A Stein Equation Approach for Solutions to the Bezout Identity and the Generalized Bezout Identity

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Abstract—This paper is concerned with the solutions to the Bezout identity and the generalized Bezout identity. Analytical solutions are obtained by solving a class of Stein equations with nilpotent coefficient matrices. As long as the Stein equations are solved by existing methods in a numerically stable way, all the elements of the Bezout identity and the generalized Bezout identity can be expressed explicitly. The problem of finding solutions to a class of generalized Sylvester equations is reconsidered to illustrate the application and efficiency of the proposed approach.

I. INTRODUCTION

Because of important applications of the Bezout identity and the generalized Bezout identity in system analysis and design (see [1]-[4]), solutions to these two identities have received a lot of attention in the past few years. Now many results have been reported in the literature (see [5]-[15] and the references therein). In [5]-[8], the authors have proposed different procedures to produce solutions to one-dimensional Diaphontine equation which plays an important role in design and synthesis of compensators. As a result, solving the Bezout identity can be seen as a special case of solving the Diaphontine equation. Fang proposed in [9] explicit formulas to express the elements of the the generalized Bezout identity in closed form. By using the concepts of state feedback and proportional plus derivative state feedback respectively, in closed form. By using the concepts of state feedback and proportional plus derivative state feedback respectively, Fang proposed new simple methods in [10] and [11] to calculate the coprime matrix fraction description (CMFD) problem of linear systems and the generalized Bezout identity. In [12], Basilio proposed algorithms for the computation of all matrices of the generalized Bezout identity based on the computation of minimal polynomial basis for the right null spaces of certain polynomial matrices and singular value decompositions of certain real matrices. Recently, we proposed a Stein equation approach for computing CMFD in [13], which can be applied to solve the Bezout identity and the generalized Bezout identity effectively. The CMFD problems and solutions to the generalized Bezout identity of descriptor linear systems were also considered in [14] and [15].

The problem considered in this paper is to present simple and analytical solutions to the Bezout identity and the generalized Bezout identity for linear systems. By adopting the basic ideas in [10], [11] and [13], the problem is transformed into solving a class of Stein equations with nilpotent coefficient matrices. The problem of finding solutions to a class of generalized Sylvester equations is reconsidered to illustrate the application and efficiency of the proposed approach.

The remainder of this paper is organized as follows. In Section II, the problem formulation and some preliminaries are given. Section III contains the main results of this paper. A numerical example is given in Section IV to illustrate the proposed method. Section V concludes the paper.

Notations: Denote $X^T$ the transpose of any matrix $X$, $R^{n \times m}[s]$ the set of all polynomial matrices of dimension $n \times m$, $U^{n \times n}[s]$ the set of all unimodular polynomial matrices of dimension $n \times n$ and $N^{n \times n}_h = \{N \in R^{n \times n} : N^h = 0 \text{ and } N^{h-1} \neq 0\}$.

II. PROBLEM FORMULATION AND PRELIMINARIES

Two polynomial matrix pairs $(X(s), Y(s)) \in (R^{n \times m}[s] \times R^{m \times n}[s])$ and $(H(s), L(s)) \in (R^{n \times n}[s] \times R^{n \times m}[s])$ are respectively said to be a right coprime polynomial matrix pair and a left coprime polynomial pair if

$$\text{rank} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = m, \text{rank} \begin{bmatrix} H(s) & L(s) \end{bmatrix} = n, \forall s \in C.$$

The problems we are interested in this paper are stated as follows.

Problem 1: Find a polynomial matrix pair $(P(s), Q(s)) \in (R^{n \times m}[s] \times R^{m \times n}[s])$ such that

$$(sI - A) P(s) + BQ(s) = I,$$  \hspace{1cm} (1)

where $A \in R^{n \times n}, B \in R^{n \times m}$ and $(A, B)$ is a given controllable matrix pair.

Equation (1) is known as the Bezout identity.

