\ell > 0$ is positive definite.
5. For every eigenvector $0 \neq V \geq 0$ of $Z (l)$, there holds

$$
V \{ C_0^T C_1^T \cdots C_l^T \} \neq 0.
$$

**Proof.** (1)$\iff$(2) follows from Lemma 11, (1)$\iff$(3) follows from Lemma 10, (3)$\iff$(4) follows from Lemma 12 and (2)$\iff$(5) is the discrete-time version of Proposition 11 in Damm (2007).

**Remark 15.** Clearly, Item 3 of Theorem 14 is the most convenient way to check the observability of the discrete-time stochastic linear system (1). It turns out that the matrix $\ell^2$ takes the role of observability matrix in discrete-time stochastic linear system.

**Remark 16.** We note that a similar concept, namely, university observability of stochastic systems, is introduced in Dragan and Morozan (2006) by adopting the concept of universal observability for time-varying linear systems. However, as commented in Remark 13, our results generalize the results in that paper. Moreover, the main contribution of this subsection is not only to provide simpler condition to test observability but also to show the equivalence of some existing concepts of observability for discrete-time stochastic linear systems. Finally, we should point out that the concept of detectability defined in this paper is essentially different from that in Dragan and Morozan (2006) where the detectability is defined as the MS-stabilizability of its dual system.
4. Applications in stochastic Lyapunov and Riccati equations

4.1. Discrete-time stochastic Lyapunov equations

In this subsection, we consider the discrete-time stochastic Lyapunov matrix equation

\[ \sum_{k=0}^{N} A_k^T P A_k - P = - \sum_{i=0}^{N} C_i^T C_i, \]  \hspace{1cm} (24) \]

It will be clear that the detectability and observability concepts studied in the previous section take the similar functions in discrete-time stochastic linear systems as the detectability and observability concepts in deterministic time-invariant linear system theory. To present our main result, we need the following technical lemmas.

Lemma 17. Assume that \( A \) is mean square stable. Then the discrete-time stochastic Lyapunov matrix equation (24) has a unique solution \( P \geq 0 \). Moreover,

\[ \mathcal{N} (P) = \mathcal{N} (\varepsilon^T \varepsilon) = \bigcap_{i=1}^{l} \mathcal{N} (\varepsilon_i), \quad \forall l \geq n^2. \]  \hspace{1cm} (25) \]

Proof. Construct the following iteration

\[ P_{k+1} = \sum_{j=0}^{N} A_j^T P A_j + \sum_{j=0}^{N} C_j^T C_j, \quad k \geq 0, \]

with \( P_0 = 0 \). Since \( A \) is mean square stable, it follows from Lemma 3 that \( \sigma (A) \subset \mathbb{C}^\infty \). Therefore, iteration (26) converges, i.e., \( \lim_{k \to \infty} P_k = P_\infty \) which satisfies Eq. (24). Since \( P_k \geq 0 \), \( l = 1, 2, \ldots \), we conclude that \( P_\infty \geq 0 \). Moreover, the uniqueness of \( P_\infty \) is guaranteed by \( \sigma (A) \subset \mathbb{C}^\infty \).

To prove (25), we first prove

\[ \mathcal{N} (P) = \mathcal{N} (\varepsilon^T \varepsilon). \]  \hspace{1cm} (27) \]

Let \( \eta \in \mathcal{N} (\varepsilon^T \varepsilon) \) be a nonzero vector. Then \( 0 = \eta^T \varepsilon \eta = \text{tr} (\varepsilon^T \eta \varepsilon) = \eta^T \varepsilon \eta \), which, by using Lemma 6 implies that \( Y (j) = 0 \), \( \forall j \geq 0 \) which corresponds to the output process of system (6) with initial condition \( X_0 = \eta \varepsilon \). Comparing (10) with (26), we clearly see that \( \varepsilon_i = P_i, i \geq 0 \). Then we have

\[ \eta^T P \eta = \lim_{i \to \infty} \text{tr} (\varepsilon_i \eta \varepsilon_i) = \sum_{j=0}^{\infty} \text{tr} (Y (j)) = 0. \]

That is to say, \( \eta \in \mathcal{N} (\varepsilon^T \varepsilon) \), i.e., \( \mathcal{N} (\varepsilon^T \varepsilon) \subset \mathcal{N} (P) \). Now we prove \( \mathcal{N} (P) \subset \mathcal{N} (\varepsilon^T \varepsilon) \). To this end, we will show

\[ \varepsilon \not\in \mathcal{N} (\varepsilon^T \varepsilon), \quad \varepsilon \not\in \mathcal{N} (\varepsilon^T \varepsilon). \]

Note that \( \varepsilon \not\in \mathcal{N} (\varepsilon^T \varepsilon) \) implies

\[ 0 \neq \varepsilon^T \varepsilon \varepsilon \Rightarrow \text{tr} (\varepsilon \varepsilon) = \sum_{i=0}^{n^2-1} \text{tr} (Y (i)) \geq 0, \]  \hspace{1cm} (29) \]

where \( Y (j), \forall j \geq 0 \) corresponds to the output process of system (6) with initial condition \( X_0 = \varepsilon \varepsilon \). Relation (29) indicates that there exists at least one integer \( k_0 \in [0, n^2 - 1] \) such that \( \text{tr} (Y (k_0)) > 0 \). Consequently, it follows from (28) that \( \varepsilon^T \varepsilon \varepsilon \geq \sum_{i=0}^{k_0} \text{tr} (Y (j)) > 0 \), which indicates that \( \varepsilon \not\in \mathcal{N} (P) \). Therefore, Eq. (27) is true.

Finally, the relation \( \mathcal{N} (\varepsilon^T \varepsilon) = \bigcap_{i=1}^{l} \mathcal{N} (\varepsilon_i), \forall l \geq n^2 \) can be proven easily owing to Lemmas 5 and 6. The proof is completed.

Lemma 18. Assume that \( (A, C) \) is detectable. If the discrete-time stochastic Lyapunov matrix equation (24) admits a solution \( P \geq 0 \), then \( A \) is mean square stable.

Proof. According to Lemma 1, there exists a matrix \( 0 \neq V_0 \geq 0 \) such that \( \sum_{k=0}^{N} A_k V_0 A_k^T = \rho (\mathcal{L}_A) V_0 \), from which we obtain

\[ \text{tr} \left( \sum_{k=0}^{N} A_k^T P A_k - P \right) V_0 = \left( \rho (\mathcal{L}_A) - 1 \right) \text{tr} \left( P V_0 \right). \]

Then by using (24) and the above equation, we have

\[ \left( \rho (\mathcal{L}_A) - 1 \right) \text{tr} (P V_0) + \text{tr} \left( \sum_{i=0}^{N} C_i^T C_i V_0 \right) = 0. \]  \hspace{1cm} (30) \]

If \( \rho (\mathcal{L}_A) \geq 1 \), then (30) clearly implies that \( \text{tr} (\sum_{i=0}^{N} C_i^T C_i V_0) = 0 \) which implies \( C_i V_0 = 0, i \in \mathbb{N} \), i.e., the system \( (A, C) \) is not detectable. A contradiction. Therefore, we must have \( \rho (\mathcal{L}_A) < 1 \), that is to say, \( A \) is mean square stable according to Lemma 3.

The main result of this subsection is stated as follows:

Theorem 19. Consider the discrete-time stochastic linear system (1) and the associated Lyapunov matrix equation (24). Define the following statements: (a) \( A \) is mean square stable; (b) The Lyapunov matrix equation (24) admits a solution \( P \geq 0 \); (c) The Lyapunov matrix equation (24) admits a solution \( P > 0 \); (d) \( (A, C) \) is detectable, and (E) \( (A, C) \) is observable. Then

1. (b) \( \Rightarrow \) (a), (c) \( \Rightarrow \) (b), (a, b) \( \Rightarrow \) (c);
2. (E, T) \( \Rightarrow \) (a, b) \( \Rightarrow \) (c).

Proof. (1) Note that (b, c) \( \Rightarrow \) (a) follows from Lemma 18, (a, c) \( \Rightarrow \) (b) follows from Lemma 17. Since \( A \) is mean square stable, \( (A, C) \) is detectable for arbitrary \( C \), that is (c, b) \( \Rightarrow \) (c).

(2) Again, (E, T) \( \Rightarrow \) (a) results from Lemma 18. Based on Lemma 17, the Lyapunov matrix equation (24) has a unique solution \( P \geq 0 \) if \( A \) is mean square stable. Moreover, \( \mathcal{N} (P) = \mathcal{N} (\varepsilon^T \varepsilon) \). Since \( (A, C) \) is observable, according to Lemma 10, we know that \( \mathcal{N} (\varepsilon^T \varepsilon) = \emptyset \). Hence \( \mathcal{N} (P) = \emptyset \), i.e., \( P > 0 \). This proves (a, b) \( \Rightarrow \) (E). Finally, using Lemma 17 again, we clearly have \( (a, b) \Rightarrow \mathbb{T} \). The proof is done.

Remark 20. Theorem 19 clearly generalizes the results for the Lyapunov matrix equation \( A^T P A - P = -C^T C \) associated with the deterministic discrete-time linear system \( x (k+1) = Ax (k), y = Cx (k) \). Theorem 19 also extends the result given in Huang et al. (2006) where it is shown that \( (E, T) \Rightarrow (a, b) \Rightarrow \mathbb{E} \) for the case without noise terms in the output equation of system (1).