Problem 2: Find the following four polynomial matrix pairs $(N(s), D(s))$, $(\overline{N}(s), \overline{D}(s))$, $(X(s), Y(s))$, $(X(s), Y(s))$ such that

$$\overline{N}(s) D^{-1}(s) = \overline{D}^{-1}(s) \overline{N}(s) = C (sI - A)^{-1} B,$$  \hspace{1cm} (2)

$$\begin{bmatrix} \overline{D}(s) & \overline{N}(s) \\ -\overline{X}(s) & \overline{Y}(s) \end{bmatrix} \begin{bmatrix} \overline{Y}(s) & -N(s) \\ \overline{X}(s) & D(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$  \hspace{1cm} (3)

where $A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}$ are given constant matrices and the linear system $(A, B, C)$ is controllable and observable.
Polynomial matrix pairs \((N(s), D(s))\) and \((\overline{N(s)}, \overline{D(s)})\) satisfying (2) are called doubly CMFD of the system \((A, B, C)\) and (3) are well-known as the generalized Bezout identity.

III. MAIN RESULTS

A. Stein Equations and Solutions to the Problems

At first we introduce a lemma which generalizes Theorem 2 in [11] and Theorem 1 in [13].

**Lemma 1:** Let \((A, B)\) be controllable. Then there exist two constant matrices \(K_1\) and \(K_2\) such that

\[
U(s) = (sI - A + sBK_2 + BK_1)^{-1},
\]

is unimodular with degree \(\omega\), i.e., \(U(s) \in U^{n \times n} [s]\) and \(\text{deg} (U(s)) = \omega\) if and only if there exists a constant matrix \(K\) such that \(A - BK\) is invertible and a matrix \(Y_1\) such that the following Stein equation

\[
AX_1N + X = BY_1,
\]

where \(N \in \mathbb{N}^{n \times n}_{\omega+1}\) has a nonsingular solution \(X_1\). Moreover, for \(i = 0, 1, \ldots, \omega\), the coefficient matrices of \(U(s)\) are expressed as follows

\[
U_i = (-1)^i X_1 N^i X_1^{-1} (-A + BK)^{-1},
\]

\[
K_1 = K,\ K_2 = (K X_1 N - Y_1) X_1^{-1}.
\]

Moreover, the matrix pair \((P(s), Q(s))\) given by

\[
P(s) = U(s),\ Q(s) = (sK_2 + K_1)U(s)
\]

satisfies (1).

**Proof:** Proof of (4)-(7) can be found in [13]. Here we only need to prove (8). Substituting (8) into (1) gives

\[
(sI - A) P(s) + BQ(s) = (sI - A) U(s) + B(sK_2 + K_1)U(s) = (sI - A + sBK_2 + BK_1)U(s) = I,
\]

which completes the proof.

Then we can get the following solution to Problem 1.

**Theorem 1:** Let \((A, B)\) be controllable. Assume that \(Y_1\) is chosen such that the solution \(X_1\) to the Stein equation (5) is nonsingular. Then the matrix pair \((P(s), Q(s))\) with coefficients given by

\[
Q_0 = -K (A - BK)^{-1},
\]

and

\[
\begin{bmatrix}
P_i \\
Q_{i+1}
\end{bmatrix} = \begin{bmatrix}(-1)^i X_1 N^i \\
(-1)^i Y_1 N^i
\end{bmatrix} M,\ i = 0, 1, \ldots, \omega,
\]

satisfies (1), where

\[
M = X_1^{-1} (A - BK)^{-1},\ K \in \{K: \text{det} (A - BK) \neq 0\}.
\]

**Proof:** The expression of \(P_i\) follows directly from (6) and (8). To show the expressions for \(Q_i, i = 0, 1, \ldots, \omega\), by substituting (6) into

\[
Q(s) = (K_1 + sK_2) U(s),
\]

and equating the coefficients of \(s^i, i = 0, 1, \ldots, \omega + 1\), on both sides of (12), we get

\[
\begin{cases}
K_1 U_0 = Q_0, \\
K_1 U_i + K_2 U_{i-1} = Q_i, i = 1, 2, \ldots, \omega, \\
K_2 U_\omega = Q_\omega + 1.
\end{cases}
\]

Notice that \(Q_0\) in (13) can be obtained by substituting \(U_0\) into \(Q_0\) in (13). For \(i = 1, 2, \ldots, \omega\), we have

\[
Q_i = K_1 U_i + K_2 U_{i-1} = K (-1)^i X_1 N^i X_1^{-1} (-A + BK)^{-1} + \\
K_2 (-1)^i X_1 N^i X_1^{-1} (-A + BK)^{-1} = (-1)^{i-1} Y_1 N^{i-1} M,
\]

where we have used the expression (7). Furthermore, by using (7) again and noticing the fact that \(N^{\omega+1} = 0\), we get

\[
Q_{\omega+1} = K_2 (-1)^\omega X_1 N^\omega X_1^{-1} (-A + BK) = (-1)^{\omega+1} (K X_1 N - Y_1) N^\omega M
\]

\[
= (-1)^\omega Y_1 N^\omega M.
\]

Equations (14) and (15) can be unified as (10). The proof is completed.

The following corollary is the dual result of Lemma 1.

**Corollary 1:** Let \((A, C)\) be observable. Then there exist two constant matrices \(L_1\) and \(L_2\) such that

\[
H(s) = (sI - A + sL_2 C + L_1 C)^{-1},
\]

is unimodular with degree \(\omega\), i.e., \(H(s) \in U^{n \times n} [s]\) and \(\text{deg} (H(s)) = \omega\) if and only if there exist a constant matrix \(L\) such that \(A - LC\) is invertible and a matrix \(Y_2\) such that the following Stein equation

\[
A^T X_2 N + X_2 = C^T Y_2,
\]

where \(N \in \mathbb{N}^{n \times n}_{\omega+1}\) has a nonsingular solution \(X_2\). Moreover, for \(i = 0, 1, \ldots, \omega\), the coefficient matrices of \(H(s)\) are given by

\[
H_i = (-1)^i (-A + LC)^{-1} X_2^T (N^T)^i X_2^T,
\]

\[
L_1 = L,\ L_2 = X_2^T (N^T X_2^T L - Y_2^T).
\]

**Proof:** We only show the expressions (18) and (19).

Since \(H(s)\) is unimodular with degree \(\omega\), so is \(H^T(s)\), i.e., \(H^T(s) \in U^{n \times n} [s]\) and \(\text{deg} (H^T(s)) = \omega\). Then it follows from (16) that

\[
(sI - A^T + sC^T L_2^T + C^T L_1^T) \left( \sum_{i=0}^{\omega} \overline{P_i} s^i \right) = I,
\]

where \(\overline{P_i} = H_i^T\). Now both sides of the above equation are matrix polynomials in \(s\). Comparing with the coefficient matrices of \(s^i, i = 0, 1, \ldots, \omega\), gives

\[
\begin{cases}
(-A^T + C^T L_1^T) \overline{P}_0 = I, \\
(I + C^T L_2^T) \overline{P}_0 + (-A^T + C^T L_1^T) \overline{P}_1 = 0, \\
(I + C^T L_2^T) \overline{P}_1 + (-A^T + C^T L_1^T) \overline{P}_2 = 0, \\
\vdots \\
(I + C^T L_2^T) \overline{P}_{\omega-1} + (-A^T + C^T L_1^T) \overline{P}_\omega = 0,
\end{cases}
\]

\[
(I + C^T L_2^T) \overline{P}_\omega = 0.
\]
Therefore, there exist two matrices $L_1$ and $L_2$ such that $H(s) \in U^{n \times n}[s]$ and $\deg (H(s)) = \omega$ if and only if (20) has solutions $\hat{H}_i$, $i = 0, 1, \ldots, \omega$, where $\hat{H}_\omega \neq 0$. Adopting a similar procedure used in the proof of Theorem 1 in [13] gives
\[
\hat{H}_i = (-1)^i X_2 N_i X_2^{-1} \left(-A^T + C^T L_1^T\right)^{-1}, i = 0, 1, \ldots, \omega.
\]
Denote $L_1 = L$. By noticing the fact that $(N_i)^T = (N^T)^i$, we get
\[
H_i = \hat{H}_i = (-1)^i (-A + LC)^{-1} X_2^{-T} (N_i^T)^i X_2^{T}.
\]

The proof is completed.

**Remark 1:** The nilpotent matrices in the Stein equations (5) and (17) can be different from each other, but here we assume that they are the same to simplify the expressions.

Next we introduce a lemma which is a generalization of the main results in [11]. The proof is straightforward and omitted here.