4.2. Discrete-time stochastic Riccati equations

The following algebraic Riccati equation

\[ \begin{align*}
\sum_{k=0}^{N} & A_k^T P A_k - P - \mathcal{U} R^{-1} \mathcal{U} + \sum_{i=0}^{N} C_i^T C_i = 0 \\
R & = R + \sum_{k=0}^{N} B_k^T P B_k > 0,
\end{align*} \hspace{1cm} (31) \]

where \( \mathcal{U} = \mathcal{U} (P) = \sum_{k=0}^{N} B_k^T P B_k \) and \( R > 0 \) is a given matrix, plays important functions in infinite horizon linear quadratic optimal regulator problem for the discrete-time stochastic linear system (1) (see, for instance, Huang et al. (2006) and Wonham (1968)). In this subsection, by using the detectability and observability concepts studied in Section 3, we give some interesting properties of the solutions of the Riccati equation (31).
Firstly, we prepare some notations. Let $P$ be a solution to (31), the associated feedback gain matrix is defined as

$$F = -R^{-1}U(P).$$

(32)

By using notations (3) and (32), it is not difficult to verify that the Riccati equation (31) can be equivalently rewritten as

$$\sum_{k=0}^{N} A_k^T(F)P A_k(F) - P = -F^T R F - \sum_{l=0}^{N} C_l^T C_l.$$  
(33)

Moreover, we say that $P$ is a stabilizing solution if the corresponding closed-loop system (1) with $u(k) = Fx(k)$ is mean square stable, or equivalently, $A_c(F)$ is mean square stable.

**Lemma 21.** Let $A_c(K) = (A_0^c(K), \ldots, A_N^c(K))$, $A_c^c(K) = A_0 + B_c K_i \in \mathbb{R}^{m \times n}$, $C_c(K) = (C_0, C_1, \ldots, C_N, R^2 K)$, where $R$ is an arbitrary positive definite matrix. Then the following statements hold true:

1. If $(A, C)$ is detectable, then $(A_c(K), C_c(K))$ is detectable for arbitrary $K \in \mathbb{R}^{m \times n}$.
2. If $(A, C)$ is observable, then $(A_c(K), C_c(K))$ is observable for arbitrary $K \in \mathbb{R}^{m \times n}$.

**Proof.** (1) Let $K \in \mathbb{R}^{m \times n}$ be a fixed matrix. The system $\Sigma_c$ refers to system (6) by replacing $(A, C)$ with $(A_c(K), C_c(K))$. For an initial condition $X_0 = x_0^0 x_0^0^T$, the symbols $X_k(k)$ and $Y_k(k)$ are respectively the state process and output process of $\Sigma_c$. Moreover, the symbol $\delta_k(l, X_0)$, $l \geq 1$ is defined as (8) associated with $\Sigma_c$. By Lemma 7, we need only to show that for any $X_0 \in \mathbb{R}^n$,

$$\delta_k(n, X_0) = 0 \Rightarrow \lim_{k \to \infty} \|X_k(k)\|_F = 0.$$  
(34)

With the help of Lemma 6, for arbitrary $l \geq 0$, we have $\delta_k(l, X_0) = 0$. Then

$$0 = \delta_k(l, X_0) = \sum_{k=0}^{l-1} \text{tr} \left( X_k(k) \left( \sum_{j=0}^{N} C_j^T C_j + R^2 K \right) \right)$$

$$= \sum_{k=0}^{l-1} \text{tr} \left( X_k(k) \left( \sum_{j=0}^{N} C_j^T C_j \right) \right) + \sum_{k=0}^{l-1} \text{tr} \left( X_k(k) K \right)$$

$$\geq 0,$$

from which we know that

$$\sum_{k=0}^{l-1} \text{tr} \left( X_k(k) \left( \sum_{j=0}^{N} C_j^T C_j \right) \right) = 0, \quad \forall l \geq 1,$$

(35)

and

$$R^2 K X_0 = 0, \quad \forall k \geq 0.$$  
(36)

In the following, we will show

$$X_k(m) = X(m), \quad \forall m \geq 0,$$  
(37)

by induction, where $X(k)$ denotes the solution of system (6) with initial condition $X_0 = x_0^0 x_0^0^T$. Obviously, for $m = 0$, we have $X_0(0) = X(0) = X_0$. Suppose that (37) is true with $m = k, k \geq 0$. Then when $m = k + 1$, by using (36), we obtain

$$X_k(k + 1) = \sum_{l=0}^{N} (A_l + B_l K) X_k(k) (A_l + B_l K)^T$$

$$= \sum_{l=0}^{N} A_l X_k(k) A_l^T$$

$$= X(k + 1),$$  
(38)

which indicates that (37) is true with $m = k + 1$. Eq. (37) follows from the induction principle.