**Lemma 2:** For a controllable and observable linear system $(A, B, C)$. Assume that there exist four constant matrices of appropriated dimensions $K_1, K_2, L_1, L_2$ such that $U(s)$ and $H(s)$ in (4) and (16) are unimodular with the same degree $\omega$, then the matrices satisfying (2) and (3) can be constructed as follows:
\[
\begin{bmatrix}
N(s) \\
D(s)
\end{bmatrix} = \begin{bmatrix}
CU(s)B \\
I - (sK_2 + K_1)U(s)B
\end{bmatrix},
\]
\[
\begin{bmatrix}
\check{X}(s) \\
\check{Y}(s)
\end{bmatrix} = \begin{bmatrix}
(sK_2 + K_1)U(s)(sL_2 + L_1) \\
I + CU(s)(sL_2 + L_1)
\end{bmatrix},
\]
\[
\begin{bmatrix}
\check{N}(s) \\
\check{D}(s)
\end{bmatrix} = \begin{bmatrix}
CH(s)B \\
I - CH(s)(sL_2 + L_1)
\end{bmatrix},
\]
\[
\begin{bmatrix}
\check{X}(s) \\
\check{Y}(s)
\end{bmatrix} = \begin{bmatrix}
(sK_2 + K_1)H(s)(sL_2 + L_1) \\
I + (sK_2 + K_1)H(s)B
\end{bmatrix}.
\]

Then we can present the following result regarding solutions to Problem 2.

**Theorem 2:** Let the linear system $(A, B, C)$ be controllable and observable. Assume that $Y_1$ and $Y_2$ are chosen such that the solutions $X_1$ and $X_2$ to the Stein equations (5) and (17) respectively are nonsingular. Then the coefficient matrices of matrix pairs $(N(s), D(s))$, $(\check{N}(s), \check{D}(s))$, $(\check{X}(s), \check{Y}(s))$, $(\check{X}(s), \check{Y}(s))$ satisfying (2)-(3) are given by
\[
\begin{aligned}
D_0 &= I + K (A - BK)^{-1} B, \\
\check{D}_0 &= I + C (A - LC)^{-T} L,
\end{aligned}
\]
\[
\begin{aligned}
\check{Y}_0 &= I - K (A - LC)^{-T} B,
\check{X}_0 &= -K (A - LC)^{-T} L,
\check{X}_1 &= K \check{M} Y_2^{T} - K_2 (A - LC)^{-T} L,
\check{X}_0 &= -K (A - BK)^{-T} L,
\check{X}_1 &= Y_1 ML - K (A - BK)^{-T} L_2,
\end{aligned}
\]
and for $i = 0, 1, \ldots, \omega$,
\[
\begin{bmatrix}
N_i \\
D_{i+1}
\end{bmatrix} = -\begin{bmatrix}
(1)^i C X_1 N_i^{T} \\
(1)^i Y_1 N_i\end{bmatrix} MB, \quad (19)
\]
\[
\begin{bmatrix}
\check{X}_{i+2} \\
\check{Y}_{i+1}
\end{bmatrix} = \begin{bmatrix}
(1)^i K (N^T)^i Y_2^{T} \\
(1)^{i+1} K (N^T)^i X_2^{T} B
\end{bmatrix}, \quad (20)
\]
\[
\begin{bmatrix}
\check{N}_i \\
\check{D}_{i+1}
\end{bmatrix} = -C \check{M} \begin{bmatrix}
(1)^i (N^T)^i X_2^{T} B \\
(1)^{i+1} (N^T)^i Y_2^{T}
\end{bmatrix}, \quad (21)
\]
\[
\begin{bmatrix}
\check{X}_{i+2} \\
\check{Y}_{i+1}
\end{bmatrix} = \begin{bmatrix}
(1)^i Y_1 N_i \check{M} \\
(1)^{i+1} C X_1 N_i \check{M}
\end{bmatrix}, \quad (22)
\]
where
\[
\check{K} = K_2 \check{M} - K \check{M} N^T, \quad \check{M} = ML_2 - NML,
\]
in which
\[
M = X_1^{-1} (A - BK)^{-1}, \quad \check{M} = (A - LC)^{-T} X_2^{-T},
\]
$K$ and $L$ are chosen such that $A - BK$ and $A - LC$ are invertible, $K_2$ and $L_2$ are chosen as in (7) and (19) respectively.