Relations (35) and (38) clearly indicate

$$\delta (n^2, X_0) = \delta_k (n^2, X_0) = 0.$$  
(39)

Since $(A, C)$ is detectable, it follows from Lemma 7 that

$$\delta (n^2, X_0) \Rightarrow \lim_{k \to \infty} \|X(k)\|_F = 0.$$  

Relation (34) is then follows from (38) and (39).

(2) We show this by contradiction. Assume that there exists a $K \in \mathbb{R}^{m \times n}$ such that $(A_c(K), C_c(K))$ is not observable. Then there exists a nonzero vector $x_0$ such that $x_0^T \phi_l^c(K) x_0 = 0$ where $\phi_l^c(K)$ is defined as (10) for system $\Sigma_c$. Therefore, we have $\text{tr} \left( \phi_l^c(K) x_0 x_0^T \right) = 0$. Then similar to the proof of item 1 of this lemma, we can also show (39) from which we have $\delta (n^2, X_0) = 0$, i.e., $\text{tr} \left( x_0 x_0^T \right) = 0$. Thus, $(A, C)$ is not observable. A contradiction. ■

**Lemma 22.** If the discrete-time stochastic Riccati equation (31) has a stabilizing solution, then it must be unique.

**Proof.** Assume that there are two stabilizing solutions, namely, $P$ and $S$. Let $F$ be defined in (32) by replacing $P$ with $S$. By denoting $P_e = S - P$, after a tedious algebraic manipulation, we get

$$\sum_{k=0}^{N} (A_k^c(F))^T P_e A_k^c(F) - P_e = - (F)^T R F.$$  
(40)

where $\Delta F = F - S$. Note that (40) is in the form of (24). Since $A_c(F)$ is mean square stable by assumption, it follows from Lemma 17 that (40) has a unique semi-positive definite solution $P_e \geq 0$. Similarly, by changing the order of $P$ and $S$, we can also show that $P_e = S - P \geq 0$. Consequently, $P = S$, i.e., the stabilizing solution $P$ is unique. ■

The main result of this subsection is presented as follows.

**Theorem 23.** Assume that $(A, B)$ is stabilizable. Then the discrete-time stochastic Riccati equation (31) has a semi-positive definite solution $P \geq 0$. Moreover, we have the following statements.

1. If $(A, C)$ is detectable, then $P$ is a stabilizing solution and is also the unique semi-positive definite solution.

2. If $(A, C)$ is observable, then $P$ is a stabilizing solution and is also the unique positive definite solution.

**Proof.** Since $\sum_{k=0}^{N} C_k^T C_k \geq 0$ and $R > 0$, by using the same techniques used in obtaining Corollary 4 in [Rami and Zhou (2000)], we can show that the discrete-time stochastic Riccati equation (31) has a semi-positive definite solution $P \geq 0$.

(1) Since $(A, C)$ is detectable, it follows from Lemma 21 that $(A_c(F), C_c(F))$ is also detectable. As $(A_c(F), C_c(F))$ and $P$ satisfy Eq. (33), it follows from Lemma 18 that $A_c(F)$ is mean square stable. The uniqueness of $P$ follows from Lemma 22.

(2) Note that the observability of $(A, C)$ implies the detectability of $(A, C)$. Therefore, it follows from item 1 of this theorem that $P$ is a stabilizing solution. Again, Lemma 22 shows that $P$ is unique. Thus we need only to show $P > 0$. Since $(A, C)$ is observable, we conclude that $(A_c(F), C_c(F))$ is also observable by using Lemma 21. Once more, as $(A_c(F), C_c(F))$ and $P$ satisfy Eq. (33), it follows from the second relation in item 2 of Theorem 19 that $P$ is positive definite. ■

**Remark 24.** The above theorem generalizes several aspects of Theorem 6.9 in [Freiling and Hochhaus (2003)]. Item 1 of Theorem 23 clearly implies that result since our concept of detectability implies the concept of $d$-detectability as pointed out in [Remark 9]. Moreover, item 2 of Theorem 23 has given conditions to guarantee the existence of a strict positive definite solution to the Riccati equation (31). Theorem 23 can also be regarded as a generalization of the discrete-time version of Corollary 13.8 in [Zhou, Doyle, and Glover (1996)] to stochastic setting.
5. Conclusion

In this paper, we have studied the detectability and observability notions of discrete-time stochastic linear systems. These new definitions for stochastic systems not only unify some recently reported detectability notion and exact observability notion but also allow us to obtain a rank criterion to test observability of stochastic linear systems which can be regarded as a generalization of the well-known rank criterion for testing the observability of deterministic linear systems. Based on these two notions, the discrete-time stochastic Lyapunov matrix equation and Riccati equations are studied. The results presented are expected to find their applications in several related control problems for stochastic systems.

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References


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