**Proof:** For simplicity, we only show the expression for $\check{X}_1$. Calculating the expression of $X(s)$ in (24) gives
\[
\check{X}(s) = (sK_2 + K_1)H(s)(sL_2 + L_1) = s^2 K_2 H(s)L_2 + s(K_2 H(s)L + KH(s)L_2) + KH(s)L_2.
\]

We see that both sides of the above equation are matrix polynomials in $s$. By comparing the coefficients of $s^i, i = 0, 1, \ldots, \omega + 2$, we obtain the following series of equations
\[
\begin{aligned}
\check{X}_0 &= KH_0 L, \\
\check{X}_1 &= KH_1 L + K_2 H_0 L + KH_0 L_2, \\
\check{X}_2 &= K_2 H_2 L + K_2 H_1 L + KH_1 L_2 + K_2 H_0 L_2,
\end{aligned}
\]
\[
\begin{aligned}
\check{X}_{\omega+1} &= K_2 H_{\omega+1} L + KH_{\omega+1} L_2 + K_2 H_{\omega-1} L_2,
\check{X}_{\omega+2} &= K_2 H_{\omega+2} L_2.
\end{aligned}
\]

The rest of the proof is similar to the proof of Theorem 1 and is omitted here for simplicity.

**Remark 2:** Theorem 1 and Theorem 2 generalize several issues of those given in [9]-[13].

1) Theorem 1 and Theorem 2 allow to produce a class of solutions to Problem 1 and Problem 2 based on numerically reliable solutions to the Stein equations as well as reducing the computation burden in practical applications (see [13]).

2) Theorem 1 gives explicit solutions to the Bezout identity, which is directly applicable to construct parametric solutions to a class of generalized Sylvester equations ([27]) (see the discussion in the next subsection).

3) Reference [13] only provides solutions to the CMFMD problem under the assumption that $C = I$ while Theorem 2 gives solutions to not only doubly CMFMD problem but also the generalized Bezout identity without such restriction.
B. Solutions to a Class of Generalized Sylvester Equations

To illustrate the application of the proposed approach, in this subsection, we consider the solutions to a class of generalized Sylvester equations, which have wide applications in many control problems such as pole/eigenstructure assignment design, Luenberger-type observer design and robust fault detection for linear systems and descriptor linear systems (see [16]-[26] and the references therein). The equation we considered is as follows

\[ AV - VF = BW + R, \]  

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \), \((A, B)\) is a known controllable matrix pair, \( F \) and \( R \) are two arbitrarily known matrices, and \((V, W)\) is a matrix pair to be determined.

We give the following result regarding solutions to (30).

**Lemma 3:** ([27]) Assume that \((A, B)\) is controllable. Let \((N(s), D(s))\) and \((P(s), Q(s))\) be two pairs of polynomial matrices such that

\[(sI - A)N(s) - BD(s) = 0, \]
and

\[(sI - A)P(s) + BQ(s) = I, \]

are satisfied. Moreover, \((N(s), D(s))\) is right coprime. Then all the solutions to (30) can be characterized as

\[
\begin{bmatrix}
V \\
W
\end{bmatrix} = \sum_{i=0}^{\omega_1} \left[ N_i \right] ZF^i + \sum_{k=0}^{\omega_2} \left[ P_k, Q_k \right] RF^k, \tag{31}
\]

where \( Z \in \mathbb{R}^{m \times n} \) is any parameter matrix and

\[
\omega_1 = \max \{ \deg (N(s)), \deg (N(s)) \}, \]
\[
\omega_2 = \max \{ \deg (P(s)), \deg (Q(s)) \}.
\]

**Proposition 1:** Assume that \((A, B)\) is controllable. Then all the solutions to (30) can be characterized as

\[
V = \sum_{i=0}^{\omega} (-1)^{i+1} X_1 N_i M (R + BZ) F^i, \tag{32}
\]

\[
W = W_0 + \sum_{i=1}^{\omega} (-1)^{i-1} Y_1 N^{i-1} M (R + BZ) F^i, \tag{33}
\]

where \( Z \in \mathbb{R}^{m \times n} \) is any parameter matrix, \( \omega = \deg (N(s)), \)

\[
W_0 = -Z - K (A - BK)^{-1} (R + BZ),
\]

and \((K, X_1, Y_1, N, M)\) are the matrices satisfying the conditions in Theorem 2.

**Proof:** It directly follows from Lemma 3 that \( V \) and \( W \) can be described as follows:

\[
V = \sum_{i=0}^{\deg (P(s))} P_i RF^i + \sum_{i=0}^{\deg (N(s))} N_i ZF^i, \tag{34}
\]

\[
W = \sum_{i=0}^{\deg (Q(s))} Q_i RF^i - \sum_{i=0}^{\deg (D(s))} D_i ZF^i. \tag{35}
\]

Then by setting \( C = I \), we deduce from Theorem 1 and Theorem 2 that

\[
V = \sum_{i=0}^{\omega} P_i RF^i + \sum_{i=0}^{\omega} N_i ZF^i, \tag{36}
\]

\[
W = \sum_{i=0}^{\omega+1} Q_i RF^i - \sum_{i=0}^{\omega+1} D_i ZF^i. \tag{37}
\]

Substituting the coefficients of \((P(s), Q(s)), (N(s), D(s)), \) i.e., (9)-(11), (25) and (26) into (36) and (37) gives (32) and (33). The proof is completed.

**Remark 3:** In [27], we only show the existence of \((N(s), D(s))\) and \((P(s), Q(s))\). However, as soon as the Stein matrix equation (5) is solved by existing algorithms (see for example [13]), solutions to \((N(s), D(s)), \)

\[(P(s), Q(s))\) and (30) can be given explicitly.

**Remark 4:** Regarding \( K \), we can find a \( K \) such that \( A - BK \) is well conditioned. \( F \) and \( R \) are determined by practical problems and are not chosen in advance (see, for example, [23]). In the following illustrative example, we only prescribe them to illustrate the method. Finally, regarding \( Y_1 \), we can simply choose it such that \((N, Y_1)\) is observable ([13]).

IV. ILLUSTRATIVE EXAMPLE

In this section, we use an example to illustrate the application and efficiency of the proposed approach. Consider a controllable and observable linear system with the following data:

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}, B = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}, C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

We first use Theorem 1 and Theorem 2 to compute matrix pairs \((P(s), Q(s))\) and \((N(s), D(s))\) for system \((A, B, C)\) with \( \deg (D(s)) = 2 = \omega + 1 \). In this case, we let the nilpotent matrix \( N \) take the form

\[
N = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Denote \( Y_1 = [y_{ij}] \). Solving the Stein matrix equation (5) gives

\[
X_1 = \begin{bmatrix}
0 & -y_{21} & 0 \\
y_{21} & y_{22} - y_{11} & y_{23} \\
y_{11} & y_{12} - y_{11} & y_{13}
\end{bmatrix}.
\]

It follows that \( X_1 \) is nonsingular if and only if \( y_{21} \neq 0 \) and \( y_{21}y_{13} \neq y_{11}y_{23} \). And we specialize \( Y_1 \) as

\[
Y_1 = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix},
\]

and simply choose \( K = 0 \) to guarantee the nonsingularity of \( A - BK \). Then we can compute \((P(s), Q(s)), (N(s), D(s))\) according to (9)-(11), (25) and (26) as follows...
We next consider explicit solutions to the generalized Sylvester equation (30). Let $F$ and $R$ be chosen as

\[
F = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix},
\quad R = \begin{bmatrix}
1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Then it follows from Proposition 1 that the closed-form solutions to (30) can be characterized as

\[
V = \begin{bmatrix}
z_{21} - z_{11} & z_{22} - z_{12} - 1 \\
z_{12} - z_{22} - 1 & z_{22} - z_{11} - 2 \\
-z_{21} & -z_{22} \\
z_{23} - z_{13} - 1 & z_{22} + z_{23} - z_{12} - z_{13} - 1 \\
-z_{23}
\end{bmatrix},
\]

\[
W = \begin{bmatrix}
z_{21} - z_{11} & z_{22} - z_{12} - 1 & z_{23} - z_{13} \\
z_{12} - 2z_{22} - 1 & z_{12} - 2z_{12} + 2 & z_{13} - 2z_{23} + 1
\end{bmatrix},
\]

in which $Z = [z_{ij}]$ is an arbitrary real matrix representing the degree of freedom in the solution and can be used to achieve some other purposes in system analysis and design.

V. Conclusion

In this paper, the problems of finding analytical solutions to the Bezout identity and the generalized Bezout identity are considered. By using the proportional plus derivative state feedback, the problems are transformed into solving a class of Stein equations, whose solutions can be obtained by analytical algorithms or numerically reliable algorithms existing in the literature. As a direct application of the proposed approach, explicit solutions to a class of generalized Sylvester equations are obtained. The approach allows one to obtain the solutions while maintaining the degree of freedom, and they can be further utilized to achieve some other control objectives.

References